

Mathematical Journal of Okayama University

Volume 34, Issue 1

1992

Article 21

JANUARY 1992

Degrees of Self-maps

Akira Sasao*

*Waseda University

Copyright ©1992 by the authors. *Mathematical Journal of Okayama University* is produced by
The Berkeley Electronic Press (bepress). <http://escholarship.lib.okayama-u.ac.jp/mjou>

DEGREES OF SELF-MAPS

AKIRA SASAO

0. Introduction. As well-known, one of the basic problems in algebraic topology is to determine the homology representation

$$H : [X, Y] \rightarrow \text{Hom}(H_*(X), H_*(Y))$$

where $[X, Y]$ denotes the set of homotopy classes of maps from X to Y .

In this paper we shall consider it for the case $X = Y$. Previously, Sullivan and Quillen proved the famous theorem for $X = \mathbb{H}\mathbb{P}^\infty$ in [7] and C. A. McGibbon determined the image of H for $X = \mathbb{R}\mathbb{P}^n, \mathbb{C}\mathbb{P}^n$ and $\mathbb{H}\mathbb{P}^n$ in the stable case in [3]. Furthermore D. M. Davis investigated it for $X = \mathbb{C}\mathbb{P}^{n+2}/\mathbb{C}\mathbb{P}^{n-1}$, the stunted projective space in [1] and also S. Sasao and M. Nagaishi determined it for $X = \mathbb{H}\mathbb{P}^3$ in [6]. In this paper we shall consider the case $X = S^n \cup e^{n+2} \cup e^{n+4}$, which is a generalization of Davis's case and contains the following:

- (1) The total spaces of S^2 -bundles over S^4 .
- (2) the Thom complexes of real n -vector bundles over a 2-cell complex $S^2 \cup e^4$, which contain the stunted complex projective spaces $\mathbb{C}\mathbb{P}^{n+2}/\mathbb{C}\mathbb{P}^{n-1}$.
- (3) the iterated suspension of (1) or (2).

Let X be a 3-cell complex of the form $S^n \cup e^{n+2} \cup e^{n+4}$, and e_j be the corresponding generator of $H_j(X) \simeq \mathbb{Z}$ for $j = n, n+2, n+4$. Then for each self-map $f \in [X, X]$, the endomorphism

$$H(f) = f_* : H_*(X) \rightarrow H_*(X)$$

is uniquely determined by a triple of degrees (d_1, d_2, d_3) which is defined by

$$f_*(e^{(n-2)+2j}) = d_j e^j$$

for $j = 1, 2, 3$. We call f a self-map of X of degrees (d_1, d_2, d_3) .

Hence our problem is reduced to characterize a triple of integers (d_1, d_2, d_3) which is a triple of degrees of a self-map of X . This note is organized as follows: In §1, we shall consider the case $n = 2$, and investigate the case $n = 3, 4$ in §2. In §3, we shall treat the case $n > 4$ which is belonging to the stable range, and some examples shall be given in §4.

Remark. *In a subsequent paper, we shall consider the kernel of H .*

Here we state a part of our results.

Theorem A. For $X = S^2 \cup e^4 \cup e^6$, there exists a self-map of X of degrees (d_1, d_2, d_3) if and only if the followings hold :

- (0) If $e^2 \cdot e^2 \neq 0$, $Sq^2(e^4) = 0$, then $d_2 = d_1^2$ and $d_3 = d_1^3 \bmod h_6(X)$.
- (1) If $e^2 \cdot e^2 \neq 0$, $Sq^2(e^4) \neq 0$, $e^2 \cdot e^4 \neq 0$, then $d_2 = d_1^2$ and $d_3 = d_1^3$.
- (2) If $e^2 \cdot e^2 \neq 0$, $Sq^2(e^4) \neq 0$, $e^2 \cdot e^4 = 0$, then $d_2 = d_1^2$ and $d_3 \equiv d_1 \bmod 2$.
- (3) If $e^2 \cdot e^2 = 0$, $e^2 \cdot e^4 \neq 0$, $Sq^2(e^4) = 0$, then $d_3 = d_1 d_2$ and $d_3 \equiv d_1 \bmod h_7(X)$.
- (4) If $e^2 \cdot e^2 = 0$, $e^2 \cdot e^4 \neq 0$, $Sq^2(e^4) \neq 0$, then $d_3 = d_1 d_2$ and $d_3 \equiv d_2 \bmod 2$.
- (5) If $e^2 \cdot e^2 = e^2 \cdot e^4 = 0$, $Sq^2(e^4) = 0$, then $d_3 \equiv d_1 \bmod h_6(X)$.
- (6) If $e^2 \cdot e^2 = e^2 \cdot e^4 = 0$, $Sq^2(e^4) \neq 0$, then $d_3 \equiv d_2 \bmod 2$.

