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Introduction

In this paper we shall establish a decomposition theory of non-commutative self-adjoint algebras of operators on Hilbert spaces, as a generalization of J. v. Neumann's, F. Moutner's and I. E. Segal's results. Their decomposition theories were established on the separability assumption for Hilbert spaces, and the algebraic properties of their decompositions were very obsqure. Then we shall construct the theory extending the algebraic method by I. Gelfund, which is applicable for every C^*-algebra on every Hilbert space.

Our main theorem is the diagonal decomposition theorem of states on C^*-algebras which is explained in the following. And applying this theorem we shall obtain several decomposition theorems of operator algebras.

A uniformly closed self-adjoint algebra $\mathbf{A}$ of bounded linear operators on a Hilbert space which contains the identity operator I is said to be a *C^*-algebra.* A linear functional p on $\mathbf{A}$ such that $p(A^*A) \geqq 0$ and $p(A^*) = \overline{p(A)}$, is said to be a state on $\mathbf{A}$. A state p is said to be *irreducible* if there is no pair (q, r) of states with $p = q + r$ other than $q = \alpha p$ and $r = (1-\alpha)p$. Given a state p. We can choose a Hilbert space $L^2(p)$, a C^*-algebra $\mathbf{A}_p$ and an element $\hat{p} \in L^2(p)$ such that (1). There exists a $*$-algebraic homomorphism $A \rightarrow A_p$ of $\mathbf{A}$ in a dense subalgebra of $\mathbf{A}_p$. (2). The set $(A_p\hat{p}: A \in \mathbf{A})$ is dense in $L^2(p)$. On $L^2(p)$ there exists at least one commutative C^*-algebra $\mathbf{E}$, so called a *diagonal algebra* on $L^2(p)$, which coincides with the set of all operators commuting simultaneously to all $A_p \in \mathbf{A}_p$ and all $A \in \mathbf{E}$. Then the main theorem reads as follows.

Theorem 1. *Let $\mathfrak{N}$ denote the topological space of all irreducible states u with $u(I) = 1$, whose topology is the functional weak topology as a subset of the conjugate space of $\mathbf{A}$. Given a state p and a diagonal algebra $\mathbf{E}$ on $L^2(p)$. There exists a non-negative Borel measure ρ on $\mathfrak{N}$ with $\rho(\mathfrak{N}) = 1$ which satisfies the next conditions.*

(1). *Let $\mathbf{M}$ denote the Banach algebra of all bounded measurable functions on $\mathfrak{N}$ so that $|\varphi| =$ ess. max. $|\varphi(\lambda)|$ and $\varphi^* = \bar{\varphi}$. Then for every $\varphi \in \mathbf{M}$ there exists $K_\varphi \in \mathbf{E}$ such that*

Mathematical Journal of Okayama University, Vol. 3 [1953], Iss. 2, Art. 8

$$(A_p K_\varphi \hat{p}, \hat{p}) = \int \varphi(\lambda)\, \lambda(A)\, d\rho(\lambda).$$

(2). *The correspondence* $\varphi \to K_\varphi$ *is a* $*$*-algebraic isometric isomorphism between two algebras* **M** *and* **E**.

Let **S** denote the full matrix algebra of order n on an n-dimensional complex Euklidean space $\mathfrak{E}^n$. All the diagonal decompositions of any state on **S** is obtained by the following method.

Define an inner-product in **S** by $(A, B) = \sum_{ij} a_{ij}\overline{b_{ij}}$ for $A = (a_{ij})$, $B = (b_{ij}) \in$ **S**. Then $(A, B) = (B^*, A^*)$, and $(AB, C) = (B, A^*C) = (A, CB^*)$. Given a state p on **S**, we can choose $P \in$ **S** so that $p(A) = (A, P)$, where P should be a definite Hermitian. Now we can choose a unitary matrix $U \in$ **S** so that $Q = U^{-1}PU$ is a diagonal matrix. Then

$$p(A) = (A, UQU^{-1}) = (U^{-1}AU, Q) = q(U^{-1}AU),$$

where $q(A)$ is the state with $q(A) = (A, Q)$. Therefore by a suitable change of the base of $\mathfrak{E}^n$, the representative matrix P is obtained as a diagonal matrix $P = \sum \alpha_i E_i$, where E_i denotes the matrix (e_{jk}) so that $e_{ii} = 1$, and $e_{jk} = 0$ otherwise. Now every state $u_i(A) = (A, E_i)$ is irreducible, and we have $p = \sum_i \alpha_i u_i$, where $\alpha_i \geqq 0$. This is a diagonal decomposition of p.

Chapter 1. Preliminaries.

Let **A** denote a C^*-algebra on a Hilbert space $\mathfrak{H}$. The set **A**$'$ of all bounded operators on $\mathfrak{H}$ which commute with all $A \in$ **A** is said to be the *commutor* of **A**. The common part of **A** and **A**$'$ is said to be the *center* of **A**, and denoted by **Z**. **A** is said to be *irreducible* if its commutor consists of αI only.

For two Hermitians A, B we shall denote by $A \geqq B$ if $A - B$ is a definite Hermitian. Also for two states p, q on **A** we shall denote by $p \geqq q$ if $p - q$ is a state.

Let p be a state on **A**. The set $\mathrm{T}_p = (A : p(A^*A) = 0)$ is a linear set. Let S_p denote the quotient space of **A** by T_p, and let $A \to A(p)$ denote the corresponding linear mapping of **A** on S_p. Then S_p has an inner-product $(A(p), B(p)) = p(B^*A)$. Completing the space S_p by this inner-product, we obtain a Hilbert space $L^2(p)$. For every $A \in$ **A** we can choose[1] $B \in$ **A** with $B^*B = |A|^2 I - A^*A$. Then

1) cf. Footnote of [19].

$$|A|^2\|X(p)\|^2 - \|(AX)(p)\|^2 = p(X^*(|A|^2I - A^*A)X) \\ = p((BX)^*BX) \geqq 0.$$

So $|A|\,\|X(p)\| \geqq \|(AX)(p)\|$. Now $X(p) \to (AX)(p)$ is a many-one bounded linear transform on S_p, and there exists a bounded linear operator A_p on $L^2(p)$ so that $A_pX(p) = (AX)(p)$. Denote by $\mathbf{A}_p$ the uniform closure of $(A_p\colon A \in \mathbf{A})$, and put $\hat{p} = I(p)$. Then

Lemma 1.1. *Given a state p. We can choose a Hilbert space $L^2(p)$, a C^*-algebra $\mathbf{A}_p$ and an element $\hat{p} \in L^2(p)$ such that* (1). *there exists a $*$-algebraic homomorphism $A \to A_p$ of $\mathbf{A}$ in a dense sub-algebra of $\mathbf{A}_p$.* (2). *The set $(A_p\hat{p}\colon A \in \mathbf{A})$ is dense in $L^2(p)$.*

Lemma 1.2. *Let p, q be two states on A so that $p \geqq q$. Then there exists $0 \leqq K \in \mathbf{A}_p'$ with $q(A) = (A_p\hat{p}, K\hat{p})$.*

Proof. The mapping $A_p\hat{p} \to A_q\hat{q}$ from $L^2(p)$ into $L^2(q)$ satisfies $\|A_p\hat{p}\| \geqq \|A_q\hat{q}\|$. Then it is a many-one bounded linear mapping. And there exists a definite Hermitian K on $L^2(p)$ with $(KA_p\hat{p}, B_p\hat{p}) = (A_q\hat{q}, B_q\hat{q}) = q(B^*A)$. So

$$(A_pKB_p\hat{p}, C_p\hat{p}) = (KB_p\hat{p}, (A^*C)_p\hat{p}) = q(C^*AB) \\ = (KA_pB_p\hat{p}, C_p\hat{p}).$$

That is, $A_pK = KA_p$. This means $0 \leqq K \in \mathbf{A}_p'$. q.e.d.

Let u be an irreducible state. Then there is no state $p \leqq u$ other than $p = \alpha u$. For every definite Hermitian $K \in \mathbf{A}_u'$ with $|K| \leqq 1$, the state $q(A) = (A_u\hat{u}, K\hat{u})$ satisfies $q \leqq u$. Then $q = \alpha u$ and $K = \alpha I$. Now $\mathbf{A}_u'$ consists of αI only. Hence $\mathbf{A}_p$ is irreducible. This concludes the next lemma.

Lemma 1.3. *A state u is irreducible if and only if $\mathbf{A}_u$ is irreducible.*

Chapter 2. Reducible states and center-reducible states.

A state r is said to be *reducible* if for every state $p \leqq r$ there exists at least one $0 \leqq K \in \mathbf{Z}$ so that $p(A) = r(KA)$. For every linear functional f on $\mathbf{A}$, the linear functional f_z on the center $\mathbf{Z}$ which coincides with f on $\mathbf{Z}$ is said to be the *part* of f in $\mathbf{Z}$. If p is a state on $\mathbf{A}$ whose part p_z in $\mathbf{Z}$ is a reducible state, then p is called *center-reducible.* Let $\mathfrak{Z}$ denote the set of all states u on $\mathbf{Z}$ so that $u(KL) = u(K)u(L)$ for $K, L \in \mathbf{Z}$. Since $\mathbf{Z}$ is a commutative C^*-algebra, by the theorem of Gelfund-Raikov $\mathfrak{Z}$ is a weakly comapct subset of the conjugate space of **Z**. And **Z** is isometrically representable as the algebra $C(\mathfrak{Z})$ of all continuous functions on $\mathfrak{Z}$. $\mathfrak{Z}$ is called the *spectrum* of **Z**, and we regard **Z** as $C(\mathfrak{Z})$. Let Ω be a compact space. Every positive functional p on $C(\Omega)$ is expressed by an indefinite integral of a non-negative Borel measure on Ω. We denote it by the same letter p. Then we have $p(f) = \int f\, dp$ for every $f \in C(\Omega)$. The *support* $\mathfrak{D}(p)$ of p is the smallest closed set X with $p(X) = p(\Omega)$. Now every state p on **Z** is a positive functional on $C(\mathfrak{Z})$. Then p denotes simultaneously its representative Borel measure on $\mathfrak{Z}$.

Theorem 2. *Let p be a center-reducible state. Let p_z denote the part of p in* **Z**, *and simultaneously its representative Borel measure on $\mathfrak{Z}$. And let $\mathfrak{D}(p)$ denote the support of p_z. Then for every $\lambda \in \mathfrak{D}(p_z)$ there corresponds a uniquely determined state λ^p on* **A** *such that*

(1). *For every fixed $A \in$ **A** and $K \in$ **Z**, $\lambda^p(A)$ is continuous in $\mathfrak{D}(p)$ and satisfies $\lambda^p(KA) = K(\lambda)\lambda^p(A)$.*

(2). $p = \int \lambda^p\, dp_z(\lambda)$,

where $\int \lambda^p\, dp_z$ is a Pettis integral so that $p(A) = \int \lambda^p(A)\, dp_z(\lambda)$.

Hereafter the integral expression $p = \int \lambda^p\, dp_z(\lambda)$ in the Theorem is said to be the *spectral decomposition* of p, and every state λ^p is said to be the *derivative state* of p at $\lambda \in \mathfrak{D}(p)$.

Proof. For every $A \in$ **A** with $I \geqq A \geqq 0$, define the functional φ_A on **Z** by $\varphi_A(K) = p(KA)$. If $0 \leqq K \in$ **Z**, we have $K \geqq KA \geqq 0$, and $p(K) \geqq p(KA) = \varphi_A(K) \geqq 0$. Then φ_A is a state on **Z** with $\varphi_A \leqq p_z$. By the center-reducibility of p there exists $0 \leqq W_A \in$ **Z** with $p(KA) = \varphi_A(K) = p_z(W_A K)$. Now we define W_A for every Hermitian $A \in$ **A** as follows: If $0 \neq A$, $B = I - A/|A|$ satisfies $I \geqq B \geqq 0$. Then putting $W_A = I - |A| W_B$ we obtain $W_A \in$ **Z** and $p(KA) = p_z(KW_A)$ for $K \in$ **Z**. Also putting $W_X = W_A + iW_B$ for every $X = A + iB \in$ **A**, we define W_X for every $X \in$ **A**. Now put $\lambda^p(A) = W_A(\lambda)$ for every $\lambda \in \mathfrak{D}(p_z)$. Then

(1)′ For every fixed A, $\lambda^p(A)$ is continuous in $\mathfrak{D}(p_z)$, and satisfies

$$p(KA) = \int_{\mathfrak{D}(p_z)} K(\lambda)\, \lambda^p(A)\, dp_z(\lambda).$$

Such $\lambda^p(A)$ under the condition (1)′ is uniquely determined.

