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ON THE GROUP $\pi(\Sigma A \times B, X)$

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INTRODUCTION

We work in the homotopy category of pointed CW -complexes, and denote the suspension functor with Σ . The group $\pi(\Sigma A \times B, X)$ is not **abelian** in general under the usual multiplication but we use the notation “+” for it. In §1 we shall describe the multiplication of the group $\pi(\Sigma A \times B, X)$, and, as a by-product, obtain from the associativity of the multiplication that the bi-additivity of the generalized Whitehead product (GWP) does not hold in general (see Proposition 3.4 of [1]). In fact, as an example, we offer the following:

Let i_1 and i_2 be inclusion maps from CP^2 and S^n into $CP^2 \vee S^n$ respectively. Then we have that

$$[2\Sigma i_1, \Sigma i_2] \neq 2[\Sigma i_1, \Sigma i_2] \text{ and } [\Sigma i_1, 2\Sigma i_2] = 2[\Sigma i_1, \Sigma i_2]$$

in $\pi(\Sigma CP^2 \wedge S^n, \Sigma CP^2 \vee S^n)$.

In §2 we investigate the group $\pi(\Sigma X, \Sigma A \times B)$ and show that any element of this group can be determined by 4-components under some assumptions. In §3 we apply §2 to the case of $X = A \times B$, i.e. the group of self-maps of the space $\Sigma A \times B$.

Specially we are interested in describing the composition of two elements with their components and give the special case of $A = B = S^n$ as an example.

1. $\pi(\Sigma A \times B, X)$

Let i_1 and i_2 be inclusion maps: $A, B \rightarrow A \times B$, and let P_A and P_B be projections: $A \times B \rightarrow A, B$ respectively.

Lemma 1.1. *Let $\pi: A \times B \rightarrow A \wedge B$ be the projection. Any element $f \in \pi(\Sigma A \times B, X)$ can be uniquely represented by the form*

$$f = \alpha \Sigma P_A + \beta \Sigma P_B + \gamma \Sigma \pi$$

for $\alpha \in \pi(\Sigma A, X)$, $\beta \in \pi(\Sigma B, X)$ and $\gamma \in \pi(\Sigma A \wedge B, X)$.

Proof. In fact a representation can be obtained from a part of Puppe exact sequence of the cofiber: $A \vee B \rightarrow A \times B$. Then clearly we have $\alpha = f \Sigma i_A$, $\beta = f \Sigma i_B$ and moreover the uniqueness of γ follows from the injectivity of $(\Sigma \pi)^*$. □

Here we give a brief account of GWP of [1]. For two maps $\alpha: \Sigma A \rightarrow X$ and $\beta: \Sigma B \rightarrow X$, GWP $[\alpha, \beta] \in \pi(\Sigma A \wedge B, X)$ is defined by

$$\alpha \Sigma P_A + \beta \Sigma P_B = \beta \Sigma P_B + \alpha \Sigma P_A + [\alpha, \beta] \Sigma \pi.$$

Proposition 1.2. *GWP has following properties:*

- (1) *Let us be $\langle \alpha, \beta \rangle$ the commutator of α and β ($\in \pi(\Sigma A, X)$) then we have*

$$\langle \alpha, \beta \rangle = [\alpha, \beta] \Sigma d_A,$$

where d_A is the diagonal map: $A \rightarrow A \wedge A$.

- (2) *If $f \in \pi(X, Y)$ then $f[\alpha, \beta] = [f\alpha, f\beta]$.*
 (3) *If $\sigma_k \in \pi(\Sigma Y_k, Z)$ and $f_k \in \pi(X_k, Y_k)$ for $k = 1, 2$ then it holds that*

$$[\sigma_1 \Sigma f_1, \sigma_2 \Sigma f_2] = [\sigma_1, \sigma_2] \Sigma f_1 \wedge f_2.$$

- (4) *For four maps $\Sigma f \in \pi(\Sigma Y, \Sigma A)$, $\Sigma g \in \pi(\Sigma Y, \Sigma B)$, $\sigma \in \pi(\Sigma A, X)$ and $\tau \in \pi(\Sigma B, X)$ we have*

$$\sigma \Sigma f + \tau \Sigma g = \tau \Sigma g + \sigma \Sigma f + [\sigma, \tau] \Sigma (f \wedge g) \Sigma d_Y.$$

- (5) *If X is a suspension (i.e. $X = \Sigma X^*$) then $d_X = 0$.*

Lemma 1.3. *For $\alpha \in \pi(\Sigma A, X)$, $\beta \in \pi(\Sigma B, X)$ and $\gamma \in \pi(\Sigma A \wedge B, X)$ we have the following:*