Here $\cdot$ denotes the cup product in the cohomology ring and Sq is the Steenrod squaring operation, and $h_n(X)$ denotes the image of the Hurewicz homomorphism at dimension n .

Corollary E. For $X = S^n \cup e^{n+2} \cup e^{n+4}$ ($5 \leq n$), a triple of integers (d_1, d_2, d_3) is realizable by a self-map of X if and only if the followings hold :

- (1) $d_3 \equiv d_2 \bmod 2$ and $2d_3 \equiv 2d_1 \bmod h_{n+4}(X)$ if $Sq^2(e^{n+2}) \neq 0$.
- (2) $d_3 \equiv d_1 \bmod h_{n+4}(X)$ if $Sq^2(e^{n+2}) = 0$ and $Sq^2(e^n) = 0$.
- (3) $d_2 \equiv d_1 \bmod 2$ and $d_3 \equiv d_1 \bmod h_{n+4}(X)$ if $Sq^2(e^{n+2}) = 0$ and $Sq^2(e^n) \neq 0$.

The author wishes to thank the referee for his profound criticism.

1. The case $n = 2$. Let X be a 3-cell complex of the form $S^2 \cup e^4 \cup e^6$. Let a and b be integers and $\varepsilon \in \{0, 1\}$. Then we call X to be of type (a, b, ε) if and only if

$$e^2 \cdot e^2 = ae^4, \quad e^2 \cdot e^4 = be^6, \quad \text{and} \quad Sq^2(e^4) = \varepsilon e^6$$

where e^j denotes the generator of $H^j(X, \mathbf{Z})$ (or $H^j(X, \mathbf{Z}/2)$). From now on in this section, we assume that X is of type (a, b, ε) .

First we consider the sub-case

1.1 $a = 0$ (i.e. $e^2 \cdot e^2 = 0$) By the assumption we may consider that X has a form $X = (S^2 \vee S^4) \cup e^6$. Then, the attaching class β for the cell e^6 is the following

$$\beta = x\iota_2(\eta_2\eta_3\eta_4) + \iota_4(\eta_4) + b[\iota_2, \iota_4]$$

where η_n denotes the Hopf class of $\pi_{n+1}(S^n)$.

Now define two maps $S^2 \vee S^4 \rightarrow S^2 \vee S^4$ as follows :

$$\begin{aligned}\phi_{k,\ell}|S^2 &= k\iota_2 \quad \text{and} \quad \phi_{k,\ell}|S^4 = \ell\iota_4, \\ \phi_{k,\ell}|S^2 &= k\iota_2 \quad \text{and} \quad \phi_{k,\ell}|S^4 = \ell\iota_4 + \iota_2\eta_2\eta_3.\end{aligned}$$

Lemma 1. *When $b \neq 0$, there exists a self-map of X of degrees (k, ℓ, m) if and only if $m = k\ell$ and $m \equiv \ell \pmod{2}$.*

Proof. By the standard argument we can easily know that there is a self-map of X of degrees (k, ℓ, m) if and only if $\phi_{k,\ell}(\beta) = m\beta$ or $\phi_{k,\ell}(\beta) = m\beta$. Since two endomorphisms

$$\phi_{k,\ell} \text{ and } \phi_{k,\ell} : \pi_5(S^2 \vee S^4) \rightarrow \pi_5(S^2 \vee S^4)$$

are clearly given by

$$\begin{aligned}\phi_{k,\ell}(\beta) &= kx\iota_2\eta_2\eta_3\eta_4 + \iota_2\eta_2\eta_3\eta_4 + \ell\iota_4\eta_4 + k\ell b[\iota_2, \iota_4], \\ \phi_{k,\ell}(\beta) &= kx\iota_2\eta_2\eta_3\eta_4 + \ell\iota_4\eta_4 + k\ell b[\iota_2, \iota_4]\end{aligned}\tag{1-1},$$

the condition is equivalent to $mb = k\ell b$, $\ell \equiv m \pmod{2}$, and $kx\eta_2\eta_3\eta_4 \equiv mx\eta_2\eta_3\eta_4 \pmod{b\eta_2\eta_3\eta_4}$. Then the assumption $b \neq 0$ completes the proof.

Now assume that $b = 0$ and $c \neq 0$. Then, from (1-1) we can obtain the following :

Lemma 2. *When $\varepsilon = 0$ and $b \neq 0$, there is a self-map of X of degrees (k, ℓ, m) if and only if $m = k\ell$ and $m \equiv k \pmod{h_7(\Sigma X)}$, where Σ denotes the suspension functor.*

Analogously we have

Lemma 3. *When $b = 0$, there exists a self-map of X of degrees (k, ℓ, m) if and only if*

- (1) $m \equiv k \pmod{h_6(X)}$ if $\varepsilon = 0$.
- (2) $m \equiv \ell \pmod{2}$ if $\varepsilon \neq 0$.