Then λ^p is a linear functional on $\mathbf{A}$ so that $\lambda^p(A) = W_A(\lambda) \geqq 0$ for $A \geqq 0$. Further if $K, L \in \mathbf{Z}$, then

$$p(KLA) = \int K(\lambda) L(\lambda) \lambda^p(A) \, dp_z(\lambda) = \int K(\lambda) \lambda^p(LA) \, dp_z(\lambda).$$

Hence $L(\lambda) \lambda^p(A) = \lambda^p(LA)$. This concludes the theorem.

The following theorem is a special form of the main Theorem 1. But as we shall observe in the latter, the main theorem may be easily derived from this theorem.

Theorem 3. *A state p is reducible if and only if it is a center-reducible state whose almost all derivative states on the support $\mathfrak{D}(p_z)$ of the measure p_z are irreducible.*

The proof of the necessity part of the theorem, which will be done in the next chapters 3-4, is very interesting but somewhat long. We prove here merely its sufficiency part. That is:

Lemma 2.1. *Let p be a center reducible state whose almost all derivative states on the support of the measure p_z are irreducible, then p is reducible.*

Proof. Let $p \geqq t \geqq 0$. Since p is center-reducible, from $p_z \geqq t_z \geqq 0$ there exists $0 \leqq T \in \mathbf{Z}$ with $t_z(K) = p_z(TK)$. Then t is also center-reducible. Let $p = \int \lambda^p \, dp_z(\lambda)$ and $t = \int \lambda^t \, dt_z(\lambda)$ be the spectral decompositions of p and t respectively. Putting $\lambda^t = 0$ for $\lambda \in \mathfrak{D}(p_z) - \mathfrak{D}(t_z)$, we can define λ^t for every $\lambda \in \mathfrak{D}(p_z)$. For every fixed A, $T(\lambda) \lambda^t(A)$ is continuous in $\mathfrak{D}(p_z)$. In fact, $\mathfrak{W} = (\lambda : T(\lambda) > 0, \lambda \in \mathfrak{D}(p_z))$ is contained in $\mathfrak{D}(t_z)$ because $dt_z(\lambda) = T(\lambda) \, dp_z(\lambda)$. And $\mathfrak{W}$ is relatively open in $\mathfrak{D}(p_z)$. Then $T(\lambda) \lambda^t(A)$ is continuous in $\mathfrak{W}$. On the other hand $T(\lambda) \lambda^t(A)$ vanishes in $\mathfrak{D}(p_z) - \mathfrak{W}$, where $\lambda^t(A)$ is bounded in $\mathfrak{D}(p_z)$. Hence $T(\lambda) \lambda^t(A)$ is also continuous in $\mathfrak{D}(p_z) - \mathfrak{W}$.

Now if $0 \leqq A \in \mathbf{A}$ and $0 \leqq K \in \mathbf{Z}$, then $KA \geqq 0$ and $t(KA) \leqq p(KA)$. So

$$\int K(\lambda) T(\lambda) \lambda^t(A) \, dp_z(\lambda) \leqq \int K(\lambda) \lambda^p(A) \, dp_z(\lambda).$$

This proves $T(\lambda) \lambda^t(A) \leqq \lambda^p(A)$ exactly on the whole $\mathfrak{D}(p_z)$. Thus $T(\lambda) \lambda^t \leqq \lambda^p$ for every $\lambda \in \mathfrak{D}(p_z)$. But almost all λ^p are irreducible, then $\alpha(\lambda) \lambda^p = T(\lambda) \lambda^t$ hold almost everywhere. Now for every $\lambda \in \mathfrak{D}(p_z)$ with $T(\lambda) > 0$ we have $\lambda^t(I) = 1$. Then $\alpha(\lambda) = \alpha(\lambda) \lambda^p(I) = T(\lambda) \lambda^t(I) = T(\lambda)$ hold almost everywhere. Hence

$$t(A) = \int T(\lambda)\, \lambda^t(A)\, dp_z(\lambda) = \int T(\lambda)\, \lambda^p(A)\, dp_z(\lambda) = p(TA).$$

This concludes the reducibility of p.

Chapter 3. Extreme decompositions of reducible states.

Let $\mathfrak{P}$ denote the set of all states p with $p(I)=1$. And let $\mathfrak{N}$ denote the set of all irreducible states u with $u(I)=1$. Let $\mathbf{A}^H$ denote the real Banach space of all Hermitian operators in $\mathbf{A}$. $\mathfrak{P}$ is a bounded regularly convex sub-set[1] of the conjugate space of $\mathbf{A}^H$. Then the weak closure $\overline{\mathfrak{N}}$ of $\mathfrak{N}$ is a weakly compact sub-set of $\mathfrak{P}$. For every $A \in \mathbf{A}$ let $A^\vee$ denote the weakly continuous function on $\overline{\mathfrak{N}}$ so that $A^\vee(\lambda) = \lambda(A)$ for every $\lambda \in \overline{\mathfrak{N}}$. For every linear functional w on $C(\overline{\mathfrak{N}})$ let $w^\vee$ denote the linear functional on $\mathbf{A}$ so that $w^\vee(A) = w(A^\vee)$. For every linear functional w on $C(\overline{\mathfrak{N}})$ and every $f \in C(\overline{\mathfrak{N}})$ let w_f denote the linear functional so that $w_f(g) = w(fg)$. Now if $A \geqq 0$, then $A^\vee \geqq 0$. And if w is a positive functional on $C(\overline{\mathfrak{N}})$, $w^\vee$ is a state on $\mathbf{A}$ such as $w^\vee(A) = \int \lambda(A)\, dw(\lambda)$, i.e. $w^\vee = \int \lambda\, dw(\lambda)$.

By Theorem 3 of my separate paper [19], for every state p on $\mathbf{A}$ there exists at least one positive functional w on $C(\overline{\mathfrak{N}})$ so that $p = w^\vee = \int \lambda\, dw(\lambda)$.

Now we shall consider a fixed reducible state r and a fixed positive functional m on $C(\overline{\mathfrak{N}})$ with $m^\vee = r$.

Lemma 3.1. *Let $K \in \mathbf{Z}$ and $A \in \mathbf{A}$. Then $(KA)^\vee = K^\vee A^\vee$.*

Proof. If $\lambda \in \mathfrak{N}$, the algebra $\mathbf{A}_\lambda$ on $L^2(\lambda)$ is irreducible. And from $K \in \mathbf{Z}$ we have $K_\lambda \in \mathbf{A}_\lambda'$. Then $K_\lambda = \alpha I_\lambda$. Now

$$\lambda(KA) = (K_\lambda A_\lambda \hat{\lambda}, \hat{\lambda}) = \alpha(A_\lambda \hat{\lambda}, \hat{\lambda}) = \alpha\lambda(A).$$

Putting $A = I$ we have $\alpha = \lambda(K)$ and $\lambda(KA) = \lambda(K)\lambda(A)$. Therefore $K^\vee(\lambda) A^\vee(\lambda) = (KA)^\vee(\lambda)$ is valid in the dense sub-space $\mathfrak{N}$ of $\overline{\mathfrak{N}}$, where $K^\vee$ etc. are continuous in $\overline{\mathfrak{N}}$. Hence $(KA)^\vee = K^\vee A^\vee$.

Lemma 3.2. *For every $f \in C(\overline{\mathfrak{N}})$ there exists at least one $F \in \mathbf{Z}$ so that $(m_f)^\vee = (m_{F^\vee})^\vee$, where we can choose $F \geqq 0$ when $f \geqq 0$. If we define such F for every $f \in C(\overline{\mathfrak{N}})$, then it satisfies further $m(f) = m(F^\vee)$ and $m_{F^\vee}(gA^\vee) = m_{F^\vee}(G^\vee A^\vee)$ for every $f, g \in C(\overline{\mathfrak{N}})$ and $A \in \mathbf{A}$.*

Proof. If $f \in C(\overline{\mathfrak{N}})$ satisfies $1 \geqq f \geqq 0$, then $m \geqq m_f \geqq 0$ and $r = m^\vee \geqq m_f^\vee \geqq 0$. By the reducibility of r there exists $0 \leqq F \in \mathbf{Z}$ so that

1) cf. Footnote of [19].

$$m_f^\nu(A) = r(FA) = m(F^\nu A^\nu) = (m_{F^\nu})^\nu(A).$$

That is, $m_f^\nu = (m_{F_\nu})^\nu$. Now we can determine F for every $f \in C(\overline{\mathfrak{N}})$ as follows. If f is real, we have $f = \alpha h - \beta k$, where $1 \geq h$, $k \geq 0$. Then put $F = \alpha H - \beta K$. If f is not real, then $f = h + ik$, where h, k are real. Hence put $F = H + iK$. This F satisfies clearly $(m_f)^\nu = (m_{F^\nu})^\nu$. Further $m(f) = m_f(I^\nu) = m_{F^\nu}(I^\nu) = m(F^\nu)$. And for f, $g \in C(\overline{\mathfrak{N}})$ and $A \in \mathbf{A}$, we have

$$m_{F^\nu}(gA^\nu) = m(gF^\nu A^\nu) = m(G^\nu F^\nu A^\nu) = m_{F^\nu}(G^\nu A^\nu). \qquad \text{q.e.d.}$$

Let λ be a point in the support $\mathfrak{D}(m)$ of m. If U is an open set which contains λ, there exists a positive continuous function f_U on $\overline{\mathfrak{N}}$ with $\int f_U\, dm = 1$, which vanishes outside of U. Let $\{U\}$ be a filter of open neighbourhoods of λ which coverges to λ. Then $\{m_{f_U}\}$ coverges weakly to the "*Dirac integral*" δ_λ which is the positive functional on $C(\overline{\mathfrak{N}})$ defined by $\delta_\lambda(f) = f(\lambda)$ for $f \in C(\overline{\mathfrak{N}})$. For each f_U choose $0 \leq F_U \in \mathbf{Z}$ which satisfies the conditions in Lemma 3.2. Since $|m_{F_U}{}^\nu| = m(F_U{}^\nu) = m(f_U) = 1$, $\{m_{F_U}{}^\nu\}$ is norm-bounded, and there exists a suitable sub-filter $\{V\}$ of $\{U\}$ such that $\{m_{F_V}\}$ converges weakly to a suitable positive functional u on $C(\overline{\mathfrak{N}})$.

Lemma 3.3. $\delta_\lambda{}^\nu = u^\nu$. $u(gA^\nu) = u(G^\nu A^\nu)$ *for* $g \in C(\overline{\mathfrak{N}})$ *and* $A \in A$.

Proof. We have

$$\delta_\lambda(A^\nu) = \lim_{V \to \lambda} m_{f_V}(A^\nu) = \lim_{V \to \lambda} m_{F_V}{}^\nu(A^\nu) = u(A^\nu).$$

Then $\delta_\lambda{}^\nu = u^\nu$. Also for $g \in C(\overline{\mathfrak{N}})$ and $A \in \mathbf{A}$,

$$u(gA^\nu) = \lim_{V \to \lambda} m_{F_V}{}^\nu(gA^\nu) = \lim_{V \to \lambda} m_{F_V}{}^\nu(G^\nu A^\nu) = u(G^\nu A^\nu).$$

This concludes the lemma.

Lemma 3.4. $u = \delta_\lambda$.

