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## Some theorems on a system of matrices and a geometrical application

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# SOME THEOREMS ON A SYSTEM OF MATRICES AND A GEOMETRICAL APPLICATION

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In this paper, we shall prove some theorems on a system of  $(n, m)$ -matrices and apply them to the proof of a theorem on isometrical imbedding of Riemann spaces in Euclidean spaces.

1. Let  $\mathfrak{M}_{n,m}$  be the set of  $(n, m)$ -matrices. We define a multiplication as follows:

$$\mathfrak{M}_{n,m} \ni M_1, M_2, \quad M_1 \circ M_2 = M_1 M_2' (\in \mathfrak{M}_{n,n}),$$

where  $M_2'$  denotes the transposed matrix of  $M_2$ . Let us denote the full linear group and the orthogonal group in  $n$  variables by  $L(n)$  and  $O(n)$  respectively. We will say that  $M_1$  and  $M_2$  is commutative if  $M_1 \circ M_2 = M_2 \circ M_1$ .

**Lemma 1.** *If  $M_1$  and  $M_2$  in  $\mathfrak{M}_{n,m}$  is commutative, then  $AM_1B$  and  $AM_2B$  is also commutative for any  $A \in L(n)$ ,  $B \in O(n)$ .*

The proof is evident from the definition of the multiplication of  $M_1$  and  $M_2$ .

Let  $\varphi_{A,B}: \mathfrak{M}_{n,m} \rightarrow \mathfrak{M}_{n,m}$  ( $A \in L(n)$ ,  $B \in L(m)$ ) be a linear transformation of  $\mathfrak{M}_{n,m}$  onto itself such that  $\varphi_{A,B}(M) = AMB$ ,  $M \in \mathfrak{M}_{n,m}$ . Then we have a corollary.

**Corollary.**  *$\varphi_{A,B}$  ( $A \in L(n)$ ,  $B \in O(m)$ ) is a linear homeomorphism on  $\mathfrak{M}_{n,m}$  preserving the commutativity with respect to the multiplication  $M_1 \circ M_2$ .*

Now, we will say that a linear subspace  $\mathfrak{N}$  of  $\mathfrak{M}_{n,m}$  is an  $\alpha$ -system if any two elements of  $\mathfrak{N}$  are commutative. Our main theorem is as follows:

**Theorem 1.** *If  $\mathfrak{N}$  is an  $\alpha$ -system in  $\mathfrak{M}_{n,m}$ , then  $\dim \mathfrak{N} \leq m$ .*

*Proof.* We shall prove inductively the theorem. In order to prove the theorem, it is sufficient that if we can prove the following propositions depending on positive integers  $n, m$ .

$(q_{n,m})$ : *Let  $\mathfrak{N}$  be any  $\alpha$ -system in  $\mathfrak{M}_{n,m}$ . If  $M_1, M_2, \dots, M_h \in \mathfrak{N}$ ,  $h \leq m$ , are linearly independent, then*

$$\text{rank} \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_h \end{pmatrix} \geq h, \quad (n, m = 1, 2, 3, \dots).$$

$(p_{n,m})$ : The dimension of any  $\alpha$ -system in  $\mathfrak{M}_{n,m} \leq m$ .

Since  $\mathfrak{M}_{1,m}$  is an  $m$ -dimensional vector space and our product of  $M_1, M_2 \in \mathfrak{M}_{1,m}$  is the inner product of  $M_1, M_2$ , the propositions  $(q_{1,m}), (p_{1,m}), m = 1, 2, 3, \dots$  are evident. We suppose that the propositions  $(q_{j,m}), (p_{j,m}) j = 1, 2, \dots, n-1, m = 1, 2, \dots$  are true.

Since  $\mathfrak{M}_{n,1}$  is an  $n$ -dimensional vector space, it is evident that the propositions  $(q_{n,1}), (p_{n,1})$  hold good.

Now, we suppose that the propositions  $(q_{n,1}), (p_{n,1}), \dots, (p_{n,m-1}), (q_{n,m})$  have already been proved.

*Proof of Proposition  $(p_{n,m})$ .* Let  $\mathfrak{N}$  be an  $\alpha$ -system in  $\mathfrak{M}_{n,m}$ . Let  $M_1, M_2, \dots, M_m, M_{m+1}$  be a set of elements of  $\mathfrak{N}$  such that any  $m$  elements of them are linearly independent. We shall prove that  $M_1, M_2, \dots, M_m, M_{m+1}$  are linearly dependent.