1.2 $a \neq 0$ (i.e. $e^2 \cdot e^2 \neq 0$). Let A_a be the 2 cell-complex $S^2 \cup e^4$ which has a $a\eta_2$ as the attaching class for the cell e^4 . First we quote the following ((2.13) of [8]):

Lemma 4.

$$\pi_5(A_a) = \begin{cases} \mathbf{Z} & a \equiv 1 \pmod{2} \\ \mathbf{Z} + \mathbf{Z}/4 & a \equiv 2 \pmod{4} \\ \mathbf{Z} + \mathbf{Z}/2 + \mathbf{Z}/2 & a \equiv 0 \pmod{4} \end{cases}$$

Let $\phi_k : A_a \rightarrow A_a$ be a map of degrees (k, k^2) , then other self-maps of A_a of degrees (k, k^2) is only one, which is given by the composite

$$\phi'_k : A_a \xrightarrow{c} A_a \vee S^4 \rightarrow A_a = \phi_k \vee \iota_2 \eta_2 \eta_3 \cdot c$$

Let β be the attaching class for the cell e^6 of X and β' be the image of β by the pinching map $A_a \rightarrow S^4 = A_a/S^2$. Clearly β' is 0 or η_4 , which is determined by $Sq^2(e^4)$. Using the fact $[\iota_2, \eta_2 \eta_3] = 0$, we can easily obtain the following :

$$\phi_{k*}(\beta) = k^3\beta + \iota_2\gamma \quad \text{and} \quad \phi'_{k*}(\beta) = \phi_{k*}(\beta) + \iota_2\eta_2\eta_4\beta' \quad (1-2)$$

where γ is an element of $\pi_5(S^2)$, i.e. 0 or $\eta_2\eta_3\eta_4$.

Lemma 5. *If $a \equiv 1 \pmod{2}$, then we have $\phi_{k*}(\beta) = \phi'_{k*}(\beta) = k^3\beta$.*

Proof. Lemma 4 implies that γ and β' in the formula (1-2) is always 0. Hence the proof is complete.

Next we investigate the case $a \equiv 0 \pmod{2}$ ($a \neq 0$). First we prove

Lemma 6. *For $a = 2$, we can choose ϕ_k satisfying $\phi_{k*}(\beta) = k^3\beta$.*

Proof. We may regard A_2 as the 4-skelton of the reduced product S^2_∞ of S^2 in [2]. The map $k\iota_2 : S^2 \rightarrow S^2$ induces the map $S^2_\infty \rightarrow S^2_\infty$ whose restriction on its 4-skelton is the desired map ϕ_k , and $\phi_{k*}(\alpha) = k^3\alpha$ holds for α , the attaching class for the 6-cell of S^2_∞ . On the other hand, from lemma 4 and the homotopy exact sequence of the pair (A_a, S^2) we can know that there is a class α_1 of $\pi_5(A_2)$ satisfying $j_*(\alpha_1) = [\chi_4, \iota_2]_r$, where $[\ , \]_r$ denotes the relative Whitehead product and χ_4 is the characteristic map for the cell e^4 of S^2_∞ . Since we have $e^2 \cdot e^4 = 3e^6$ in $H(S^2_\infty)$ (see [2]), the following relations hold

$$3\alpha_1 = \alpha \quad \text{or} \quad 3\alpha_1 = \alpha + \eta_2\eta_3\eta_4.$$

Then we have

$$\begin{aligned} 3\phi_{k*}(\alpha_1) &= \phi_{k*}(\alpha) = k^3\alpha = 3k^3\alpha_1 \quad \text{or} \\ 3\phi_{k*}(\alpha_1) &= \phi_{k*}(\alpha) + k\eta_2\eta_3\eta_4 = k^3\alpha + k\eta_2\eta_3\eta_4 = k^3(\alpha + \eta_2\eta_3\eta_4) = 3k^3\alpha_1 \quad \text{i.e.} \end{aligned}$$

$$\psi_{k*}(\alpha_1) = k^3 \alpha_1.$$

Hence, for $\beta (\in \pi_5(A_2))$ with $\beta' = 0$, it holds $\psi_{k*}(\beta) = k^3 \beta$. Moreover, if $\beta' = 0$, we obtain from the formula (1-2) that $\psi_{k*}(\beta) = k^3 \beta$ or $\psi'_{k*}(\beta) = k^3 \beta$. Thus the proof is complete.

Secondly, we prove the general case.