Proof. Let $g \in C(\overline{\mathfrak{N}})$ and $A \in \mathbf{A}$. Then by lemma 3.3,

$$u(gA^\nu) = u^\nu(GA) = \delta_\lambda(G^\nu A^\nu) = \delta_\lambda(G^\nu)\,\delta_\lambda(A^\nu).$$

Putting $A = I$, we have $u(g) = \delta_\lambda(G^\nu)$. Then $u(gA^\nu) = u(g)\,\delta_\lambda(A^\nu)$. Now if $A_1, A_2, \cdots\cdots, A_n \in \mathbf{A}$, then

Mathematical Journal of Okayama University, Vol. 3 [1953], Iss. 2, Art. 8

$$\begin{aligned} u(A_1^\vee \cdot A_2^\vee \cdot \cdots\cdots \cdot A_n^\vee) &= u(A_1^\vee \cdot A_2^\vee \cdot \cdots\cdots \cdot A_{n-1}^\vee)\,\delta_\lambda(A_n^\vee) \\ &= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &= u(A_1^\vee)\,\delta_\lambda(A_2^\vee)\ \cdots\cdots\ \delta_\lambda(A_n^\vee) \\ &= \delta_\lambda(A_1^\vee)\,\delta_\lambda(A_2^\vee)\ \cdots\cdots\ \delta_\lambda(A_n^\vee) \\ &= \delta_\lambda(A_1^\vee \cdot A_2^\vee \cdot \cdots\cdots \cdot A_n^\vee). \end{aligned}$$

Now $u(f) = \delta_\lambda(f)$ is valid for every function f in the smallest Banach ring $\mathbf{C}$ which contains all functions $A^\vee (A \in \mathbf{A})$, where $\mathbf{C}$ contains $1 = I^\vee$, and self-involutive, i.e. $f \in \mathbf{C}$ implies $\bar{f} \in \mathbf{C}$ since $\overline{A^\vee} = A^{*\vee}$ holds. Further for $\lambda \neq \mu \in \overline{\mathfrak{N}}$ there exists $A \in \mathbf{A}$ with $A^\vee(\lambda) \neq A^\vee(\mu)$, Then by the theorem of Gelfund-Silov $\mathbf{C}$ coincides with $C(\overline{\mathfrak{N}})$. Hence $u(f) = \delta_\lambda(f)$ for every $f \in C(\overline{\mathfrak{N}})$. This proves $u = \delta_\lambda$.

Lemma 3.5. *For every $f \in C(\overline{\mathfrak{N}})$ there exists $F \in \mathbf{Z}$ such that $F^\vee$ coincides with f on $\mathfrak{D}(m)$.*

Proof. Let $\lambda \in \mathfrak{D}(m)$. By lemma 3.3-4 we have $\delta_\lambda(fA^\vee) = \delta_\lambda(F^\vee A^\vee)$. Putting $A = I$ we have $f(\lambda) = F^\vee(\lambda)$. Then F is a desired one.

By lemma 3.1 the part λ_z of every $\lambda \in \overline{\mathfrak{N}}$ in the center $\mathbf{Z}$ satisfies $\lambda_z(KS) = \lambda_z(K)\lambda_z(S)$ for $K, S \in \mathbf{Z}$. Then λ_z belongs to the spectrum $\mathfrak{Z}$ of $\mathbf{Z}$. And $\lambda \to \lambda_z$ is a weakly continuous mapping of $\overline{\mathfrak{N}}$ in $\mathfrak{Z}$. Further

Lemma 3.6. *$\lambda \to \lambda_z$ mapps $\mathfrak{D}(m)$ homeomorphicly on a suitable compact subset $\mathfrak{E}$ of $\mathfrak{Z}$.*

Proof. Let $\lambda \neq \mu \in \mathfrak{D}(m)$, then there exists $f \in C(\overline{\mathfrak{N}})$ with $f(\lambda) \neq f(\mu)$. Now choose $F \in \mathbf{Z}$ so that $F^\vee$ coincides with f on $\mathfrak{D}(m)$. Then $F^\vee(\lambda) \neq F^\vee(\mu)$ and $\lambda_z(F) \neq \mu_z(F)$. So $\lambda \to \lambda_z$ mapps $\mathfrak{D}(m)$ one-to-one continuously on a suitable $\mathfrak{E} \subset \mathfrak{Z}$. Since $\mathfrak{D}(m)$ is compact, the mapping is a homeomorphism between $\mathfrak{D}(m)$ and $\mathfrak{E}$. q.e.d.

Let Φ denote the family of all bounded non-negative Borel measures whose supports are contained in $\mathfrak{D}(m)$. Also let Ψ denote the family of all bounded non-negative Borel measures in $\mathfrak{Z}$ whose supports are contained in $\mathfrak{E}$. Let $\lambda \to \lambda^r$ denote the inverce mapping of the homeomorphism $\lambda \to \lambda_z$ between $\mathfrak{D}(m)$ and $\mathfrak{E}$. Then $\lambda \to \lambda^r$ induces a one-to-one correspondence $w \to w^r$ between Φ and Ψ in the following way. Given $w \in \Phi$, w^r is defined by $w^r(X) = w((X \cap \mathfrak{E})^r)$ for every Borel set $X \subset \mathfrak{Z}$. Also given $w^r \in \Psi$, w is determined by $w(Y) = w^r((Y \cap \mathfrak{D}(m))_z)$.

On the other hand Φ and Ψ represent simultaneously the respective families of positive functionals on $C(\overline{\mathfrak{N}})$ and $\mathbf{Z}$ which correspond with these measures writen by the same letters.

Lemma 3.7. *$w^r = w^\nu{}_z$ for every $w \in \Phi$. Then $w \to w^r$ preserves the order $\geqq$. Especially $m^r = r_z$. And the support of r_z is $\mathfrak{E}$.*

Proof. For every $K \in \mathbf{Z}$,

$$w^r(K) = \int_{\mathfrak{E}} \lambda(K)\, dw^r(\lambda) = \int_{\mathfrak{D}(m)} \tau(K)\, dw(\tau) = w(K^\nu).$$

Then $w^r = w^\nu{}_z$. Especially $m^r = m^\nu{}_z = r_z$. And the support of r_z is $\mathfrak{D}(m^r) = \mathfrak{E}$.

Lemma 3.8. *r is center-reducible.*

Proof. Let s be a state on $\mathbf{Z}$ with $r_z \geqq s$. The support of s is contained in $\mathfrak{D}(r_z) = \mathfrak{E}$. Then $s \in \Psi$, and we can choose $t \in \Phi$ with $t^r = s$. From $m^r = r_z \geqq t^r$ we have $m \geqq t$ and $r = m^\nu \geqq t^\nu$. Since r is reducible, there exists $0 \leqq K \in \mathbf{Z}$ with $t^\nu(A) = r(KA)$. Then for every $X \in \mathbf{Z}$,

$$s(X) = t^\nu{}_z(X) = t^\nu(X) = r_z(KX).$$

This means the reducibility of r_z. Hence r is center reducible. q.e.d.

From $r_z = m^r$ we obtain

$$r(A) = \int_{\mathfrak{D}(m)} \omega(A)\, dm(\omega) = \int_{\mathfrak{D}(p_z)} \lambda^r(A)\, dr_z(\lambda).$$

That is, $r = \int \lambda^r\, dp_z(\lambda)$. This expression satisfies all the conditions of the spectral decomposition of r in the Theorem 2. Then

Theorem 3′. *Every reducible state r is center-reducible, and the set $\mathfrak{D}$ of all derivative states is contained in $\overline{\mathfrak{R}}$. There exists a Borel measure m on $\overline{\mathfrak{R}}$ with $\mathfrak{D}(m) = \mathfrak{D}$ so that $m(\mathrm{X}) = r_z((\mathrm{X} \cap \mathfrak{D})_z)$ for every Borel set $\mathrm{X} \subset \overline{\mathfrak{R}}$. This m is the only one positive functional w on $\mathrm{C}(\overline{\mathfrak{R}})$ which satisfies $w^\nu = r$.*

Chapter 4. Derivative states of reducible states.

Now the Theorem 3 may be completed if we prove the irreducibility of almost all derivative states of the given reducible state r. In first we shall prepare an extension theorem of positive functionals.

Let Ω be a compact space. Let $\mathrm{B}(\Omega)$ denote the real Banach space of all real bounded functions on Ω. And let $\mathrm{R}(\Omega)$ denote the real Banach space of all real continuous functions in Ω. A positive functional p on $\mathrm{B}(\Omega)$ is called a *Radon-integral* on Ω. If p is a positive functional on $\mathrm{R}(\Omega)$, then the *Daniell's outer integral* of p is the functional p^* on $\mathrm{B}(\Omega)$ defined by $p^*(f) = \inf_{f \leqq h \in \mathrm{R}(\Omega)} p(h)$.

Mathematical Journal of Okayama University, Vol. 3 [1953], Iss. 2, Art. 8

Theorem 4. *Let p be a positive functional on* $R(\Omega)$, *and let p^* be its Daniell's outer integral. For every $h \in B(\Omega)$ and every $p^*(h) \geqq t \geqq -p^*(-h)$ there exists at least one extended Radon-integral q of p with $q(h) = t$.*

Proof. We have $p^*(\alpha f) = \alpha p^*(f)$ for $\alpha \geqq 0$, and $p^*(f+g) \leqq p^*(f) + p^*(g)$. Then by the theorem of Hahn-Banach there exists a linear functional q on $B(\Omega)$ with $q(h) = t$. We show that q is a desired positive functional. If $f \in R(\Omega)$, we have $p^*(f) = -p^*(-f) = p(f)$. Then $q(f) = p(f)$. If $0 \leqq g \in B(\Omega)$, then $0 = p(0) \geqq \inf_{-g \leqq k \in R(\Omega)} p(k) = p^*(-g)$ and $q(g) \geqq -p^*(-g) \geqq 0$. Therefore q is a Radon-extension of p. q.e.d.

Let p be a positive functional on $R(\Omega)$. A function (or a set) in Ω is called *p-measurable*, if it is measurable by the representative Borel measure p. And we denote $p(f) = \int f\,dp$ if f is a p-integrable function.

Lemma 4.1. *Let p be a positive functional on* $R(\Omega)$, *and p^* denote its outer integral. If a real bounded function h satisfies $p^*(h) = -p^*(-h)$, then h is p-measurable and $p(h) = p^*(h)$.*

Proof. We can choose a sequence of $R(\Omega)$: $f_1 \geqq f_2 \geqq \cdots\cdots \geqq h$ so that $\lim_n p(f_n) = p^*(h)$. Also we can choose another sequence $g_1 \leqq g_2 \leqq \cdots\cdots \leqq h$ so that $\lim_n p(g_n) = -p^*(-h)$. Put $f = \lim_n f_n$ and $g = \lim_n g_n$, then $f \geqq h \geqq g$ and $\int f\,dp = p^*(h) = -p^*(-h) = \int g\,dp$. Then h coincides almost everywhere with the measurable function f, where $p(h) = p(f) = p^*(h)$. q.e.d.

Let us consider once more a reducible state r and the positive functional m on $C(\overline{\mathfrak{N}})$ with $m^\vee = r$.

Lemma 4.2. *If a state p is not irreducible, there exists at least one positive functional p' on $C(\overline{\mathfrak{N}})$ with $p'^\vee = p$ whose support contains at least two points.*

Proof. If p is not irreducible, there exists a state $s \leqq p$ so that $s \neq \alpha p$. Put $t = p - s$, and choose two positive functionals s', t' on $C(\overline{\mathfrak{N}})$ so that $s'^\vee = s$ and $t'^\vee = t$. Then $p' = s' + t'$ satisfies $p'^\vee = p$. We show that p' is a desired positive functional. If $\mathfrak{D}(p')$ consists of only one point λ. Then $p'(f) = \int_\lambda f(\tau)\,dp'(\tau) = f(\lambda)p'(\lambda) = \alpha\delta_\lambda(f)$, where $\alpha = p'(\lambda)$, and δ_λ is the Dirac integral at λ. Thus $p' = \alpha\delta_\lambda$. Now $p' \geqq s'$ implies $\mathfrak{D}(s') \subseteqq \mathfrak{D}(p')$. Then we have also $s' = \beta\delta_\lambda$. This means $s' = rp'$ and $s = rp$. It contradicts $s \neq rp$. Then $\mathfrak{D}(p')$ contains at least two points. q.e.d.

We can choose for every $\lambda \in \overline{\mathfrak{N}}$ a positive functional λ' in the following way. If $\lambda \in \mathfrak{N}$, we put $\lambda' = \delta_\lambda$. And if $\lambda \in \overline{\mathfrak{N}} - \mathfrak{N}$, λ' is a positive functional on $C(\overline{\mathfrak{N}})$ with $\lambda'^\nu = \lambda$, whose support contains at least two points. For every $f \in R(\overline{\mathfrak{N}})$ we define a function f' on $\overline{\mathfrak{N}}$ by $f'(\lambda) = \lambda'(f)$. f' belongs to $B(\overline{\mathfrak{N}})$ since $|f'(\lambda)| = |\lambda'(f)| \leq \lambda'(1)|f| = |f|$. Further $f \to f'$ is a bounded linear transformation which mapps every positive function to a positive function.

