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An Example of an Indecomposable Module which is not Injective

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AN EXAMPLE OF AN INDECOMPOSABLE MODULE WHICH IS NOT INJECTIVE

Dedicated to Professor Manabu Harada on his 60th birthday

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It is well-known that every torsionfree divisible module is injective over a commutative integral domain. First we show that this theorem can be improved by using the concept of weakly σ-injective modules over a left Ore domain. Also in [2], we show that for a non-singular module M if M has no nonzero injective submodule, then so does M^{Λ} for all index sets Λ. Finally, we give an example to show that the above proposition is faulse in general.

Throughout this note R is a ring with identity and modules are unitary left R-modules unless otherwise stated. We denote the category of modules by R-*mod* and the injective hull of a module M by $E(M)$. As for terminologies and basic properties concerning torsion theories and preradicals, we refer to [3]. Let ρ be a preradical. We call it *stable* if $T(\rho)$ is closed under essential extensions. Also the left linear topology corresponding to a left exact preradical ρ is denoted by $\mathscr{L}(\rho)$. Now for two preradicals ρ and τ, we shall say that ρ *is larger than* τ if $\rho(M) \supseteq \tau(M)$ for all modules M.

We put $\mathscr{D} = \{\mathrm{r} \in R \mid \mathrm{rs} \neq 0 \text{ and } \mathrm{sr} \neq 0 \text{ for all } s\ (\neq 0) \in R\}$. We call a ring R *left Ore* if for each $\mathrm{r} \in R$ and $\mathrm{s} \in \mathscr{D}$ there exist $\mathrm{r}' \in R$ and $\mathrm{s}' \in \mathscr{D}$ such that $\mathrm{s}'\mathrm{r} = \mathrm{r}'\mathrm{s} \neq 0$. Also we put $\sigma(M) = \{\mathrm{x} \in M \mid \mathrm{rx} = 0 \text{ for some } \mathrm{r} \in \mathscr{D}\}$. In general, $\sigma(M)$ is not a submodule of M.

However

Proposition 1. *If R is a left Ore domain, then σ is the Goldie torsion functor G.*

Proof. By [3, p. 138, Example 2], σ is a left exact radical. By assumption, every non-zero ideal of R is essential in R. Thus $\mathscr{L}(Z)$ is the set of non-zero ideal of R, where Z is the singular torsion functor. Thus $\sigma \leq Z$. Conversely let r be a non-zore element of R and let $\mathrm{s}+R\mathrm{r}$ be in $R/R\mathrm{r}$. Then $\mathrm{s}'\mathrm{s} = \mathrm{r}'\mathrm{r}$ for some r' and s' in R by assumption. Thus $R/R\mathrm{r}$ is in $T(\sigma)$ and so $R\mathrm{r}$ is in $\mathscr{L}(\sigma)$. Hence every non-zero ideal of R belongs to $\mathscr{L}(\sigma)$. Since σ is a radical, so is Z, namely, $Z = G$. Hence $\sigma = G$.

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Definition. Let τ be a preradical. We call a module H (resp. *weakly*) *τ-injective* if for all exact sequences of modules $O \to A \to B \to C \to O$ with $C \in T(\tau)$ (resp. $B \in T(\tau)$), the functor $Hom_R(-, H)$ preserves the exactness.

Lemma 2 [1, Theorem 1. 11.]. *Let τ be a left exact preradical. Then the following conditions are equivalent:*

(i) *Every τ-injective module is injective.*

(ii) *$\bar{\tau}$ is larger than the Goldie torsion functor G, where $\bar{\tau}$ is the smallest radical larger than τ.*

By the above lemma, every σ-injective module is injective.

Theorem 3. *Let R be a left Ore domain. For a module M, the following conditions are equivalent:*

(i) *M is divisible and weakly σ-injective.*

(ii) *M is injective.*

Proof. (ii) $\Rightarrow$ (i) is clear. (i) $\Rightarrow$ (ii). By Proposition 1, $\sigma = G$. Thus it is sufficient to show that M is σ-injective by Lemma 1.2. We assume that $\sigma(E(M)/M) \neq O$. Then there exists $\bar{x} = x + M$ (x is in $E(M)$ and is not in M) such that $r\bar{x} = 0$ for some $r(\neq 0)$ in R. Since M is divisible and rx is in M, rx $=$ rm for some m in M, namely $r(x-m) = 0$. Thus $x-m$ is in $\sigma(E(M))$. Since M is weakly σ-injective, $\sigma(E(M)) = \sigma(M)$ and so $x-m$ is in M. Hence x is in M. This is a contradiction. Thus M is σ-injective.

Since σ is left exact, every σ-torsionfree module is weakly σ-injective. Thus we have the following famous result:

Corollary 4. *Let R be a commutative integral domain and M a torsionfree module. Then M is injective if and only if it is divisible.*

We call a ring R *left hereditary* if every left ideal of R is projective. From Theorem 3 and [3, Proposition 4.5], we have

Corollary 5. *Let R be a left Ore domain. Then the following coditions are equivalent:*

(i) *Every divisible module is weakly σ-injective.*

(ii) *Every divisible module is σ-injective.*

(iii) *Every divisible module is injective.*

(iv) *R is left hereditary.*

First we give an example of a module which is divisible but not injective.