Lemma 7. *If $a \equiv 0 \pmod{2}$, then there is a map $\phi_k : A_a \rightarrow A_a$ satisfying $\phi_{k*}(\beta) = k^3 \beta$ for any β of $\pi_5(A_a)$.*

Proof. For β with $\beta' \neq 0$, the proof follows from the formula (1-2). If $\beta' = 0$ we put $a = 2a'$. Since, for any map $\phi(1, a') : A_a \rightarrow A_2$ of degrees $(1, a')$, there exists a map $\phi_k : A_a \rightarrow A_2$ which makes the following diagram commutative :

$$\begin{array}{ccc} A_a = A_{2a} & \xrightarrow{\phi(1, a')} & A_2 \\ \downarrow \phi_k & & \downarrow \phi_k \\ A_a & \xrightarrow{\phi(1, a')} & A_2 \end{array}$$

Then, the proof follows from the formula (1-1), lemma 6, and the restriction $\phi(1, a)|S^2 = \text{the identity}$.

Proposition 1. *For $a \neq 0$, a triple (d_1, d_2, d_3) is realizable by a self-map of X if and only if the followings hold :*

- (0) $d_2 = k^2$.
- (1) If $\varepsilon = 0$, then $d_3 \equiv d_1^3 \pmod{h_6(X)}$.
- (2) If $\varepsilon \neq 0$ and $b \neq 0$, then $d_3 = d_1^3$.
- (3) If $\varepsilon \neq 0$ and $b = 0$, then $d_3 \equiv d_1^3 \pmod{2}$.

Proof. (0) is easy by the assumption $a \neq 0$. Using lemma 5 and 7, we may take $\gamma = 0$ in the formula (1-1). Then our desired condition is

$$m\beta = k^3 \beta \quad \text{or} \quad m\beta = k^3 \beta + t_2 \eta_2 \eta_3 \beta'.$$

Thus the proof of (1), (2), and (3) follows from using that β' is determined by $Sq^2(e^4)$.

Now we prove Theorem A. Namely, (3) follows from lemma 2. (4) follows from lemma 1. (5) and (6) follow from lemma 3. The others follow from lemma 5 and proposition 1.

2. The case $n = 3, 4$. The case can be divided into two ones by the formula

$$Sq^2 Sq^2 = Sq^3 Sq^1 :$$

(1) $Sq^2(e^{n+2}) \neq 0$, then $X = (S^n \vee S^{n+2}) \cup e^{n+4}$ because of $Sq^2(e^n) = 0$.

(2) $Sq^2(e^{n+2}) = 0$, then $X = e^{n+4} \cup S^n \cup e^{n+2}$.

Let β be the attaching class for the $(n+4)$ -cell of X . First we consider the case (1). Define two maps $\psi_{k,\ell}$ and $\psi'_{k,\ell}$ as follows :

$$\begin{aligned} \psi_{k,\ell}, \psi'_{k,\ell} : S^n \vee S^{n+2} &\rightarrow S^n \vee S^{n+2}, \\ \psi_{k,\ell}|S^n &= k\iota_n \quad \text{and} \quad \psi_{k,\ell}|S^{n+2} = \ell\iota_{n+2}, \\ \psi'_{k,\ell}|S^n &= k\iota_n \quad \text{and} \quad \psi'_{k,\ell}|S^{n+2} = \ell\iota_{n+2} + \iota_n\eta_n\eta_{n+1}. \end{aligned}$$

Then we have

$$\psi_{k,\ell*}(\beta) = (k\iota_n)_*(\beta_1) + \iota_{n+2}(\ell\eta_{n+2}),$$

$$\text{and} \quad \psi'_{k,\ell*}(\beta) = (k\iota_n)_*(\beta_1) + \iota_n\eta_n\eta_{n+1}\eta_{n+2} + \iota_{n+2}(\ell\eta_{n+2}),$$

where $\beta = \beta_1 + \iota_{n+2}\eta_{n+2} (\in \pi_{n+3}(S^n \vee S^{n+2}) = \pi_{n+3}(S^n) + \pi_{n+3}(S^{n+2}))$. Hence, a triple (k, ℓ, m) is realizable by a self-map of X if and only if the following equality holds :

$$m\beta_1 + m\iota_{n+2}\eta_{n+2} = m\beta = (k\iota_n)_*\beta_1 + \ell\iota_{n+2}\eta_{n+2} \pmod{\iota_n\eta_n\eta_{n+1}\eta_{n+2}}. \quad (2-1)$$

Lemma 8. For $n = 3$ ($Sq^2(e^5) \neq 0$), there exists a self-map of X of degrees (k, ℓ, m) if and only if $\ell \equiv m \pmod{2}$ and $2m \equiv 2k \pmod{h_7(X)}$.