- (1) $\alpha \Sigma P_A + \beta \Sigma P_B = \beta \Sigma P_B + \alpha \Sigma P_A + [\alpha, \beta] \Sigma \pi$.
 (2) $\alpha \Sigma P_A + \gamma \Sigma \pi = \gamma \Sigma \pi + \alpha \Sigma P_A + [\alpha, \gamma] \Sigma \varphi_A \Sigma \pi$, where φ_A is defined by $\varphi_A(a \wedge b) = a \wedge a \wedge b$.
 (3) $\beta \Sigma P_B + \gamma \Sigma \pi = \gamma \Sigma \pi + \beta \Sigma P_B + [\beta, \gamma] \Sigma \psi_B \Sigma \pi$, where ψ_B is defined by $\psi_B(a \wedge b) = b \wedge a \wedge b$.

Proof. (1) is just the definition of GWP. Next, by applying (1) and (4) of lemma 1.2 to the diagram:

$$\begin{array}{ccccc} \Sigma A \times B & \xrightarrow{\Sigma P_A} & \Sigma A & \xrightarrow{\alpha} & X \\ \parallel & & & & \parallel \\ \Sigma A \times B & \xrightarrow{\Sigma \pi} & \Sigma A \wedge B & \xrightarrow{\gamma} & X, \end{array}$$

we have that

$$[\alpha \Sigma P_A, \gamma \Sigma \pi] \Sigma d_A \times 1_B = [\alpha, \gamma] \Sigma (P_A \wedge \pi) \Sigma d_A \times 1_B = [\alpha, \gamma] \Sigma \varphi_A \Sigma \pi.$$

The case (3) is analogous to the case (2). Thus the proof is completed. \square

Now let us represent $f = \alpha \Sigma P_A + \beta \Sigma P_B + \gamma \Sigma \pi$ with the triple (α, β, γ) .

Theorem 1.4. $(\alpha_1, \beta_1, \gamma_1) + (\alpha_2, \beta_2, \gamma_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2, \delta + \gamma_2)$ for $\delta = \delta_1 - [\beta_2, \delta_1] \Sigma \psi_B$ and $\delta_1 = -[\alpha_2, \beta_1] + \gamma_1 - [\alpha_2, \gamma_1] \Sigma \varphi_A$.

Proof. For abbreviation we use notations: $\bar{\alpha} = \alpha \Sigma P_A$, $\bar{\beta} = \beta \Sigma P_B$, $\bar{\gamma} = \gamma \Sigma \pi$ and so on. Now by using lemma 1.3 we have equalities

$$\begin{aligned}
 f_1 + f_2 &= (\alpha_1, \beta_1, \gamma_1) + (\alpha_2, \beta_2, \gamma_2) \\
 &= \bar{\alpha}_1 + \bar{\beta}_1 + \bar{\gamma}_1 + \bar{\alpha}_2 + \bar{\beta}_2 + \bar{\gamma}_2 \\
 &= \bar{\alpha}_1 + \bar{\beta}_1 + \bar{\alpha}_2 + \bar{\gamma}_1 - [\alpha_2, \gamma_1] \Sigma \varphi_A \Sigma \pi + \bar{\beta}_2 + \bar{\gamma}_2 \\
 &= \bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\beta}_1 - [\alpha_2, \beta_1] + \bar{\gamma}_1 - [\alpha_2, \gamma_1] \Sigma \varphi_A + \bar{\beta}_2 + \bar{\gamma}_2 \\
 &= \bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\beta}_1 + \bar{\delta}_1 + \bar{\beta}_2 + \bar{\gamma}_2 \\
 &= \bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\beta}_1 + \bar{\beta}_2 + \bar{\delta}_1 - [\beta_2, \delta_1] \Sigma \psi_B + \bar{\gamma}_2 \\
 &= \bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\beta}_1 + \bar{\beta}_2 + \bar{\delta} + \bar{\gamma}_2 \\
 &= (\alpha_1 + \alpha_2, \beta_1 + \beta_2, \delta + \gamma_2).
 \end{aligned}$$

Thus the proof is completed. \square

Corollary 1.5. *If A and B are both suspensions then it holds that*

$$(\alpha_1, \beta_1, \gamma_1) + (\alpha_2, \beta_2, \gamma_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2, -[\alpha_2, \beta_1] + \gamma_1 + \gamma_2).$$