Let $\mathbf{Z}$ be the ring of integers, $\mathbf{Q}$ the field of rational numbers and $\mathbf{Z}_p$ the localization of $\mathbf{Z}$ with respect to $p\mathbf{Z}$ for a prime number p.

Example 6. *Let R be the polynomial ring over $\mathbf{Z}_p$ and K the quotient field of R. Then K/R is not injective.*

Proof. We put $I = pR+xR$. Let f be a map from I to K/R with $f(pa_0 +a_1x+ \cdots\cdots +a_nx^n) = (p/x)(a_0+a_1x+ \cdots\cdots +a_nx^n)+((1-p)/p)(a_1+a_2x+ \cdots\cdots +a_nx^{n-1})+R$, where $a_i(i = 0, 1, \cdots\cdots, n)$ are in $\mathbf{Z}_p$. Then $f(p) = p/x+R$ and $f(x) = 1/p+R$. Suppose that K/R is injective. Then there exists an R-homomorphism $g: R \to K/R$ such that $g(a) = f(a)$ for all $a \in I$. We put $g(1) = k+R(k \in K)$. Then $g(p) = pk+R = p/x+R = f(p)$ and $g(x) = xk +R = 1/p+R = f(x)$. Since $pk-p/x \in R$ and $xk-x/p \in R$, k =

$$\frac{p+c_0x+c_1x^2+ \cdots\cdots +c_mx^{m+1}}{px} = \frac{1+pd_0+pd_1x+ \cdots\cdots +pd_nx^n}{px}$$

for some $c_i \in \mathbf{Z}_p$ $(i = 0, 1, \cdots\cdots, m)$ and $d_j \in \mathbf{Z}_p$ $(j = 0, 1, \cdots\cdots, n)$. Thus $1+pd_0 = p$ and so $d_0 = (p-1)/p$ does not belong to $\mathbf{Z}_p$. This is a contradiction. Hence K/R is not injective.

If a module M is nonsingular, then M^{Λ} has no nonzero injective submodule if and only if M has no nonzero injective submodule, where M^{Λ} is a direct product of copies of M for an index set Λ [2, Theorem 2.9]. But this is not true for some singular module M.

Lemma 7. *Let R be a commutative integral domain with quotient field $K \neq R$. Then the following assertions hold.*

(1) *K is an injective R-module.*

(2) *$(K/R)^{R-\{0\}}$ has a nonzero injective submodule.*

Proof. (1). Since K is divisible and is in $F(\sigma)$, it is injective. (2). We consider a correspondence $\phi: K \to (K/R)^{R-\{0\}}$ defined by $\phi(k) = (\cdots\cdots, k/r_\alpha, \cdots\cdots)$. Then ϕ is an R-homomorphim and $Ker(\phi) = \{k \in K \mid k/r_\alpha \in R$ for all $r_\alpha \in R-\{O\}\}$. Clearly ϕ is a monomorphism. By (1), $(K/R)^{R-\{0\}}$ has a nonzero injective submodule.

Example 8. *Let $R = \mathbf{Z}_p+x\mathbf{Q}[[x]]$, where $\mathbf{Q}[[x]]$ is the ring of formal power series over $\mathbf{Q}$ and K the quotient field of R. Then K/R has no nonzero* *injective submodule but* $(K/R)^{R-\{0\}}$ *has a nonzero injective submodule.*