Proof. Since we can replace $(k\iota_n)_*(\beta_1)$ with $k\beta_1$ in the formula (2-1) we have that $\ell \equiv m \pmod{2}$ and $(m-k)\beta_1 \equiv 0 \pmod{\eta_3\eta_4\eta_5}$. On the other hand, $\eta_3\eta_4\eta_5$ is the only one element of order 2 in $\pi_6(S^3)$. Therefore the latter condition is equivalent to $2(m-k)\beta_1 = 0$. Thus the proof is completed by $2\beta_1 = 2\beta$.

Lemma 9. For $n = 4$ ($Sq^2(e^6) \neq 0$), our conditions are as follows :

(1) If $e^4 \cdot e^4 = 0$, then $\ell \equiv m \pmod{2}$ and $2(m-k) \equiv 0 \pmod{h_8(X)}$.

(2) If $e^4 \cdot e^4 \neq 0$, then $m = k^2$, $\ell \equiv m \pmod{2}$, and $m \equiv k \pmod{h_9(\Sigma X)}$.

Proof. Since β_1 has a representation

$$\beta_1 = x\nu + y\Sigma\omega \quad \text{for some integers } x \text{ and } y,$$

where ν denotes the Hopf map $S^7 \rightarrow S^4$ and ω is the Blaker-Massey map, we get

$$\begin{aligned} (k\iota_4)_*(\beta_1) &= k\beta_1 + k(k-1)/2[\iota_4, \iota_4]H(\beta_1) \\ &= k^2x\nu + \{ky + k(k-1)x/2\}\Sigma\omega. \end{aligned}$$

Hence, the formula (2-1) is equivalent to

$$mx\nu + my\Sigma\omega = k^2x\nu + \{2ky + k(k-1)x\}/2\Sigma\omega \pmod{\eta_4\eta_5\eta_6}.$$

Moreover this gives that

if $x \neq 0$ (i.e. $e^4 \cdot e^4 \neq 0$), then $m = k^2$ and $\{2my - 2ky - k(k-1)x\}/2\Sigma\omega \equiv 0 \pmod{\eta_4\eta_5\eta_6}$, and that

if $x = 0$ (i.e. $e^4 \cdot e^4 = 0$), then $(m-k)y\Sigma\omega \equiv 0 \pmod{\eta_4\eta_5\eta_6}$.

Thus (1) is obtained from the same argument as lemma 8. Next, we consider the case (2). From $m = k^2$ we have that

$$2my - 2ky - k(k-1)x = (m-k)2y - k(k-1)x = k(k-1)(2y-x).$$

On the other hand, we know that $\Sigma\beta_1 = (x-2y)\nu$. Thus the proof of (2) follows from $k(k-1)\Sigma\beta_1 = k(k-1)\Sigma\beta$.

Secondly, we prove the case $Sq^2(e^{n+2}) = 0$. Let A be the subcomplex, $S^n \cup e^{n+2}$, of X and let $\phi_k(\gamma)$ be the map defined by

$$\phi_k(\gamma) = (k1_A \vee \gamma)C_A : A \rightarrow A \vee S^{n+2} \rightarrow A$$

for $\gamma \in \pi_{n+2}(A)$ where $k1_A$ denotes the k time of the identity of A in the sense of the suspension-addition and C_A is the co-action map of A . Here we note that

- (0) $\beta = i_*(\beta')$, where i is the inclusion $S^n \rightarrow A$.
- (1) $\phi_k(\gamma)$ is of degrees $(k, k + h_{n+2}(X))$.
- (2) $\phi_k(\gamma)_*(\beta) = i_*((k\iota_n)_*(\beta'))$.

Now consider the following diagram :

$$\begin{array}{ccc} \pi_{n+3}(S^n) & \xrightarrow{i_*} & \pi_{n+3}(A) \\ (k\iota_n)_* \downarrow & & \downarrow (k1_A)_* \\ \pi_{n+4}(A, S^n) & \xrightarrow{\partial} \pi_{n+3}(S^n) & \xrightarrow{i_*} \pi_{n+3}(A) \end{array}$$

Then, it is easy from $\phi_k(\gamma)_*(\beta) = (k1_A)_*(\beta)$ to obtain the following :

Lemma 10. *There exists a self-map of X of degrees (k, ℓ, m) if and only if $\ell \equiv k \pmod{h_{n+2}(X)}$ and $m(\beta') \equiv (k\iota_n)_*(\beta') \pmod{\partial\text{-image}}$.*

Remark. $h_{n+2}(X) = \mathbf{Z}$ if $Sq^2(e^n) = 0$ and $h_{n+2}(X) = 2\mathbf{Z}$ if $Sq^2(e^n) \neq 0$.

Since $(k\iota_3)_*(\beta') = k\beta'$ (for $n = 3$), we have

Lemma 11. *For $n = 3$, a triple (k, ℓ, m) is realizable by a self-map of X*

if and only if $\ell \equiv k \pmod{h_5(X)}$ and $m \equiv k \pmod{h_7(X)}$.