Lemma 4.3. *If $f \in R(\overline{\mathfrak{N}})$, then f' is m-measurable, and $m(f) = m(f')$.*

Proof. m is considerable as a positive functional on $R(\overline{\mathfrak{N}})$. Let m^* denote the Daniell's outer integral of m. If $f \in R(\overline{\mathfrak{N}})$, then by Theorem 4 there exists an extended Radon-integral n of m in $B(\overline{\mathfrak{N}})$ so that $n(f') = m^*(f')$. Let n' denote the positive functional on $C(\overline{\mathfrak{N}})$ defined by $n'(f+ig) = n(f') + in(g')$ for every $f + ig \in C(\overline{\mathfrak{N}})$, (where $f, g \in R(\overline{\mathfrak{N}})$). Then for every Hermitian $A \in \mathbf{A}$,

$$n'(A^\nu) = n(A^{\nu\prime}) = n(A^\nu) = m(A^\nu) = r(A).$$

That is, $n'^\nu = m^\nu = r$. But by Theorem 3′ there is no positive functional w on $C(\overline{\mathfrak{N}})$ with $w^\nu = r$ other than $w = m$. Then $n' = m$, and especially $m^*(f') = n'(f) = m(f)$. Analogously we have $m^*(-f') = m(-f)$. Then $m^*(f') = -m^*(-f') = m(f)$. Hence by lemma 4.1 f' is m-measurable, and satisfies $m(f) = m(f')$.

Lemma 4.4. *Every real bounded m-measurable function f on $\overline{\mathfrak{N}}$ is λ-measurable except for those λ'-measures whose λ belong to a set of m-measure 0. And the function f' in $\overline{\mathfrak{N}}$ defined by $f'(\lambda) = \lambda'(f)$ is m-measurable and satisfies $m(f) = m(f')$.*

Proof. This lemma will be proved by an analogous consideration with the Fubini's theorem.

Let L denote the smallest family of bounded real functions which satisfies the next two conditions.

(L_1). $C(\overline{\mathfrak{N}}) \subseteqq L$. ($L_2$). *If $\{f_n\}$ is a sub-sequence of* L *so that $|f_n| < r$, and* $\lim f_n(\lambda)$ *exists at each* $\lambda \in \overline{\mathfrak{N}}$, *then* $\lim_n f_n \in L$.

On the other hand let M denote the family of all real bounded m-measurable functions k on $\overline{\mathfrak{N}}$ which satisfy the next two conditions.

(M_1). *Each k is λ'-measurable except for those λ'-measures whose λ belongs to a set of m-measure* 0.

Mathematical Journal of Okayama University, Vol. 3 [1953], Iss. 2, Art. 8

(M_2). *The fuuction k' in $\mathfrak{N}$ so that $k'(\lambda) = \lambda'(k)$ is m-measurable, and satisfies $m(k) = m(k')$.*

Clearly M satisfies the conditions L_1 and L_2. Then M contains L. And every $t \in L$ satisfies M_1 and M_2.

As is well-known, for every bounded m-measurable function f there exists $g, h \in L$ so that $g \geqq f \geqq h$ and $m(g) = m(h) = m(f)$. From $g' \geqq h'$ and $m(g') = m(h')$, it follows $g'(\lambda) = h'(\lambda)$ except for a set $\mathfrak{T}$ of m-measure 0. Let $\lambda \in \mathfrak{N} - \mathfrak{T}$, then we have $\lambda'(g) = \lambda'(h)$, where $g \geqq f \geqq h$. So that f is λ'-measurable, and satisfies $\lambda'(g) = \lambda'(f) = \lambda'(h)$. Hence f satisfies the condition M_1. Further from $g' \geqq f' \geqq h'$ and $m(g') = m(h')$, f' coincides with g' except for a set of m-measure 0. Then f' is m-measurable and satisfies $m(f') = m(g') = m(g) = m(f)$. Hence f satisfies the condition M_2. This completes the lemma.

Lemma 4.5. *Every bounded m-measurable function k coincides with k' except for a set of m-measure* 0.

Proof. If $0 \lneqq K \in \mathbf{Z}$, the state r_K of $\mathbf{A}$ defined by $r_K(A) = r(KA)$ is clearly reducible. Let $0 \lneqq f \in C(\mathfrak{N})$. By lemma 3.2 we can choose $0 \lneqq F \in \mathbf{Z}$ with $(m_f)^\vee = (m_{F^\vee}) = r_F$. Then Lemma 4.4 should be valid even if we replace r by r_F and m by m_f. And every real bounded m-measurable function k satisfies $m_f(k') = m_f(k)$, i.e. $\int f(\lambda)\, k(\lambda)\, dm = \int f(\lambda)\, k'(\lambda)\, dm$ for every $0 \lneqq f \in R(\mathfrak{N})$. Hence k coincides with k' except for a set of m-measure 0.

Lemma 4.6. *Every closed subset $\mathfrak{X}$ of $\mathfrak{N}$ satisfies $m[\mathfrak{X} - \sum_{\mathfrak{D}(\lambda') \subseteqq \mathfrak{X}} \mathfrak{D}(\lambda')] = 0$.*

Proof. Since the characteristic function $\varphi_{\mathfrak{X}}$ of $\mathfrak{X}$ is m-measurable, we have $\varphi_{\mathfrak{X}}(\lambda) = \lambda'(\varphi_{\mathfrak{X}}) = \lambda'(\mathfrak{X})$, except for a set $\mathfrak{T}$ of m-measure 0. If $\lambda \in \mathfrak{X} - \mathfrak{T}$, then $\lambda'(\mathfrak{X}) = 1 = \lambda'(\mathfrak{N})$. This means $\mathfrak{X} \geqq \mathfrak{D}(\lambda')$. Then $\mathfrak{X} - \sum_{\mathfrak{D}(\lambda') \subseteqq \mathfrak{X}} \mathfrak{D}(\lambda') \subseteqq \mathfrak{T}$. This concludes the lemma.

Lemma 4.7. *Every $\lambda \in \mathfrak{D}(m) - \mathfrak{N}$ satisfies $\mathfrak{D}(\lambda') - \mathfrak{D}(m) \neq 0$.*

Proof. Let $\lambda \in \mathfrak{N}$ and $F \in \mathbf{Z}$. Putting $\alpha = \lambda(F)$ we have $\lambda'(|\alpha 1 - F^\vee|^2) = 0$. Then $\alpha 1 - F^\vee$ vanishes on the whole $\mathfrak{D}(\lambda')$. Hence $F^\vee(\tau) = F^\vee(\lambda)$ holds for every $\tau \in \mathfrak{D}(\lambda')$. If λ, μ are different two points in $\mathfrak{D}(m)$, we can choose $f \in C(\mathfrak{N})$ with $f(\lambda) \neq f(\mu)$. By Lemma 3.5 there exists $F \in \mathbf{Z}$ so that $F^\vee$ coincides with f on $\mathfrak{D}(m)$. Then $F^\vee(\lambda) \neq F^\vee(\mu)$. Now if $\tau \in \mathfrak{D}(\lambda')$, we have $F^\vee(\tau) = F^\vee(\lambda) \neq F^\vee(\mu)$ and $\tau \neq \mu$.

Then for every $\lambda \in \mathfrak{D}(m)$ the common element of $\mathfrak{D}(m)$ and $\mathfrak{D}(\lambda')$ is at most λ only. But if $\lambda \in \mathfrak{D}(m) - \mathfrak{R}$, $\mathfrak{D}(\lambda')$ contains at least two points. Then $\mathfrak{D}(\lambda') - \mathfrak{D}(m) \neq 0$.

Lemma 4.8. $m(\overline{\mathfrak{R}} - \mathfrak{R}) = 0$.

Proof. By Lemma 4.7 we have

$$\mathfrak{D}(m) - \mathfrak{R} \leqq \mathfrak{D}(m) - \sum_{\mathfrak{D}(\lambda') \subseteqq \mathfrak{D}(m)} \mathfrak{D}(\lambda').$$

Then by Lemma 4.6 we have

$$m(\overline{\mathfrak{R}} - \mathfrak{R}) = m(\mathfrak{D}(m) - \mathfrak{R}) \leqq m[\mathfrak{D}(m) - \sum_{\mathfrak{D}(\lambda') \subseteqq \mathfrak{D}(m)} \mathfrak{D}(\lambda')] = 0.$$

Theorem 3''. *Every reducible state is center reducible, and its almost all derivative states on the support of the measure r_z are irreducible.*

Proof. Let $\mathfrak{X}$ denote the set of all $\lambda \in \mathfrak{D}(r_z)$ so that the derivative states λ^r are not irreducible. Then we have $\mathfrak{D}(m) - \mathfrak{R} = \mathfrak{X}^r = (\lambda^r : \lambda \in \mathfrak{X})$. Hence by lemma 3.7, we have

$$r_z(\mathfrak{X}) = m^r(\mathfrak{X}) = m(\mathfrak{X}^r) = m(\mathfrak{D}(m) - \mathfrak{R}) = 0.$$

This concludes the theorem.

From Theorem 3'' and Lemma 2.1, we obtain the Theorem 3.

Chapter 5. Diagonal decompositions of states.

Let $\mathbf{M} \cup \mathbf{N}$ denote the smallest C^*-algebra which contains two algebras $\mathbf{M}$ and $\mathbf{N}$ on $\mathfrak{H}$, and $\mathbf{M} \cap \mathbf{N}$ denote the common part of two algebras $\mathbf{M}$ and $\mathbf{N}$.

Lemma 5.1. *For every C^*-algebra $\mathbf{A}$ there exists a commutative C^*-algebra $\mathbf{E}$ on $\mathfrak{H}$ so that $(\mathbf{A} \cup \mathbf{E})' = \mathbf{E}$.*

Proof. Consider the family Π of all commutative C^*-subalgebras of $\mathbf{A}'$. Π is non-empty, and by the Zorn's principle it contains at least one maximal algebra $\mathbf{E}$. Let A be an Hermitian in $(\mathbf{A} \cup \mathbf{E})'$. The smallest C^*-algebra $\mathbf{S}$ which contains A and $\mathbf{E}$ belongs to Π. Then we have $\mathbf{S} = \mathbf{E}$ and $A \in \mathbf{E}$. This proves $(\mathbf{A} \cup \mathbf{E})' \leqq \mathbf{E}$. But $(\mathbf{A} \cup \mathbf{E})' \geqq \mathbf{E}$ is clear. Hence $(\mathbf{A} \cup \mathbf{E})' = \mathbf{E}$.

Given a state p on $\mathbf{A}$. A commutative C^*-algebra $\mathbf{E}$ on $L^2(p)$ so that $(\mathbf{E} \cup \mathbf{A}_p)' = \mathbf{E}$ is called a *diagonal algebra* on $L^2(p)$. By lemma 5.1, there exists at least one diagonal algebra $\mathbf{E}$ on $L^2(p)$. Put $\mathbf{R} = \mathbf{E} \cup \mathbf{A}_p$. Then the linear functional t on $\mathbf{R}$ so that $t(X) = (X\hat{p}, \hat{p})$

Mathematical Journal of Okayama University, Vol. 3 [1953], Iss. 2, Art. 8

is a state, where $L^2(t) = L^2(p)$, $\hat{t} = \hat{p}$ and $\mathbf{R}_t = \mathbf{R}$ satisfy the conditions of Lemma 1.1.

Lemma 5.2. *t is reducible.*

Proof. For every state $s \lesssim t$ on $\mathbf{R}$, there exists $0 \lesssim K \in \mathbf{R}_t' = \mathbf{E}$ with $s(A) = (A_p\hat{p}, K\hat{p}) = t(KA)$, where $\mathbf{E}$ is the center of $\mathbf{R}$. Hence t is reducible.

Lemma 5.3. *The support of the part t_E of t in $\mathbf{E}$ coincides with the spectrum $\mathfrak{E}$ of $\mathbf{E}$.*

Proof. It is sufficient to show that

$$t_E(K^*K) = \int |K(\lambda)|^2 dt_E \neq 0$$

for every $0 \neq K \in \mathbf{E}$. If $K \in \mathbf{E}$ satisfies $t_E(K^*K) = 0$. Then for every $A \in \mathbf{A}$ we have

$$\| KA_p\hat{p} \|^2 = \| A_pK\hat{p} \|^2 \leqq |A|^2 \| K\hat{p} \|^2 = |A|^2 t_E(K^*K) = 0.$$

Then $KA_p\hat{p} = 0$. This means $K = 0$. Hence the support of t_E coincides with $\mathfrak{E}$.