Proof. It is sufficient to show that K/R is indecomposable and it is not injective. First we show that K/R is indecomposable. We assume that $K/R = A/R \oplus B/R$, where A and B are R-submodules of K containing R with $K = A+B$ and $A \cap B = R$. We claim that $1/x$ belongs either to A or to B. Since $1/x$ is in $A+B$, there exist $\alpha \in A$ and $\beta \in B$ such that $1 = \alpha x + \beta x$. Thus $\alpha x = 1-\beta x$ is in B and so αx is in R. Simillary βx is in R. Therefore $\alpha x = a_0 + a_1 x + \cdots\cdots + a_n x^n + \cdots\cdots$ and $\beta x = b_0 + b_1 x + \cdots\cdots + b_n x^n + \cdots\cdots$ for some a_0 and b_0 is $\mathbf{Z}_p$ and a_i and b_i $(i = 1, 2, \cdots\cdots)$ in $\mathbf{Q}$. Since $1 = \alpha x + \beta x$, $a_0 + b_0 = 1$ and $a_i + b_i = 0$ for $i \geq 1$. Thus either a_0 or b_0 is a unit. If a_0 (resp. b_0) is a unit, then αx (resp. βx) is unit in R and so $1/x = \alpha(\alpha x)^{-1}$ (resp. $\beta(\beta x)^{-1}$) is in A (resp. B). Thus we may assume that $1/x$ is in A. Then $\mathbf{Q}[[x]]$ is an R-submodule of A. In fact take $\gamma = c_0 + c_1 x + c_2 x^2 + \cdots\cdots$ be in $\mathbf{Q}[[x]]$. If $c_0 = 0$, then γ is in $R \subset A$. On the other hand, if $c_0 \neq 0$, then $c_0 = (1/x)\cdot c_0 x \in A$. Since $c_1 x + c_2 x^2 + \cdots\cdots$ is in R, γ is in A and so $\mathbf{Q}[[x]]$ is an R-submodule of A. Next we show that α/x belongs to A for every $\alpha \in A$. Indeed, let $\alpha/x = \alpha' + \beta'$ ($\alpha' \in A$ and $\beta' \in B$). Then $\alpha = x\alpha' + x\beta'$ and so $x\beta' = \alpha - x\alpha'$. Thus $x\beta'$ is in $A \cap B = R$. We put $x\beta' = d_0 + d_1 x + d_2 x^2 + \cdots\cdots$, where $d_0 \in \mathbf{Z}_p$ and $d_i \in \mathbf{Z}$ $(i = 1, 2, \cdots\cdots)$. Then $\beta' = c_0/x + \sum_{i=1}^{\infty} d_i x^{i-1}$. Since c_0/x is in A and $\sum_{i=1}^{\infty} d_i x^{i-1}$ is in $\mathbf{Q}[[x]]$, β' is in A. Hence α/x is in A. As is easily seen, each element of K is of the form $h(x)/x^m$, where $h(x)$ is in $\mathbf{Q}[[x]]$ and m is a non-negative integer. Thus $K = A$, namely K/R is indecomposable. Secondly we show that K/R is not injective. We put $I = x\mathbf{Q}[[x]]$ and $I_n = (x/p^n)R$ for $n = 0, 1, 2, \cdots\cdots$. Then we have an infinite ascending chain $I_0 \subsetneqq I_1 \subsetneqq I_2 \subsetneqq \cdots\cdots \subsetneqq I_n \subsetneqq I_{n+1} \subsetneqq \cdots\cdots$ with $\bigcup_{n=0}^{\infty} I_n = I$. Note that any homomorphism φ_n from I_n to K/R is uniquely determined by an element α_n of K/R which can be arbitrarily chosen and by the equation $\varphi_n(a) = a\alpha_n$ for all $a \in I_n$. Let $\{a_i\}$ be a sequence of integers such that $0 \leqq a_i < p$ for all $i \geqq 0$. For each n, let $f_n = (1/x)\sum_{i=0}^{n} a_i p^i$ be an element of K and $\alpha_n = f_n + R$ an element of K/R. Then we have $\alpha_{n+1} - \alpha_n = (1/x)a_{n+1}p^{n+1} + R$. Since $a_{n+1}p$ is in $\mathbf{Z}$, $(x/p^n)(\alpha_{n+1} - \alpha_n) = 0$ for $n = 0, 1, 2, \cdots\cdots$. Let φ_n be the R-homomorphism from I_n to K/R corresponding to α_n as noted above. Then the above equations imply that the system (I_n, φ_n) forms a direct system and $\varinjlim I_n = I$, we have the following commutative diagram

$$\begin{tikzcd}
& I_n \arrow[d] \arrow[ddl, "\psi_n"'] \arrow[ddr, "\varphi_n"] & \\
& I_m \arrow[dl, "\psi_m"'] \arrow[dr, "\varphi_m"] & \\
\varinjlim I_n \arrow[rr, "\varphi = \varinjlim \varphi_n"'] & & K/R
\end{tikzcd}$$

where $\varphi(a) = \varphi_n(a) = a\alpha_n$ for all $a \in I_n$.

Assume that K/R is injective. Then there exists an element α of K/R such that $\varphi(s) = s\alpha$ for all $s \in I$. If we take $s = x/p^n \in I_n$, then $\varphi(s) = s\alpha = s\alpha_n = \varphi_n(s)$ and so $(x/p^n)(\alpha - \alpha_n) = 0$. If we put $\alpha = f + R$, where $f \in K$, it follows that $(x/p^n)(f - f_n) \in R$ for all n, namely, $xf - \sum_{i=0}^{n} a_i p^i \in p^n R$. Hence we have xf is in R. We put $xf = \sum_{i=0}^{\infty} b_i p^i$, where $b_0 \in \boldsymbol{Z}_p$ and $b_i \in \boldsymbol{Q}$ for $i \geq 1$. Then we have $b_0 = \sum_{i=0}^{n} a_i p^i + p^n c_0$ for some $c_0 \in \boldsymbol{Z}_p$. Thus the sequence $\{\sum_{i=0}^{n} a_i p^i\}_n$ converges to b_0 with respect to the $p\boldsymbol{Z}_p$-adic topology on $\boldsymbol{Z}_p$, that is, the p-adic number $\sum_{i=0}^{\infty} a_i p^i$ represents a rational number. This is absurd because the sequence $\{a_i\}_i$ of integers can be arbitrarily chosen. It follows that K/R is not injective.

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