If $n = 4$ we can take $x\nu + y\Sigma\omega$ as β' and use the formula,

$$(k\iota_4)_*(\beta') = k(\beta') + \{k(k-1)/2\}x[\iota_4, \iota_4]. \quad (2-2)$$

Lemma 12. *If $e^4 \cdot e^4 = 0$, then there is a self-map of X of degrees (k, ℓ, m) if and only if $\ell \equiv k \pmod{h_6(X)}$ and $m \equiv k \pmod{h_8(X)}$.*

Proof. Since $e^4 \cdot e^4 = 0$ is equivalent to $x = 0$, we have

$$(k\iota_4)_*(\beta') = k(\beta'), \quad \text{i.e.} \quad (k1_A)_*(\beta) = k\beta$$

from (2-2). Hence $(m-k)(\beta) = 0$, which completes the proof.

Lemma 13. *If $x \neq 0$, then there is a self-map of X of degrees (k, ℓ, m) if and only if $\ell \equiv k \pmod{h_6(X)}$, $m = k^2$, and $m \equiv k \pmod{h_9(\Sigma X)}$.*

Proof. First, we have $m = k^2$ from (2-2) and $[\iota_4, \iota_4] = 2\nu + \Sigma\omega$. Then, it holds $i_*\{(m-k)(x-2y)/2\}\Sigma\omega = 0$, which gives

$$i_*\{(m-k)(x-2y)\nu\} = i_*\{(m-k)\Sigma\beta'\} = (m-k)\Sigma\beta = 0$$

from applying the suspension functor. Thus the proof is complete.

Now from lemmas 8, 9, 11, 12, and 13 we have

Theorem B. *Let X be a complex of the form $S^3 \cup e^5 \cup e^7$. Then a triple (d_1, d_2, d_3) is realizable by a self-map of X if and only if*

- (1) *If $Sq^2(e^5) \neq 0$, then $d_3 \equiv d_2 \pmod{2}$ and $2d_3 \equiv 2d_2 \pmod{h_7(X)}$.*
- (2) *If $Sq^2(e^5) = 0$ and $Sq^2(e^3) = 0$, then $d_3 \equiv d_1 \pmod{h_7(X)}$.*
- (3) *If $Sq^2(e^5) = 0$ and $Sq^2(e^3) \neq 0$, then $d_3 \equiv d_1 \pmod{h_7(X)}$ and $d_2 \equiv d_1 \pmod{2}$.*

Theorem C. *Let X be a complex of the form $S^4 \cup e^6 \cup e^8$. Then a triple (d_1, d_2, d_3) is realizable by a self-map of X if and only if*

- (1) *If $Sq^2(e^6) \neq 0$ and $e^4 \cdot e^4 = 0$, then $d_3 \equiv d_2 \pmod{2}$ and $2d_3 \equiv 2d_1 \pmod{h_8(X)}$.*
- (2) *If $Sq^2(e^6) \neq 0$ and $e^4 \cdot e^4 \neq 0$, then $d_3 \equiv d_2 \pmod{2}$, $d_3 \equiv d_1^2$, and $d_3 \equiv d_2 \pmod{h_9(\Sigma X)}$.*
- (3) *If $Sq^2(e^6) = 0$ and $e^4 \cdot e^4 = 0$, then $d_2 \equiv d_1 \pmod{h_6(X)}$ and $d_3 \equiv d_1 \pmod{h_8(X)}$.*

- (4) If $Sq^2(e^6) = 0$ and $e^4 \cdot e^4 \neq 0$, then $d_2 \equiv d_1 \pmod{h_6(X)}$, $d_3 = d_1^2$, and $d_3 \equiv d_1 \pmod{h_9(\Sigma X)}$.

3. The case $X = \Sigma X'$. First we note that this case contains the case $5 \leq n$. Next, let Y be the complex X/S^n and let us consider maps from Y to X .

Lemma 14. *If $Sq^2(e^{n+2}) = 0$ in $H^*(X; \mathbb{Z}/2)$, then there exists a map from Y to X with degrees (ℓ, m) if and only if $\ell \equiv 0 \pmod{h_{n+2}(X)}$ and $m \equiv 0 \pmod{h_{n+4}(X)}$.*

Proof. Since the assumption implies $Y = S^{n+2} \vee S^{n+4}$, the proof is clear.