Let $t = \int_{\mathfrak{E}} \lambda^t \, dt_E(\lambda)$ denote the spectral decomposition of t. For $\lambda \in \mathfrak{E}$ denote by $\hat{\lambda}$, X_λ, A_λ, $\mathbf{R}_\lambda$ and $\mathbf{A}_\lambda$ respectively instead of $\hat{\lambda}^t$, X_{λ^t} (for $X \in \mathbf{R}$), $A_{p\lambda^t}$ (for $A \in \mathbf{A}$), $\mathbf{R}_{\lambda^t}$ and the uniform closure of the set $(A_\lambda : A \in \mathbf{A})$.

Lemma 5.4. *If $\lambda \in \mathbf{E}$, the linear functional λ^p on $\mathbf{A}$ so that $\lambda^p(A) = \lambda^t(A_p)$ is clearly a state on $\mathbf{A}$. It satisfies $L^2(\lambda^p)$, $L^2(\lambda)$, $\hat{\lambda}^p = \hat{\lambda}$ and $\mathbf{A}_{\lambda^p} = \mathbf{A}_\lambda$. Further $\mathbf{A}_\lambda$ coincides with $\mathbf{R}_\lambda$.*

Proof. The set $\mathbf{X} = (\sum E_i A_i : E_i \in \mathbf{E},\ A_i \in \mathbf{A})$ is uniformly dense in $\mathbf{R} = \mathbf{E} \cup \mathbf{A}$. If $\lambda \in \mathbf{E}$, every $E \in \mathbf{E}$ satisfies $E_\lambda = E(\lambda) I_\lambda$. Now $\mathbf{X}_\lambda = (\sum E_i(\lambda) A_{i\lambda} : E_i \in \mathbf{E},\ A_i \in \mathbf{A}) = (A_\lambda : A \in \mathbf{A})$ is uniformly dense in $\mathbf{R}_\lambda$. Then $\mathbf{R}_\lambda = \mathbf{A}_\lambda$. And $(A_\lambda \hat{\lambda} : A \in \mathbf{A})$ is dense in $L^2(\lambda^t)$.

Hence concerning the state λ^p, the conditions of Lemma 1.1 are satisfied by $L^2(\lambda^p) = L^2(\lambda^t)$, $\hat{\lambda}^p = \hat{\lambda}$ and $\mathbf{A}_{\lambda^p} = \mathbf{A}_\lambda$.

Lemma 5.5. *λ^p is irreducible if and only if λ^t is irreducible. Almost all λ^p (concerning the measure t_E) are irreducible.*

Proof. By Lemma 1.3 λ^t and λ^p are irreducible if and only if $\mathbf{R}_\lambda = \mathbf{A}_\lambda$ is irreducible. Hence λ^p is irreducible if and only if λ^t is irreducible. Since almost all λ^t are irreducible, almost all λ^p are irreducible.

Now putting $\tau = t_E$ we obtain the next theorem.

Theorem 5. *Given a state p on* $\mathbf{A}$ *and a diagonal algebra* $\mathbf{E}$ *on* $L^2(p)$. *There exists a reducible Borel measure τ on the spectrum $\mathfrak{E}$ of* $\mathbf{E}$, *and for each $\lambda \in \mathbf{E}$ there corresponds a state λ^p on* $\mathbf{A}$ *with* $\lambda^p(I) = 1$ *such that*:

(1). *$\lambda^p(A)$ is continuous in $\mathfrak{E}$ for every fixed $A \in \mathbf{A}$.*

(2). $(A_p K\hat{p}, \hat{p}) = \int_{\mathfrak{E}} K(\lambda)\,\lambda^p(A)\,d\tau(\lambda)$ *for every $K \in \mathbf{E}$ and $A \in \mathbf{A}$.*

(3). *Almost all λ^p (concerning the measure τ) are irreducible.*

Let $\mathfrak{P}$ denote the set of all states p with $p(I) = 1$, and let $\mathfrak{N}$ denote the set of all irreducible states u with $u(I) = 1$. $\lambda \to \lambda^p$ is a weakly continuous mapping of $\mathfrak{E}$ into $\mathfrak{P}$. And almost all λ^p belong to $\mathfrak{N}$. Then the image $\mathfrak{D}$ of the mapping is contained in the weak closure $\bar{\mathfrak{N}}$ of $\mathfrak{N}$. For every set $\mathfrak{X} \subseteqq \mathfrak{D}$ let $\mathfrak{X}^q$ denote the inverse image $(x:\ x^p \in \mathfrak{X})$. Then the Borel measure ρ in $\mathfrak{D}$ relative to τ is defined by $\rho(\mathfrak{X}) = \tau(\mathfrak{X}^q)$ for every Borel set $\mathfrak{X} \subseteqq \mathfrak{D}$. Clearly we can regard ρ as a Borel measure on $\bar{\mathfrak{N}}$ whose support is $\mathfrak{D}$. Then necessarily ρ denotes the corresponding positive functional on $C(\bar{\mathfrak{N}})$.

Lemma 5.6. *Let $\mathfrak{X} \subseteqq \mathfrak{D}$ be a set so that $\mathfrak{X}^q$ is τ-measure* 0. *Then $\mathfrak{X}$ is ρ-measure* 0.

Proof. Every Borel measure which corresponds with a positive functional is regular. Then from $\tau(\mathfrak{X}^q) = 0$, there exists an open set $\mathfrak{U} \supset \mathfrak{X}^q$ with $\tau(\mathfrak{U}) < \varepsilon$, where ε is any given positive number. Now $\mathfrak{W} = \mathfrak{E} - \mathfrak{U}$ is compact, then $\mathfrak{W}^p = (x^p:\ x \in \mathfrak{W})$ is also compact, and $\mathfrak{V} = \mathfrak{D} - \mathfrak{W}^p$ contains $\mathfrak{X}$. Hence $\mathfrak{U} \geqq \mathfrak{V}^q \geqq \mathfrak{X}^q$, and $\rho(\mathfrak{V}) = \tau(\mathfrak{V}^q) \lesssim \tau(\mathfrak{U}) < \varepsilon$. This proves $\rho(\mathfrak{X}) = 0$.

Lemma 5.7. *Almost all states in $\bar{\mathfrak{N}}$ (concerning the measure ρ) are irreducible.*

Proof. Let $\mathfrak{T}$ denote the set of all non-irreducible states in $\mathfrak{D}$. $\mathfrak{T}^q$ is the set of all $\lambda \in \mathfrak{E}$ so that λ^p are non-irreducible. Then we have $\tau(\mathfrak{T}^q) = 0$ and $\rho(\mathfrak{T}) = 0$. q.e.d.

Let $\mathbf{M}$ denote the Banach algebra of all bounded ρ-measurable functions on $\bar{\mathfrak{N}}$ whose norms are defined by $|\varphi| = \text{ess. max } |\varphi(\lambda)|$. $\mathbf{M}$ is a $*$-algebra so that $\varphi^* = \bar{\varphi}$. For every $\varphi \in \mathbf{M}$ let φ^p denote the ρ-measurable function on $\mathbf{E}$ so that $\varphi^p(\lambda) = \varphi(\lambda^p)$ for $\lambda \in \mathbf{E}$. Then

Lemma 5.8. *Given $\varphi \in \mathbf{M}$. There exists $K_\varphi \in \mathbf{E}$ so that $K_\varphi(\lambda)$ coincides with $\varphi^p(\lambda)$ almost everywhere. This K_φ satisfies*

$$(A_p K_\varphi \hat{p}, \hat{p}) = \int_{\bar{\mathfrak{N}}} \lambda(A)\,\varphi(\lambda)\,d\rho(\lambda).$$

Mathematical Journal of Okayama University, Vol. 3 [1953], Iss. 2, Art. 8

Proof. Suffice it to prove the lemma on the assumption of $0 \leqq \varphi \leqq 1$. Let τ_φ denote the state on $\mathbf{E} = C(\mathfrak{E})$ so that

$$\tau_\varphi(Z) = \int_{\mathfrak{E}} \varphi^p(\omega) Z(\omega)\, d\tau(\omega)$$

for every $Z \in \mathbf{E}$. Then $\tau_\varphi \leqq \tau$. Since τ is reducible, there exists $K_\varphi \in \mathbf{E}$ so that

$$\tau_\varphi(Z) = \tau(K_\varphi Z) = \int_{\mathfrak{E}} K_\varphi(\omega) Z(\omega)\, d\tau(\omega).$$

Now K_φ coincides with $\varphi^p(\omega)$ almost everywhere. Further

$$(A_p K_\varphi \hat{p}, \hat{p}) = \int_{\mathfrak{E}} \omega^p(A) \varphi^p(\omega)\, d\tau(\omega) = \int_{\mathfrak{D}} \lambda(A) \varphi(\lambda)\, d\rho(\lambda).$$

This concludes the lemma.

Lemma 5.9. *The correspondence $\varphi \to K_\varphi$ is a *algebraic isometric isomorphism between $\mathbf{M}$ and $\mathbf{E}$.*

Proof. For every $\varphi \in \mathbf{M}$ we have

$$|\varphi| = \operatorname{ess.max}_{\omega} |\varphi^p(\omega)| = \sup_{\omega} |K_\varphi(\omega)| = |K_\varphi|.$$

Then $\varphi \to K_\varphi$ is a *-algebraic isometric isomorphism between $\mathbf{M}$ and a C^*-subalgebra $\mathbf{G}$ of $\mathbf{E}$. Now in order to show $\mathbf{G} = \mathbf{E}$, it is sufficient to show that for every $K \in \mathbf{E}$ with $0 \leqq K \leqq I$ there exists $\varphi \in \mathbf{M}$ with $K_\varphi = K$. Given $K \in \mathbf{E}$ with $0 \leqq K \leqq I$. Let ψ denote a positive functional on $C(\overline{\mathfrak{N}})$ so that

$$\psi(f) = \int f^p(\omega) K(\omega)\, d\tau(\omega).$$

From $I \geqq K(\omega) \geqq 0$ we have $\rho \geqq \psi \geqq 0$. Then ψ is absolutely continuous with respect to the measure ρ. And there exists a bounded ρ-measurable function φ on $\mathfrak{D} \subset \overline{\mathfrak{N}}$ so that $\psi(f) = \int \varphi(\lambda) f(\lambda)\, d\rho(\lambda)$. We show $K = K_\varphi$. Since K_φ coincides with $\varphi^p(\omega)$ almost everywhere, we have

$$\int f^p(\omega) K(\omega)\, d\tau(\omega) = \int f^p(\omega) K_\varphi(\omega)\, d\tau(\omega),$$

for every $f \in C(\overline{\mathfrak{N}})$.

Let $A \in \mathbf{A}$, then $A^\vee \in C(\overline{\mathfrak{N}})$ is defined by $A^\vee(\lambda) = \lambda(A)$. Now from $A^{\vee p}(\omega) = \omega^p(A)$ it follows

$$\int \omega^p(A) K_\varphi(\omega)\, d\rho(\omega) = \int \omega^p(A) K(\omega)\, d\rho(\omega).$$

That is

$$(A_p K_\varphi \hat{p}, \hat{p}) = (A_p K \hat{p}, \hat{p}).$$

Since $(K_\varphi A_p \hat{p}, B_p \hat{p}) = (K A_p \hat{p}, B_p \hat{p})$ holds for every $A, B \in \mathbf{A}$, we have $K_\varphi = K$. Hence $\mathbf{G}$ coincides with $\mathbf{E}$. q.e.d.

From Lemma 5.7 - 9 we conclude that

Theorem 6. *Let $\mathfrak{N}$ denote the set of all irreducible states u with $u(I) = 1$, and $\overline{\mathfrak{N}}$ denote its weak closure. Given a state p on $\mathbf{A}$ and a diagonal algebra $\mathbf{E}$ on $L^2(p)$. There exists a Borel measure ρ on $\overline{\mathfrak{N}}$ which satisfies the next conditions.*

(1). *$\overline{\mathfrak{N}} - \mathfrak{N}$ is ρ-measure* 0.

(2). *Let $\mathbf{M}$ denote the Banach algebra of all bounded measurable functions on $\overline{\mathfrak{N}}$ such that $|\varphi| = \text{ess. max} \, |\varphi(\lambda)|$ and $\varphi^* = \bar{\varphi}$. Then for every $\varphi \in \mathbf{M}$ there exists $K_\varphi \in \mathbf{E}$ so that*

$$(A_p K_\varphi \hat{p}, \hat{p}) = \int \varphi(\lambda) \lambda(A) \, d\rho(\lambda).$$

(3). *The correspondence $\varphi \to K_\varphi$ is a $*$-algebraic isometric isomorphism between two algebras $\mathbf{M}$ and $\mathbf{E}$.*

Now Theorem 1 in the introduction is obtained by a little change of the expression of Theorem 6.