Taking any transformation

$$N_i = \sum_{j=1}^{m+1} c_{ij} M_j, \quad i = 1, 2, \dots, m+1,$$

where  $|c_{ij}| \neq 0$ , we may investigate the linear dependency of  $N_1, \dots, N_m, N_{m+1}$  instead of the one of  $M_1, M_2, \dots, M_{m+1}$ . Accordingly, we may consider that  $M_{m+1}$  is of the form  $\begin{pmatrix} 0 & \dots & 0 \\ * \end{pmatrix}$ . The relation  $M_i \circ M_{m+1} = M_{m+1} \circ M_i (i = 1, 2, \dots, m)$  implies that if we consider each row of any element of  $\mathfrak{M}_{n,m}$  indicates a vector in an  $m$ -dimensional Euclidean vector space, then the first rows of  $M_1, \dots, M_m$  are orthogonal to each rows of  $M_{m+1}$ . Since  $M_{m+1} \neq 0$ , the first rows of  $M_1, \dots, M_m$  lie in an  $(m-1)$ -dimensional linear subspace in the  $m$ -dimensional Euclidean vector space. Hence, we may also put the first row of  $M_m$  is the zero vector. The first rows of  $M_1, \dots, M_{m-1}$  are orthogonal to each rows of  $M_m, M_{m+1}$  by the same reason above. By virtue of  $(q_{n,m})$ ,  $\text{rang} \begin{pmatrix} M_m \\ M_{m+1} \end{pmatrix} \geq 2$ . Accordingly, the first rows of  $M_1, \dots, M_{m-1}$  lie in an  $(m-2)$ -dimensional linear subspace in the  $(m-1)$ -dimensional Euclidean vector space above. Hence, we may also put the first row of  $M_{m-1}$  is the zero vector. By virtue of  $(q_{n,m})$ , we can repeat this process, hence we may con-

sider the first rows of  $M_1, \dots, M_{m+1}$  are the zero vector. If we perform inductively the same consideration with respect to  $n$ , we see that we may put  $M_{m+1} = 0$ . This contradicts to  $M_{m+1} \neq 0$ . The proof of  $(p_{n,m})$  is complete.

*Proof of Proposition  $(q_{n,m+1})$ .* Let  $\mathfrak{N}$  be any  $\alpha$ -system in  $\mathfrak{M}_{n,m+1}$ . Let us suppose that  $M_1, \dots, M_h \in \mathfrak{N}$ ,  $h \leq m+1$ , are linearly independent and

$$\text{rank} \begin{pmatrix} M_1 \\ \vdots \\ M_h \end{pmatrix} = r < h.$$

Then, if we take a suitable  $B \in O(m+1)$  and we transform  $\mathfrak{N}$  to  $\varphi_{1,n}(\mathfrak{N}) = \mathfrak{N}B$ , then we may put  $M_i B = (N_i \ 0)$  where  $N_i \in \mathfrak{M}_{n,r}$ . Hence we may put  $M_i = (N_i \ 0)$  from the beginning. Then,  $N_1, \dots, N_h$  belong to an  $\alpha$ -system in  $\mathfrak{M}_{n,r}$ . Since  $r \leq m$ , the proposition  $(p_{n,r})$  holds good by induction and the proposition  $(p_{n,m})$  which has been proved above. Accordingly, it follows that  $N_1, N_2, \dots, N_h$  are linearly dependent. Hence  $M_1, M_2, \dots, M_h$  must be also linearly dependent. This contradicts to the assumption above. Thus Proposition  $(q_{n,m+1})$  holds good. The proof of Theorem 1 is complete.

2. Let  $\mathfrak{V}$  be a vector space over the real field  $\mathfrak{k}$ . An exterior form of order 2 on  $\mathfrak{V}$  is defined by a form  $\sum_{i,j=1}^r \alpha_{ij} u_i \wedge u_j$ , where  $\alpha_{ij} = -\alpha_{ji} \in \mathfrak{V}$  and  $u_i$  are real variables. We define  $\sum \alpha_{ij} u_i \wedge u_j = \sum \beta_{ij} v_i \wedge v_j$  if  $u_i = \sum a_{ik} v_k$ ,  $a_{ik} \in \mathfrak{k}$ ,  $|a_{ik}| \neq 0$  and  $\sum_{i,j} \alpha_{ij} a_{ik} a_{jh} = \beta_{kh}$ ,  $h, k = 1, 2, \dots, r$ . Then, we have a lemma as a simple generalization of the ordinary one as follows.

**Lemma 2.** *Let  $k$  be the minimum of numbers of variables such that an exterior form  $\sum_{i,j=1}^r \alpha_{ij} u_i \wedge u_j$  of order 2 can be expressed by them, then  $r - k$  is the dimension of the set of solutions of the system of linear equations over  $\mathfrak{V}$  with respect to real variables  $x_1, \dots, x_r$ ;  $\sum_{j=1}^r \alpha_{ij} x_j = 0$ ,  $i = 1, 2, \dots, r$ .*

We can easily prove the lemma in the same manner as the ordinary case in which  $\mathfrak{V}$  is  $\mathfrak{k}$ .