Lemma 15. *If $Sq^2(e^{n+2}) \neq 0$, then there is a map from Y to X with degrees (ℓ, m) if and only if $m \equiv \ell \pmod{2}$ and $2m \equiv 0 \pmod{h_{n+4}(X)}$.*

Proof. Since $Sq^2(e^{n+2}) \neq 0$ implies $Sq^2(e^n) = 0$, X has a decomposition

$$X = (S^n \vee S^{n+2}) \cup e^{n+4}.$$

Let $f: S^{n+2} \rightarrow X$ be the map defined by $f = n\gamma + \ell\iota_{n+2}$ for $\gamma \in \pi_{n+2}(S^n)$ and $\beta = \iota_n\beta' + \iota_{n+2}\eta_{n+2}$ be the attaching class for the $(n+4)$ -cell of X . Then, we have

$$f_*(\eta_{n+2}) = \iota_n(\gamma\eta_{n+2}) + \ell\iota_{n+2}\eta_{n+2}.$$

Hence the proof follows from $m\beta \equiv m\iota_n\beta' + m\iota_{n+2}\eta_{n+2} = f_*(\eta_{n+2})$,

i.e. $m \equiv \ell \pmod{2}$ and $m\beta' \equiv 0 \pmod{\iota_n\eta_n\eta_{n+1}\eta_{n+2}}$.

Secondly, using the group structure of $[X, X] = [\Sigma X', \Sigma X']$ we have the map $k1_X$, which is of degrees (k, k, k) . Let $g: X \rightarrow X$ be a self-map of degrees (k, ℓ, m) . Then the map $g - k1_X$ is of degrees $(0, \ell - k, m - k)$, and lemmas 14 and 15 give the following

Theorem D. *Assume that X is a suspended space. Then a triple (d_1, d_2, d_3) is realizable by a self-map of X if and only if*

- (1) *If $Sq^2(e^{n+2}) = 0$, then $d_2 \equiv d_1 \pmod{h_{n+2}(X)}$ and $d_3 \equiv d_1 \pmod{h_{n+4}(X)}$.*
- (2) *If $Sq^2(e^{n+2}) \neq 0$, then $d_2 \equiv d_1 \pmod{2}$ and $2d_3 \equiv 2d_1 \pmod{h_{n+4}(X)}$.*

Remark. *If $Sq^2(e^n) = 0$, then $h_{n+2}(X) = \mathbb{Z}$, and if $Sq^2(e^n) \neq 0$, then $h_{n+2}(X) = 2\mathbb{Z}$.*

4. Examples.

Example 1. Let X_n be the space $\mathbb{C}P^{n+2}/\mathbb{C}P^{n-1}$ ($1 \leq n$). Then, using the results in [5] and $Sq^2(e^i) = ie^{i+1}$ in $H^*(\mathbb{C}P^N)$, we have that degrees of self-maps of X_n , (d_1, d_2, d_3) , is characterized as follows:

(1) For $3 \leq n$,

if $n \equiv 1 \pmod{2}$, then $d_2 \equiv d_1 \pmod{2}$ and $d_3 \equiv d_1 \pmod{24/(n+3, 24)}$,

if $n \equiv 0 \pmod{8}$, then $d_2 \equiv d_3 \pmod{2}$ and $2d_3 \equiv 2d_1 \pmod{96/(n, 48)}$,

if $n \equiv 2, 4, 6 \pmod{8}$, then $d_2 \equiv d_3 \pmod{2}$ and $2d_3 \equiv 2d_1 \pmod{48/(n, 48)}$,

where $(\ , \)$ denotes the greatest common divisor of integers.

Remark. These results also hold for the iterated-suspension $\Sigma^s X_n$.

(2) For $n = 2$, $d_3 \equiv d_2 \pmod{2}$ and $d_3 \equiv d_1^2$, i.e. $(k, k^2 + 2\mathbb{Z}, k^2)$, and for the space $\Sigma^s X_2$ ($1 \leq s$) we have $(d_1, d_2, d_3) = (k, k + 2\mathbb{Z}, k + 12\mathbb{Z})$.

(3) For $n = 1$, $(d_1, d_2, d_3) = (k, k^2, k^3)$, and for $\Sigma^s X_1$ ($1 \leq s$) we have $(d_1, d_2, d_3) = (k, k + 2\mathbb{Z}, k + 6\mathbb{Z})$.

Example 2. Let X be the 6-skelton of the reduced product of S^2 . Since it is clear that $\Sigma^s X$ is decomposed into $S^{2+s} \vee S^{4+s} \vee S^{6+s}$ ($1 \leq s$) we can know that

(1) For $s = 0$, $(d_1, d_2, d_3) = (k, k^2, k^3)$,

(2) For $1 \leq s$, (d_1, d_2, d_3) for any d_1, d_2 and d_3 .