Let $\mathfrak{K}$ denote the support of the measure ρ in Theorem 6. Then we obtain the integral expression $p = \int_{\mathfrak{K}} \lambda \, d\rho(\lambda)$ of p. This expression is called the *diagonal decomposition* of the state p (concerning $\mathbf{E}$), $\mathfrak{K}$ is called the *kernel space*, and the C^*-algebra $\mathbf{P} = (K_\varphi : \varphi \in C(\mathfrak{K}))$ is called a *kernel algebra* on $L^2(p)$. The concept of the kernel space and the kernel algebra may be important to study the properties of diagonal decompositions of states.

The kernel algebra $\mathbf{P}$ is isometrically isomorphic to $C(\mathfrak{K})$ by the correspondence $\varphi \to K_\varphi$. Now by the correspondence the spectrum of $\mathbf{P}$ is homeomorphicly representative for the kernel space $\mathfrak{K}$. Then we express every point of the kernel space by the same letter with the corresponding point of the spectrum of $\mathbf{P}$. Then $\mathfrak{K}$ denotes simultaneously the spectrum of $\mathbf{P}$. Every functions in $C(\mathfrak{K})$ and the corresponding operators in $\mathbf{P}$ are denoted by the same letters. Then for every $A \in \mathbf{A}$ and $K \in \mathbf{P}$ we have

$$(A_p K \hat{p}, \hat{p}) = \int K(\lambda) \lambda(A) \, d\rho(\lambda).$$

Theorem 7. *Given a state p and a kernel algebra* $\mathbf{P}$ *on* $L^2(p)$ *which corresponds with a diagonal decomposition* $p = \int_{\mathfrak{R}} \lambda d\rho(\lambda)$ *concerning a diagonal algebra* $\mathbf{E}$ *on* $L^2(p)$. *Then* (1). $\mathbf{E}$ *is the weak closure of* $\mathbf{P}$. (2). *We can regard the kernel space* $\mathfrak{R}$ *as the spectrum of* $\mathbf{P}$. (3). *Then* ρ *is the measure which corresponds with the state* $\rho(K) = (K\hat{p}, \hat{p})$ *on* $\mathbf{P}$. (4). *For every* $K \in \mathbf{P}$ *and* $A \in \mathbf{A}$,

$$(A_p K\hat{p}, \hat{p}) = \int_{\mathfrak{R}} K(\lambda)\lambda(A)\, d\rho(\lambda).$$

Proof. We need to prove (1) only. Since $\mathbf{E}$ is weakly closed, it is sufficient to show that $\mathbf{P}$ is weakly dense in $\mathbf{E}$. Let $\mathbf{L}$ denote the Banach space of all absolutely integrable functions on $\mathbf{P}$. The conjugate space of $\mathbf{L}$ is the space $\mathbf{M}$ of all bounded measurable functions on $\mathbf{P}$, in which $C(\mathfrak{R})$ is weakly dense. Given $\varphi \in \mathbf{M}$, we can choose a filter $\{F\} \subset C(\mathfrak{R})$ with $|F_\alpha| \leqq r < +\infty$, which converges weakly to φ. Then for every $A, B \in \mathbf{A}$ we have

$$\lim_\alpha \int F_\alpha(\lambda)\lambda(B^*A)\, d\rho(\lambda) = \int \varphi(\lambda)\lambda(B^*A)\, d\rho(\lambda).$$

That is,

$$\lim_\alpha (F_\alpha A_p\hat{p}, B_p\hat{p}) = (K_\varphi A_p\hat{p}, B_p\hat{p}),$$

where F_α express the bounded linear operators on $L^2(p)$ with $|F_\alpha| \leqq r$ which correspond with $F_\alpha \in C(\mathfrak{R})$. Since the set $(A_p\hat{p} : A \in \mathbf{A})$ is dense in $L^2(p)$, $\{F_\alpha\}$ converges weakly to K_φ. Hence the weak closure of $\mathbf{P}$ is $\mathbf{E}$.

Theorem 8. *Let* $p = \int_{\mathfrak{R}} \lambda\, d\rho(\lambda)$ *denote a diagonal decomposition of a state p on* $\mathbf{A}$.

(1). *If* $\mathbf{A}$ *is separable, then* $\mathfrak{R}$ *is a separable compact space.*

(2). *If* $\mathbf{A}$ *is commutative and has the spectrum* $\mathfrak{A}$, *then* ρ *is the usual representative measure for p, and the kernel space* $\mathfrak{R}$ *is the support of p. Then the diagonal decomposition coincides with the usual measure representation of p*:

$$p(f) = \int \lambda(f)\, d\rho(\lambda) = \int f(\lambda)\, d\rho(\lambda).$$

Proof. (1). As is well-known, the conjugate space of a separable Banach space is weakly separable. Since $\mathfrak{R}$ is a weakly compact sub-space of the conjugate space of the separable Banach space $\mathbf{A}$, $\mathfrak{R}$ is a separable compact space.

(2). Let **A** be commutative. Then the set $\mathfrak{N}$ of all irreducible state u with $u(I)=1$ coincides with the spectrum $\mathfrak{A}$ of **A**. Therefore $\overline{\mathfrak{N}}=\mathfrak{N}=\mathfrak{A}$. By the definition $\mathfrak{K}$ is a closed sub-space of $\overline{\mathfrak{N}}=\mathfrak{A}$. And ρ is a Borel measure on $\overline{\mathfrak{N}}=\mathfrak{A}$, whose support is $\mathfrak{K}$. And it satisfies

$$p(f) = \int \lambda(f)\, d\rho(\lambda) = \int f(\lambda)\, d\rho(\lambda).$$

Hence ρ is the representative measure for p.

Chapter 6. Decompositions of operator algebras.

Given a state p and its diagonal decomposition $p=\int_{\mathfrak{K}}\lambda\, d\rho(\lambda)$, concerning a kernel algebra **P** on $L^2(p)$. Let $\mathfrak{S}$ denote the set $(\sum_{i=1}^{n} K_i A_{i_p}\hat{p} :\ K_i \in \mathbf{P},\ A_i \in \mathbf{A})$. For every $f=\sum_{i=1}^{n} K_i A_{i_p}\hat{p}$ consider the function $f(\lambda)$ on $\mathfrak{K}$ so that $f(\lambda)=\sum_{i=1}^{n} K_i(\lambda) A_{i\lambda}\hat{\lambda}$. As we shall show in Lemma 6.3, the function $f(\lambda)$ is uniquely determined for every $f\in\mathfrak{S}$. We denote by $f=\int\oplus f(\lambda)$ whenever $f(\lambda)$ be constructed from an expression of $f\in\mathfrak{S}$ by the method as we stated above.

Lemma 6.1. *$f=\int\oplus f(\lambda)$ and $g=\int\oplus g(\lambda)$ imply $\alpha f+\beta g=\int\oplus(\alpha f(\lambda)+\beta g(\lambda))$.*

This is clear.

Lemma 6.2. *$f=\int\oplus f(\lambda)$ and $g=\int\oplus g(\lambda)$ imply $(f, g)=\int (f(\lambda), g(\lambda))\, d\rho(\lambda)$.*

Proof. By the definition $f(\lambda)$ and $g(\lambda)$ are represented as follows: $f(\lambda)=\sum_{i=1}^{m} K_i(\lambda) A_{i\lambda}\hat{\lambda}$ for $f=\sum_{i=1}^{m} K_i A_{i_p}\hat{p}$, and $g(\lambda)=\sum_{j=1}^{n} L_j(\lambda) B_{j\lambda}\hat{\lambda}$ for $g=\sum_{j=1}^{n} L_j B_{j_p}\hat{p}$, where $A_i, B_j \in \mathbf{A}$ and $K_i, L_j\in\mathbf{P}$. Hence

$$\begin{aligned}(f, g) = \sum_{ij}(K_i A_{i_p}\hat{p}, L_j B_{j_p}\hat{p}) &= \sum_{ji}\int K_i(\lambda)\overline{L_j(\lambda)}\,\lambda(B_j^* A_i)\, d\rho(\lambda)\\ &= \int (\sum_i K_i(\lambda) A_{i\lambda}\hat{\lambda}, \sum_j L_j(\lambda) B_{j\lambda}\hat{\lambda})\, d\rho(\lambda)\\ &= \int (f(\lambda), g(\lambda))\, d\rho(\lambda).\end{aligned}$$

Lemma 6.3. *$f\to f(\lambda)$ is a one-to-one linear correspondence.*

Proof. Let $f=\int\oplus f(\lambda)=\int\oplus g(\lambda)$. By Lemma 6.1 we have $0=f-f=\int\oplus(f(\lambda)-g(\lambda))$. Then $0=\int\|f(\lambda)-g(\lambda)\|^2d\rho(\lambda)$, where $\|f(\lambda)-g(\lambda)\|$ is continuous in $\mathfrak{R}$. Hence $f(\lambda)-g(\lambda)$ vanishes on $\mathfrak{R}$. And the function $f(\lambda)$ is uniquely determined for every $f\in\mathfrak{S}$. Further if $f\neq 0$, $\int\|f(\lambda)\|^2d\rho=\|f\|^2\neq 0$. Thus $f\to f(\lambda)$ is a one-to-one linear correspondence.

We now define a topological direct integral of Hilbert spaces as follows.

Definition. Let Ω be a locally compact space with a positive Borel measure ρ. Let $C(\Omega)$ denote the space of all bounded continuous functions on Ω. For each $\lambda\in\Omega$ a Hilbert space $\mathfrak{H}(\lambda)$ be given. A set $\mathfrak{S}$ of functions on Ω so that $f(\lambda)\in\mathfrak{H}(\lambda)$, is called a *base of a topological direct integral* if it satisfies the next conditions.

(1). For every $f, g\in\mathfrak{S}$, $(f(\lambda), g(\lambda))$ is continuous and integrable in Ω.

(2). $\mathfrak{S}$ is linear, i.e. $f(\lambda), g(\lambda)\in\mathfrak{S}$ imply $\alpha f(\lambda)+\beta g(\lambda)\in\mathfrak{S}$. For every $\varphi\in C(\Omega)$ and every $f\in\mathfrak{S}$, $\varphi f=\varphi(\lambda)f(\lambda)\in\mathfrak{S}$.

(3). The set $(f(\lambda): f\in\mathfrak{S})$ is dense in $\mathfrak{H}(\lambda)$.

In $\mathfrak{S}$ we can define an inner-product by

$$(f, g) = \int(f(\lambda), g(\lambda))\,d\rho(\lambda).$$

The Hilbert space $\mathfrak{H}$ which is the completion of $\mathfrak{S}$ is called the *topological direct integral* of $\mathfrak{H}(\lambda)$ and denoted by $\mathfrak{H}=\int_\Omega\mathfrak{H}(\lambda)\sqrt{d\rho}$.

Let $\mathfrak{H}=\int\mathfrak{H}(\lambda)\sqrt{d\rho}$ denote a topological direct integral with a base $\mathfrak{S}$. For each $\lambda\in\Omega$ given a bounded linear operator A_λ on $\mathfrak{H}(\lambda)$. If there exists a bounded linear operator A on $\mathfrak{H}$ so that $A\mathfrak{S}\subset\mathfrak{S}$ and that $(Af)(\lambda)=A_\lambda f(\lambda)$ for every $f\in\mathfrak{S}$, then A is called the direct integral of these A_λ, and denoted by $A=\int\oplus A_\lambda$.

Lemma 6.4. *If* $A=\int\oplus A_\lambda$, *then* $|A|=\sup_\lambda|A_\lambda|$.

Proof. For every $\varphi\in C(\Omega)$ we have $\|A\varphi f\|^2\leqq|A|^2\|\varphi f\|^2$. Then

$$\int|\varphi(\lambda)|^2\|A_\lambda f(\lambda)\|^2d\rho\leqq|A|^2\int|\varphi(\lambda)|^2\|f(\lambda)\|^2d\rho.$$

This proves $\|A_\lambda f(\lambda)\|^2\leqq|A|^2\|f(\lambda)\|^2$. Hence $|A_\lambda|\leqq|A|$.

Now put $r = \sup_\lambda |A_\lambda|$. Tnen

$$\int \| A_\lambda f(\lambda) \|^2 d\rho \leqq r^2 \int \| f(\lambda) \|^2 d\rho.$$

That is, $\| Af \|^2 \leqq r^2 \| f \|^2$. This proves $r \geqq |A|$ and $|A| = \sup |A_\lambda|$.