Now let  $\mathfrak{M} \subset \mathfrak{M}_{n,m}$  be a linear subspace. Let  $M_1, \dots, M_r$  be a base of  $\mathfrak{M}$  and put  $N_{ij} = M_i \circ M_j - M_j \circ M_i (\in \mathfrak{M}_{n,n})$ . We construct

an exterior form of the vector space  $\mathfrak{M}_{n,n}$  over the real field  $\mathfrak{k}$  such that

$$\phi(\mathfrak{M}, u) = \sum N_{ij} u_i \wedge u_j.$$

This form is dependent on  $\mathfrak{M}$  but independent on the choice of bases. For, we have

$$(\sum x_i M_i) \circ (\sum y_j M_j) - (\sum y_i M_i) \circ (\sum x_j M_j) = \frac{1}{2} \sum N_{ij} (x_i y_j - x_j y_i).$$

We call the exterior form  $\phi(\mathfrak{M}, u)$  the *exterior form associated with*  $\mathfrak{M}$ .

Now let  $(x_{i(\lambda)}), \lambda = k+1, \dots, r$  be a base of the system of solutions of the following linear equations in  $r$  variables

$$\sum N_{ij} x_j = 0, \quad i = 1, 2, \dots, r.$$

Then,  $K_\lambda = \sum x_{i(\lambda)} M_i, \lambda = k+1, \dots, r$ , are linearly independent. For otherwise, there exist constants  $c_\lambda, \lambda = k+1, \dots, r$ , such that they are not all zero and  $\sum c_\lambda K_\lambda = 0$ . This implies  $\sum c_\lambda x_{i(\lambda)} M_i = 0$ . Since  $M_1, \dots, M_r$  are linearly independent, it follows  $\sum_\lambda c_\lambda x_{i(\lambda)} = 0$ . This contradicts to the fact that  $(x_{i(\lambda)}), \lambda = k+1, \dots, r$  are linearly independent.

On the other hand, we have

$$\begin{aligned} K_\lambda \circ K_\mu - K_\mu \circ K_\lambda &= (\sum x_{i(\lambda)} M_i) \circ (\sum x_{j(\mu)} M_j) - (\sum x_{i(\mu)} M_i) \circ (\sum x_{j(\lambda)} M_j) \\ &= N_{ij} x_{i(\lambda)} x_{j(\mu)} = 0. \end{aligned}$$

Hence,  $K_{k+1}, \dots, K_r$  become a base of an  $\alpha$ -system in  $\mathfrak{M}$ . We denote the  $\alpha$ -system by  $\alpha(\mathfrak{M})$  and call it the  $\alpha$ -system associated with  $\mathfrak{M}$ . We prove the appropriateness of this notation.

Let  $\bar{M}_i = \sum M_j a_{ji}, |a_{ji}| \neq 0$ , be another base of  $\mathfrak{M}$ . Let us put  $\bar{N}_{ij} = \bar{M}_i \circ \bar{M}_j - \bar{M}_j \circ \bar{M}_i = N_{ik} a_{ki} a_{hj}$ . Then, the system of linear equations  $\sum \bar{N}_{ij} \bar{x}_j = 0, i = 1, 2, \dots, r$ , becomes  $\sum_{k,j} N_{ik} a_{kj} \bar{x}_j = 0, i = 1, 2, \dots, r$ . Accordingly, we may put  $x_{i(\lambda)} = \sum_j a_{ij} \bar{x}_{j(\lambda)}$ . It follows that  $K_\lambda = \sum x_{i(\lambda)} M_i = \sum a_{ij} \bar{x}_{j(\lambda)} M_j = \sum \bar{x}_{j(\lambda)} \bar{M}_j = \bar{K}_\lambda$ . This shows that the above stated  $\alpha$ -system  $\alpha(\mathfrak{M})$  is independent of the choice of base of  $\mathfrak{M}$ .

Thus, by means of Theorem 1, Lemma 2 and the argument above, we obtain a theorem as follows.

**Theorem 2.** Let  $\mathfrak{M}$  be a  $r$ -dimensional linear subspace in  $\mathfrak{M}_{n,m}$  and  $k$  be the minimum of numbers of variables such that  $\phi(\mathfrak{M})$  can be expressed by them. Then  $\dim \alpha(\mathfrak{M}) = r - k \leq m$ .

3. In this section, we shall apply Theorem 3 to a geometrical problem.

Let  $V_n$  be a Riemann manifold of dimension  $n$  whose line element is given by

$$ds^2 = \sum_{i,j=1}^n g_{ij}(x) dx_i dx_j,$$

in local coordinates  $x_1, x_2, \dots, x_n$ . Let us put

$$\begin{aligned} \sum g_{ij}(x) dx_i dx_j &= \sum_{i=1}^n \omega_i(x, dx) \omega_i(x, dx), \\ d\omega_i &= \sum_j \omega_j \wedge \omega_{ji}, \\ d\omega_{ij} &= \sum \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}, \end{aligned}$$

where  $\Omega_{ij}$  are the curvature forms of  $V_n$  as is well known.