Example 3. Let X_r be the 2-sphere bundle over S^4 whose characteristic class is r times of a generator ($\in \pi_3(SO(3)) = \mathbb{Z}$). Then it is easy to see

$$e^2 \cdot e^2 = re^4 \quad \text{and} \quad e^2 \cdot e^4 = e^6.$$

Furthermore the suspension $\Sigma^N X_r$ has a decomposition

$$\Sigma^N X_r = S^{N+2} \cup e^{N+4} \cup e^{N+6}$$

in which the attaching class β_N for the $(N+6)$ -cell is given by

$$\beta_N = i_*(\Sigma^{N-1} J(\xi_r)) = i_*(r \Sigma^{N-1} \omega) = ri_*(\Sigma^{N-1} \omega)$$

where ξ_r is the real vector bundle associated with X_r .

On the other hand, it is easy to show that

$$Sq^2(e^{N+4}) = 0 \quad (0 \leq N) \quad \text{and} \quad Sq^2(e^{N+2}) = 0 \quad \text{for even } r, \neq 0 \quad \text{for odd } r \quad (1 \leq N).$$

Since $Sq^2(e^{N+4}) = 0$ implies that i^* is injective, we get

$$h_{N+6}(\Sigma^N X_r) = (r, 12)\mathbb{Z}.$$

These facts give the following :

A triple (d_1, d_2, d_3) is realizable by a self-map of $\Sigma^N X_r$ if and only if

(1) The case $1 \leq N$.

$$d_3 \equiv d_1 \pmod{(r, 12)} \quad \text{if} \quad r \equiv 0 \pmod{2},$$

$$d_2 \equiv d_1 \pmod{2} \text{ and } d_3 \equiv d_1 \pmod{(r, 12)} \quad \text{if} \quad r \equiv 1 \pmod{2}.$$

(2) The case $N = 0$.

$$d_3 \equiv d_1 d_2 \quad \text{if} \quad r = 0,$$

$$d_2 = d_1^2 \text{ and } d_3 = d_1^3 \quad \text{if} \quad r \neq 0.$$

Example 4. Let Y_a be the CW -complex $S^2 \cup e^4$ which has $a\eta_2$ as the attaching class for the cell e^4 . Assume $5 \leq N$ and consider an N -dim real vector bundle ξ over Y_a . For simplicity we suppose that its Stiefel-Whitney class $w_2(\xi)$ is trivial. Since $w_2(\xi) = 0$ implies that the restriction $\xi|S^2$ is trivial, there is a commutative diagram

$$\begin{array}{ccc} \xi & \rightarrow & \xi' \\ \downarrow & & \downarrow \\ Y_a & \rightarrow & S^4 = Y_a/S^2 \end{array}$$

for some ξ' .

Then, from $Sq^2(e^{N+2}) = ae^{N+4}$ ([4]), we can know that the Thom complex $T(\xi)$ has a decomposition

$$T(\xi) = (S^N \vee S^{N+2}) \cup e^{N+4}$$

in which $\beta = \iota_N J(\xi') + \iota_{N+2}(a\eta_{N+2}) \in \pi_{N+3}(S^N) + \pi_{N+3}(S^{N+2})$. Thus, from Theorem D, we have that a triple (d_1, d_2, d_3) is realizable by a self-map of $T(\xi)$ if and only if

(1) if $a \equiv 0 \pmod{2}$, then $d_3 \equiv d_1 \pmod{(b, 24)}$,

(2) if $a \equiv 1 \pmod{2}$, then $d_3 \equiv d_2 \pmod{2}$ and $2d_3 \equiv 2d_1 \pmod{(b, 12)}$,

where $2be^4 = p_1(\xi)$, the first Pontrjagin class of ξ .

REFERENCES

- [1] D. M. DAVIS : BP-operations and mappings of stunted complex projective spaces, *Lec. Note of Math.* Springer, **741** (1979), 373—393.
- [2] I. M. JAMES : Reduced product spaces, *Ann. Math.* (2) **62** (1955), 170—197.
- [3] C. A. MCGIBBON : Self-maps of projective spaces, *Trans. A.M.S.* **271** (1982), 325—346.
- [4] J. W. MILNOR and J. D. STASHEFF : *Characteristic Classes*, *Ann. Math. Study* **76** (1970).
- [5] H. OSHIMA : On stable James numbers of stunted complex or quaternion projective spaces, *Osaka J. Math.* **16** (1979), 476—504.
- [6] S. SASAO and M. NAGAISHI : Self-maps of the quaternion projective 3-space, *Sugaku* **24** (1972). (In Japanese).

- [7] D. SULLIVAN : Geometric Topology, M.I.T. 1970.
- [8] K. YAMAGUCHI : The group of self-homotopy equivalences of S^2 -bundles over S^4 , Kodai Math. J. 9 (1986), 308—326.
- [9] K. YAMAGUCHI : Self-maps of $\Sigma^k \mathbb{C}P^3$ for $1 \leq k$, Kodai Math. J. 14 (1991), 144—162.

DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCE AND ENGINEERING
WASEDA UNIVERSITY
SHINJUKU-KU TOKYO 162, JAPAN

(Received June 19, 1992)