Lemma 6.5. *Given a topological direct integral* $\mathfrak{H} = \int_\Omega \mathfrak{H}(\lambda) \sqrt{d\rho}$ *with a base* $\mathfrak{S}$. *Then for each* $\varphi \in C(\Omega)$, $K_\varphi = \int \oplus \varphi(\lambda) I_\lambda$ *exists, where* I_λ *denotes the identity on* $\mathfrak{H}(\lambda)$. *The set* $\mathbf{K} = (K_\varphi : \varphi \in C(\Omega))$ *is a* C^**-algebra isometrically isomorphic to* $C(\Omega)$. $\mathbf{K}$ *is called the kernel algebra of the direct integral.*

Proof. Let $\varphi \in C(\Omega)$, then for every $f \in \mathfrak{S}$ we have $\| \varphi f \| \leqq |\varphi| \| f \|$. Therefore $f \to \varphi f$ is a bounded linear operator on $\mathfrak{S}$, and there exists a bounded linear operator K_φ on $\mathfrak{H}$ so that $K_\varphi f = \varphi f$ for every $f \in \mathfrak{S}$, where $K_\varphi = \int \oplus \varphi(\lambda) I_\lambda$. Further

$$|K_\varphi| = \sup_\lambda |\varphi(\lambda) I_\lambda| = \sup_\lambda |\varphi(\lambda)| = |\varphi|.$$

Hence $\varphi \to K_\varphi$ is an isometric isomorphism between two algebras $\mathbf{K}$ and $C(\Omega)$.

Theorem 9. *Given a topological direct integral* $\mathfrak{H} = \int \mathfrak{H}(\lambda) \sqrt{d\rho}$ *with a base* $\mathfrak{S}$. *Let* $\mathbf{A}$ *be a* C^**-algebra on* $\mathfrak{H}$ *so that every* $A \in \mathbf{A}$ *is a direct integral* $A = \int \oplus A(\lambda)$. *Let* $\mathbf{A}_\lambda$ *denote the uniform closure of the set* $(A_\lambda : A \in \mathbf{A})$. *Then* $A \to A_\lambda$ *is a* $*$*-algebraic homomorphism of* $\mathbf{A}$ *into a dense sub-algebra of* $\mathbf{A}_\lambda$. *We denote this fact by* $\mathbf{A} = \int \oplus \mathbf{A}_\lambda$, *and say that* $\mathbf{A}$ *is a direct integral of* $\mathbf{A}_\lambda$.

Proof. We only need to prove that the mapping preserves the $*$-operation. Let $A \in \mathbf{A}$, then $A = \int \oplus A_\lambda$ and $A^* = \int \oplus B_\lambda$. For every $\varphi \in C(\Omega)$ we have $(A\varphi f, g) = (\varphi f, A^* g)$. Then

$$\int \varphi(\lambda) (A_\lambda f(\lambda), g(\lambda)) d\rho = \int \varphi(\lambda) (f(\lambda), B_\lambda g(\lambda)) d\rho.$$

This proves $(A_\lambda f(\lambda), g(\lambda)) = (f(\lambda), B_\lambda g(\lambda))$. Then $B_\lambda = A_\lambda^*$. Hence $A \to A_\lambda$ is a $*$-algebraic homomorphism of $\mathbf{A}$. q.e.d.

Now from Lemma 6.1 - 3 we have immediately

Theorem 10. *Given a state* p *on a* C^**-algebra* $\mathbf{A}$, *and its diago-*

nal decomposition $p = \int_{\mathfrak{K}} \lambda d\rho$ *concerning a kernel algebra* $\mathbf{P}$ *on* $L^2(p)$. *Put* $\mathfrak{S}_p = (\sum K_i A_{ip}\hat{p}: K_i \in \mathbf{P}, A_i \in \mathbf{A})$, *and for* $f = \sum K_i A_{ip}\hat{p} \in \mathfrak{S}_p$ *let* $f(\lambda)$ *denote the function on* $\mathfrak{K}$ *so that* $f(\lambda) = \sum K_i(\lambda) A_{i\lambda}\hat{\lambda} \in L^2(\lambda)$. *Then by the expression* $f \to f(\lambda)$ *for* $f \in \mathfrak{S}_p$, $L^2(p)$ *is a topological direct integral* $L^2(p) = \int_{\mathfrak{K}} L^2(\lambda)\sqrt{d\rho}$ *with the base* $\mathfrak{S}_p$. *The kernel algebra on* $L^2(p)$ *is the kernel algebra of the direct integral. Every representative operator* A_p *on* $L^2(p)$ *for* $A \in \mathbf{A}$ *is a direct integral* $A_p = \int \oplus A_\lambda$, *where* A_λ *denotes the representative operator on* $L^2(\lambda)$ *for* A. *And the representative algebra* $\mathbf{A}_p$ *on* $L^2(p)$ *is a direct integral* $\mathbf{A}_p = \int \oplus \mathbf{A}_\lambda$, *where almost all representative algebras* $\mathbf{A}_\lambda$ *are irreducible.*

Let $\mathfrak{H}$ be a Hilbert space and $\mathbf{A}$ be a C^*-algebra on $\mathfrak{H}$. Given $g \in \mathfrak{H}$, let $[\mathbf{A}g]$ denote the closure of the set $(Ag: A \in \mathbf{A})$. For every $A \in \mathbf{A}$ let A_g denote the bounded linear operator on $[\mathbf{A}g]$ which coincides with A in $[\mathbf{A}g]$. Then the state g^s defined by $g^s(A) = (Ag, g)$ satisfies clearly $L^2(g^s) = [\mathbf{A}g]$, $\hat{g}^s = g$ and $A_{g^s} = A_g$ for every $A \in \mathbf{A}$. Further $\mathbf{A}_{g^s}$ coincides with the uniform closure $\mathbf{A}_g$ of the set $(A_g: A \in \mathbf{A})$. Then from Theorem 10 we have immediately.

Theorem 11. *Let* $\mathbf{A}$ *be a* C^**-algebra on* $\mathfrak{H}$. *Given* $g \in \mathfrak{H}$, $[\mathbf{A}g]$ *is a direct integral* $[\mathbf{A}g] = \int_{\mathfrak{G}} \mathfrak{H}(\lambda)\sqrt{d\rho}$ *which satisfies the next conditions.*

(1). $\mathfrak{G}$ *is the spectrum of the kernel algebra* $\mathbf{K}$.

(2). *The corresponding base* $\mathfrak{S}$ *contains* g, *and* $\|g(\lambda)\| = 1$.

(3). $\mathbf{A}_g$ *is a direct integral* $\mathbf{A}_g = \int \oplus \mathbf{A}_\lambda$ *so that almost all* $\mathbf{A}_\lambda$ *are irreducible.*

(4). *For every* $\lambda \neq \mu \in \mathfrak{G}$ *there exists at least one* $A \in \mathbf{A}$ *so that* $((Ag)(\lambda), g(\lambda)) \neq ((Ag)(\mu), g(\mu))$.

Lemma 6.6. $\mathfrak{H}$ *is a direct sum* $\mathfrak{H} = \sum \oplus [\mathbf{A}g_\alpha]$[1].

Proof. Consider the family Θ of all sets $\{g_\alpha\} \subseteq \mathfrak{H}$ so that any two of $[\mathbf{A}g_\alpha]$ are mutually orthogonal. Then Θ satisfies the Zorn's maximal condition, and there exists a maximal set $\{g_\alpha\}$. Assume that $\mathfrak{H} \neq \sum \oplus [\mathbf{A}g_\alpha]$. Then there exists $h \neq 0$ which is orthogonal to all $[\mathbf{A}g_\alpha]$. Now $(Ah, Bg_\alpha) = (h, A^*Bg_\alpha) = 0$ for every $A, B \in \mathbf{A}$. Then $[\mathbf{A}h]$ is orthogonal to all $[\mathbf{A}g_\alpha]$. So $\{h, g_\alpha\} \in \Theta$. It contradicts the maximality of $\{g_\alpha\}$. Hence $\mathfrak{H} = \sum \oplus [\mathbf{A}g_\alpha]$. q.e.d.

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1) $\mathfrak{M} = \Sigma \oplus \mathfrak{M}_\alpha$ expresses the fact that these $\mathfrak{M}_\alpha$ are mutually orthogonal, and $\mathfrak{M}$ is the smallest closed linear set which contains all $\mathfrak{M}_\alpha$.

Consider a decomposition $\mathfrak{H} = \sum \oplus [\mathbf{A}g_\alpha]$. Then each $[\mathbf{A}g_\alpha]$ is decomposed by Theorem 11. Combining these two decompositions we obtain the next decomposition theorem.

Theorem 12. *Let $\mathbf{A}$ be a C^*-algebra on $\mathfrak{H}$. Given a direct sum decomposition $\mathfrak{H} = \sum \oplus [\mathbf{A}g_\alpha]$ of $\mathfrak{H}$. Then each $[\mathbf{A}g_\alpha]$ is a topological direct integral $[\mathbf{A}g_\alpha] = \int_{\mathfrak{G}_\alpha} \mathfrak{H}(\lambda)\sqrt{d\rho_\alpha}$ which satisfies the conditions of Theorem 11. Let $\mathfrak{S}_\alpha$ denote the concerning bases to these decompositions, and $\mathfrak{S}$ denote the smallest linear set which contains all $\mathfrak{S}_\alpha$. Let Ω denote the discrete sum of the spaces $\mathfrak{G}_\alpha$, and ρ denote the measure on Ω so that $\rho(X) = \sum \rho_\alpha(X \cap \mathfrak{G}_\alpha)$ for every Borel set $X \subseteqq \Omega$. Then $\mathfrak{H}$ is a topological direct integral $\mathfrak{H} = \int_\Omega \mathfrak{H}(\lambda)\sqrt{d\rho}$ with the base $\mathfrak{S}$. And correspondingly $\mathbf{A}$ is a direct integral $\mathbf{A} = \int \oplus \mathbf{A}_\lambda$ whose almost all $\mathbf{A}_\lambda$ are irreducible.*

Proof. For every $f = \sum f_\alpha \in \mathfrak{S}$ we define the function $f(\lambda)$ on Ω by $f(\lambda) = f_\alpha(\lambda)$ for $\lambda \in \mathfrak{G}_\alpha$. We show that $\mathfrak{S}$ is a base of a direct integral by this expression $f \to f(\lambda)$. $\mathfrak{S}$ is clearly linear, and for every $\varphi \in C(\Omega)$ and $f = \sum f_\alpha \in \mathfrak{S}$, $\varphi f = \sum \varphi f_\alpha \in \mathfrak{S}$. Further if $f = \sum f_\alpha$, $g = \sum g_\alpha \in \mathfrak{S}$, then only finite numbers of these $f_\alpha, g_\alpha \in \mathfrak{S}_\alpha$ are $\neq 0$. Then

$$(f, g) = \sum_\alpha (f_\alpha, g_\alpha) = \sum_\alpha \int_{\mathfrak{G}_\alpha} (f_\alpha(\lambda), g_\alpha(\lambda))\, d\rho_\alpha$$
$$= \int_\Omega (f(\lambda), g(\lambda))\, d\rho.$$

Hence $\mathfrak{H}$ is a direct integral $\mathfrak{H} = \int \mathfrak{H}(\lambda)\sqrt{d\rho}$ with the base $\mathfrak{S}$. Let $f = \sum f_\alpha \in \mathfrak{S}$ and $A \in \mathbf{A}$, then $Af = \sum A_{g_\alpha} f_\alpha \in \mathfrak{S}$. Now put $A_\lambda = A_{g_\alpha \lambda}$ for every $\lambda \in \mathfrak{G}_\alpha$, then clearly $A = \int \oplus A_\lambda$ and $\mathbf{A} = \int \oplus \mathbf{A}_\lambda$, where almost all $\mathbf{A}_\lambda$ are irreducible.

The J. v. Neumann's decomposition.

Let $\mathbf{M}$ be a C^*-algebra on $\mathfrak{H}$, and $\mathbf{N}$ be a weakly dense C^*-subalgebra of the commutor $\mathbf{M}'$. Consider a decomposition of $\mathfrak{H}$: $\mathfrak{H} = \int \mathfrak{H}(\lambda)\sqrt{d\rho}$ in Theorem 12, concerning the algebra $\mathbf{R} = \mathbf{M} \cup \mathbf{N}$. Then $\mathbf{R}$ is a direct integral $\mathbf{R} = \int \oplus \mathbf{R}_\lambda$ so that almost all $\mathbf{R}_\lambda$ are irreducible. Let $\mathbf{M}_\lambda$ and $\mathbf{N}_\lambda$ denote the uniform closures of sets $(A_\lambda:\ A \in \mathbf{M},\ A = \int \oplus A_\lambda)$ and $(B_\lambda:\ B \in \mathbf{N},\ B = \int \oplus B_\lambda)$ respectively.