Let  $k(p)$  be the minimum number of linear differential forms in terms of which the curvature forms at  $p \in V_n$  can be expressed, and let  $k = \max_{p \in V_n} k(p) = k(V_n)$ . According to S. S. Chern and N. H. Kuiper,  $n - k(p)$  is called the *index of nullity at  $p$* .

**Theorem 3.** A compact Riemann manifold  $V_n$  of dimension  $n$  can not be isometrically imbedded in an Euclidean space of dimension  $2n - k(V_n) - 1$ .

*Proof.* We suppose that such an imbedding of  $V_n$  in  $E^{2n-k-1}$  exists. Let  $(p, e_1, \dots, e_n, e_{n+1}, \dots, e_{2n-k-1})$  be a field of orthonormal frames of  $E^{2n-k-1}$  defined on a coordinate neighborhood of  $V_n$  such that  $e_1, e_2, \dots, e_n$  are tangent vectors at  $p$  to  $V_n$ . Then we have

$$\begin{aligned} dp &= \sum_i \omega_i e_i, \quad de_i = \sum_j \omega_{ij} e_j + \sum_\alpha \omega_{i\alpha} e_\alpha, \\ de_\alpha &= \sum_i \omega_{\alpha i} e_i + \sum_\beta \omega_{\alpha\beta} e_\beta, \\ i, j &= 1, 2, \dots, n; \quad \alpha, \beta = n+1, \dots, 2n-k-1 \end{aligned}$$

and  $ds^2 = dp dp = \sum \omega_i \omega_i$ . These relations give

$$0 = \sum_i d\omega_i e_i - \sum_i \omega_i \wedge de_i = \sum_i (d\omega_i - \sum_k \omega_k \wedge \omega_{ki}) e_i - \sum_k \omega_k \wedge \omega_{k\alpha} e_\alpha,$$

that is,

$$d\omega_i = \sum_k \omega_k \wedge \omega_{ki}, \quad \sum_k \omega_k \wedge \omega_{k\alpha} = 0.$$

The second equation implies

$$\omega_{in+\lambda} = \sum A_{\lambda ij} \omega_j, \quad A_{\lambda ij} = A_{\lambda ji}, \quad \lambda = 1, 2, \dots, n-k-1.$$

The quadratic differential forms  $\phi_\lambda = \sum A_{\lambda ij} \omega_i \omega_j$ ,  $\lambda = 1, 2, \dots, n-k-1$ , are the so-called second fundamental forms of  $V_n$ . We have analogously

$$\begin{aligned} 0 &= \sum d\omega_{ij} e_j + \sum d\omega_{i\alpha} e_\alpha - \sum \omega_{ij} \wedge de_j - \sum \omega_{i\alpha} \wedge de_\alpha \\ &= \sum_j (d\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj} - \sum_\alpha \omega_{i\alpha} \wedge \omega_{\alpha j}) e_j \\ &\quad + \sum (d\omega_{i\alpha} - \sum_k \omega_{ik} \wedge \omega_{k\alpha} - \sum_\beta \omega_{i\beta} \wedge \omega_{\beta\alpha}) e_\alpha. \end{aligned}$$

It follows that

$$\varrho_{ij} = d\omega_{ij} - \sum \omega_{ik} \wedge \omega_{kj} = \sum \omega_{i\alpha} \wedge \omega_{\alpha j} = -\sum A_{\lambda it} A_{\lambda jt} \omega_t \wedge \omega_h.$$

Accordingly,  $k(p)$  is the number of linearly independent forms of

$$\sum_h (A_{\lambda it} A_{\lambda jt} - A_{\lambda jt} A_{\lambda it}) \omega_h.$$

Let  $M_h$  be the  $(n, n-k-1)$ -matrix whose  $(i, \lambda)$ -element is  $A_{\lambda ih}$ . Then,  $\sum (A_{\lambda it} A_{\lambda jt} - A_{\lambda jt} A_{\lambda it})$  is  $(i, j)$ -element of  $M_t \circ M_h - M_h \circ M_t = N_{th}$ . Hence  $n-k(p)$  is the dimension of solutions  $(y_1, \dots, y_n)$  of linear equations

$$\sum_h N_{th} y_h = 0, \quad t = 1, 2, \dots, n.$$

On the other hand,  $M_1, \dots, M_n$  are linearly independent at a point  $p$ , since  $V_n$  is compact and  $V_n \subset E^{2n-k-1}$ . By Theorem 3, we must have, at the point  $p$ ,  $n-k(p) \leq n-k-1$ . This contradicts to the definition of  $k(p)$ . The proof is complete.

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