Mathematical Journal of Okayama University, Vol. 3 [1953], Iss. 2, Art. 8

Then $\mathbf{M}$ and $\mathbf{N}$ are direct integrals $\mathbf{M}=\int\oplus\mathbf{M}_\lambda$ and $\mathbf{N}=\int\oplus\mathbf{N}_\lambda$. Let $\mathbf{B}_\lambda$ denote the algebra of all bounded linear operators on $\mathfrak{H}(\lambda)$, and $\mathbf{O}_\lambda$ denote the algebra (αI_λ). Whenever $\mathbf{R}_\lambda$ be irreducible, we have $\mathbf{M}_\lambda''\geqq\mathbf{N}_\lambda'$. Further

$$\mathbf{M}_\lambda''\cup\mathbf{M}_\lambda',\ \mathbf{N}_\lambda''\cup\mathbf{N}_\lambda'\geqq\mathbf{M}_\lambda''\cup\mathbf{N}_\lambda''\geqq\mathbf{R}_\lambda''=\mathbf{B}_\lambda,$$

and

$$\mathbf{M}_\lambda''\cap\mathbf{M}_\lambda'=\mathbf{N}_\lambda''\cap\mathbf{N}_\lambda'=\mathbf{M}_\lambda''\cap\mathbf{N}_\lambda''=\mathbf{O}_\lambda.$$

Following v. Neumann, we call such a pair of algebras $(\mathbf{M}_\lambda'', \mathbf{N}_\lambda'')$ a factorized pair. Then the v. Neumann's decomposition theorem is generalized as follows.

Theorem 13. *Let* $\mathbf{M}$ *be a* C^**-algebra, and* $\mathbf{N}$ *be a* C^**-subalgebra of* $\mathbf{M}'$ *which is weakly dense in* $\mathbf{M}'$. *Then* $\mathfrak{H}$ *is a topological direct integral* $\mathfrak{H}=\int\mathfrak{H}(\lambda)\sqrt{d\rho}$, *which induces the decomposition of* $\mathbf{M}$ *and* $\mathbf{N}$: $\mathbf{M}=\int\oplus\mathbf{M}_\lambda$ *and* $\mathbf{N}=\int\oplus\mathbf{N}_\lambda$ *so that almost all pairs* $(\mathbf{M}_\lambda'', \mathbf{N}_\lambda'')$ *are factorized pairs.*

When the space $\mathfrak{H}$ is separable, we can refine the Theorem 13 as follows.

Theorem 14. *Let* $\mathbf{M}$ *be a* C^**-algebra on a separable Hilbert space* $\mathfrak{H}$, *and* $\mathbf{N}$ *be a weakly dense* C^**-subalgebra of the commutor* $\mathbf{M}'$. *Then we can choose* $g\in\mathfrak{H}$ *with* $\mathfrak{H}=[(\mathbf{M}\cup\mathbf{N})g]$. $\mathfrak{H}$ *is a direct integral* $\mathfrak{H}=\int_{\mathfrak{G}}\mathfrak{H}(\lambda)\sqrt{d\rho}$ *which satisfies the next conditions.*

(1). $\mathfrak{G}$ *is the spectrum of the kernel algebra* $\mathbf{K}$.

(2). $\mathbf{K}$ *is weakly dense in* $\mathbf{M}''\cap\mathbf{M}'$.

(3). *The concerning base* $\mathfrak{S}$ *contains* g, *and* g *satisfies* $\|g(\lambda)\|=1$.

(4). $\mathbf{M}$ *and* $\mathbf{N}$ *are direct integrals* $\mathbf{M}=\int\oplus\mathbf{M}_\lambda$ *and* $\mathbf{N}=\int\oplus\mathbf{N}_\lambda$ *so that almost all pairs* $(\mathbf{M}_\lambda'', \mathbf{N}_\lambda'')$ *are factorized pairs.*

(5). *For every* $\lambda\neq\mu\in\mathfrak{G}$ *there exist two operators* $A\in\mathbf{M}$ *and* $B\in\mathbf{N}$ *which satisfy*

$$((Ag)(\lambda), (Bg)(\lambda))\neq((Ag)(\mu), (Bg)(\mu)).$$

Proof. Choose a direct sum decomposition $\mathfrak{H}=\sum_\alpha\oplus[(\mathbf{M}\cup\mathbf{N})g_\alpha]$ by lemma 6.6. Since $\mathfrak{H}$ is separable, these $g_\alpha\neq0$ are at most countable. That is, $\mathfrak{H}=\sum_{i=1}^{\infty}\oplus[\mathbf{R}g_i]$, where we put $\mathbf{R}=\mathbf{M}\cup\mathbf{N}$. We can assume here $\|g_i\| \leqq 2^{-i}$ since we can replace g_i by $2^{-i}g_i/\|g_i\|$ whenever $g_i \neq 0$. Then $g = \sum_i g_i$ converges. Now we show $\mathfrak{H} = [\mathbf{R}g]$. Let P and P_i denote the projections of $[\mathbf{R}g]$ and $[\mathbf{R}g_i]$ respectively. Then we can easily shown that P and P_i belong to $\mathbf{R}' = \mathbf{N}' \cap \mathbf{M}' = \mathbf{M}'' \cap \mathbf{M}'$. Then these are mutually commutative. Further $Pg = g$, $P_i g_i = g_i$ and $P_i P_j = 0$ for $i \neq j$. Now

$$Pg_i = PP_i g = P_i Pg = P_i g = g_i .$$

This proves $g_i \in [\mathbf{R}g]$ and $[\mathbf{R}g_i] \subseteqq [\mathbf{R}g]$. Then $\mathfrak{H} = \sum \oplus [\mathbf{R}g_i] \subseteqq [\mathbf{R}g] = [(\mathbf{M} \cup \mathbf{N})g]$.

Now in the proof of Theorem 13, we can replace the use of Theorem 12 by the direct use of Theorem 11. Then $\mathfrak{H} = [\mathbf{R}g]$ is a direct integral $\mathfrak{H} = \int_{\mathfrak{S}} \mathfrak{H}(\lambda)\sqrt{d\rho}$ which satisfies (1), (3), (4) of the Theorem, and the condition

(5′). *For every* $\lambda \neq \mu \in \mathfrak{S}$ *there exists* $X \in \mathbf{M} \cup \mathbf{N}$ *so that* $((Xg)(\lambda), g(\lambda)) \neq ((Xg)(\mu), g(\mu))$.

Since the set $(\sum_{i=1}^{n} A_i B_i : A_i \in \mathbf{M}, B_i \in \mathbf{N})$ is uniformly dense in $\mathbf{M} \cup \mathbf{N}$, we obtain the condition (5). Finally we show the condition (2). The kernel algebra $\mathbf{K}$ of the direct integral is the kernel algebra on $L^2(g^s)$, where g^s is the state on $\mathbf{R} = \mathbf{M} \cup \mathbf{N}$ defined by $g^s(X) = (Xg, g)$. But by Theorem 7, the kernel algebra $\mathbf{K}$ is weakly dense in a diagonal algebra $\mathbf{E}$ on $L^2(g^s) = \mathfrak{H}$. Now $\mathbf{E}$ is a maximal commutative C^*-subalgebra of the commutative C^*-algebra $\mathbf{R}' = \mathbf{M}'' \cap \mathbf{M}'$. Then $\mathbf{E}$ coincides with $\mathbf{R}' = \mathbf{M}'' \cap \mathbf{M}'$. This concluds the condition (2).

The Moutner's decomposition.

Let $\mathbf{A}$ be a C^*-algebra on $\mathfrak{H}$. By Lemma 5.1 we can choose a commutative C^*-algebra $\mathbf{E}$ so that $(\mathbf{A} \cup \mathbf{E})' = \mathbf{E}$. Consider a decomposition $\mathfrak{H} = \int_{\Omega} \mathfrak{H}(\lambda)\sqrt{d\rho}$ in Theorem 12 concerning the algebra $\mathbf{A} \cup \mathbf{E}$. Then $\mathbf{A} \cup \mathbf{E}$ is a direct integral $\mathbf{A} \cup \mathbf{E} = \int \oplus (\mathbf{A} \cup \mathbf{E})_\lambda$ so that almost all $(\mathbf{A} \cup \mathbf{E})_\lambda$ are irreducible. Now for every $\varphi(\lambda) \in C(\Omega)$, $\int \oplus \varphi(\lambda) I_\lambda$ commutes with all $A = \int \oplus A_\lambda \in \mathbf{A} \cup \mathbf{E}$. Then $\int \oplus \varphi(\lambda) I_\lambda \in (\mathbf{A} \cup \mathbf{E})' = \mathbf{E}$. So $\mathbf{E}$ contains the kernel algebra $\mathbf{K}$ of the direct integral. Conversely let $E = \int \oplus E_\lambda \in \mathbf{E}$. Then E commutes with all $A \in \mathbf{A} \cup \mathbf{E}$. And E_λ commutes with all $A_\lambda \in (\mathbf{A} \cup \mathbf{E})_\lambda$. Therefore, whenever

$(\mathbf{A} \cup \mathbf{E})_\lambda$ be irreducible, E_λ should be denoted as $E_\lambda = \alpha(\lambda) I_\lambda$. Then we have $E_\lambda = \alpha(\lambda) I_\lambda$ almost everywhere. Let $\mathfrak{S}$ denote the concerning base of the direct integral. Then for every $\lambda \in \Omega$, there exists $f \in \mathfrak{S}$ so that $f(\lambda) \neq 0$. Now $(E_\lambda f(\lambda), f(\lambda)) = \alpha(\lambda) \| f(\lambda) \|^2$ holds almost everywhere. Let $U \ni \lambda$ denote an open set in which $f(\lambda) \neq 0$. Then $\alpha(\lambda)$ coincides with the continuous function $\psi(\lambda) = (E_\lambda f(\lambda), f(\lambda)) / \| f(\lambda) \|^2$ in a dense subset of U. So for every $g, h \in \mathfrak{S}$, $(E_\lambda g(\lambda), h(\lambda))$ and $\psi(\lambda)(g(\lambda), h(\lambda))$ coincide with each other in a dense subset of U. Since these functions are continuous, these must coincide in the whole U. Hence we have $E_\lambda = \psi(\lambda) I_\lambda$ on U. Now E_λ is denoted as $\psi(\lambda) I_\lambda$ on the whole Ω, where $\psi(\lambda)$ is a bounded continuous function on Ω. Thus $E = \int \oplus \psi(\lambda) I_\lambda$ belongs to the kernel algebra $\mathbf{K}$.

Therefore $\mathbf{E}$ coincides with the kernel algebra $\mathbf{K}$.

Finally, $\mathbf{T} = (\sum E_i A_i : E_i \in \mathbf{E}, A \in \mathbf{A})$ is uniformly dense in $\mathbf{A} \cup \mathbf{E}$, then $\mathbf{T}_\lambda = (\sum E_i(\lambda) A_{i\lambda} : E_i \in \mathbf{E}, A_i \in \mathbf{A}) = (A_\lambda : A \in \mathbf{A})$ is uniformly dense in $(\mathbf{A} \cup \mathbf{E})_\lambda$. Therefore $\mathbf{A}$ is a direct integral $\mathbf{A} = \int \oplus (\mathbf{A} \cup \mathbf{E})_\lambda$. Hence putting $\mathbf{A}_\lambda = (\mathbf{A} \cup \mathbf{E})_\lambda$ we obtain the next theorem.

Theorem 15. *Let $\mathbf{A}$ be a C^*-algebra and $\mathbf{E}$ be a commutative C^*-algebra so that $(\mathbf{A} \cup \mathbf{E})' = \mathbf{E}$. Then $\mathfrak{H}$ is a topological direct integral $\mathfrak{H} = \int_\Omega \mathfrak{H}(\lambda) \sqrt{d\rho}$ so that $\mathbf{E}$ is the kernel algebra, and $\mathbf{A}$ is a direct integral $\mathbf{A} = \int \oplus \mathbf{A}_\lambda$ whose almost all $\mathbf{A}_\lambda$ are irreducible.*

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