Proof. Since φ_A and ψ_B are trivial by (5) of lemma 1.2 the proof follows from Theorem 1.4. \square

Corollary 1.6. *For $\alpha_1, \alpha_2 \in \pi(\Sigma A, X)$ and $\beta \in \pi(\Sigma B, X)$ it holds that*

$$[\alpha_1 + \alpha_2, \beta] = [\alpha_2, -[\alpha_1, \beta]] \Sigma \varphi_A + [\alpha_1, \beta] + [\alpha_2, \beta].$$

Proof.

$$\begin{aligned}
 &(0, \beta, 0) + \{(\alpha_1, 0, 0) + (\alpha_2, 0, 0)\} \\
 &= (0, \beta, 0) + (\alpha_1 + \alpha_2, 0, 0) \\
 &= (\alpha_1 + \alpha_2, \beta, -[\alpha_1 + \alpha_2, \beta]).
 \end{aligned}$$

On the other hand

$$\begin{aligned}
 &\{(0, \beta, 0) + (\alpha_1, 0, 0)\} + (\alpha_2, 0, 0) \\
 &= (\alpha_1, \beta, -[\alpha_1, \beta]) + (\alpha_2, 0, 0) \\
 &= (\alpha_1 + \alpha_2, \beta, -[\alpha_2, \beta] - [\alpha_1, \beta] - [\alpha_2, -[\alpha_1, \beta]] \Sigma \varphi_A).
 \end{aligned}$$

Thus the proof follows from the associativity of the addition of the group $\pi(\Sigma A \times B, X)$. \square

By (5) of lemma 1.2 and the above corollary 1.6 it is easy to obtain the following:

Corollary 1.7 (Proposition 3.4 of [1]). *If A is a suspension then it holds that*

$$[\alpha_1 + \alpha_2, \beta] = [\alpha_1, \beta] + [\alpha_2, \beta]$$

for $\alpha_1, \alpha_2 \in \pi(\Sigma A, X)$ and $\beta \in \pi(\Sigma B, X)$.

Now we consider a special case: $A = CP^2$, $B = S^n$ and $X = \Sigma CP^2 \vee S^n$. Let i_1 and i_2 be inclusions: $CP^2, S^n \rightarrow CP^2 \vee S^n$ respectively, and let us be $\alpha = \Sigma i_1$ and $\beta = \Sigma i_2$. Since $\varphi_A: CP^2 \wedge S^n \rightarrow CP^2 \wedge CP^2 \wedge S^n$ is defined by $\varphi_A(a \wedge b) = a \wedge a \wedge b$, φ_A can be regarded as $\Sigma^n d$ for the diagonal map $d: CP^2 \rightarrow CP^2 \wedge CP^2$.

Since $d^*: H^4(CP^2 \wedge CP^2) \rightarrow H^4(CP^2)$ is clearly an isomorphism $\Sigma^n d$ is non-trivial. On the other hand, accordingly to Hilton-Milnor Theorem ([3]) $[\alpha, [\alpha, \beta]]_*$ is injective. Hence $[\alpha, [\alpha, \beta]]\Sigma\varphi_A$ is non-trivial. Then these show

$$[2\Sigma i_1, \Sigma i_2] \neq 2[\Sigma i_1, \Sigma i_2] \text{ and } [\Sigma i_1, 2\Sigma i_2] = 2[\Sigma i_1, \Sigma i_2].$$

Remark. The second equality follows from Corollary 1.7.

2. ON THE GROUP $\pi(\Sigma X, \Sigma A \times B)$

First by using lemma 1.1 we define $\xi_{AB} \in \pi(\Sigma A \wedge B, \Sigma A \times B)$ as follows:

$$1_{\Sigma A \times B} = \Sigma i_A \Sigma P_A + \Sigma i_B \Sigma P_B + \xi_{AB} \Sigma \pi.$$

Corollary 2.1. $\Sigma \pi \xi_{AB} = 1_{\Sigma A \wedge B}$.

Proof. Apply $(\Sigma \pi)_*$ to the above equality. Then we have

$$\begin{aligned} \Sigma \pi &= \Sigma \pi (\Sigma i_A \Sigma P_A + \Sigma i_B \Sigma P_B + \xi_{AB} \Sigma \pi) \\ &= 0 + 0 + \Sigma \pi \xi_{A \wedge B} \Sigma \pi. \end{aligned}$$

Since $(\Sigma \pi)^*$ is injective the proof is completed. \square

Here we note that the representation of $f \in \pi(\Sigma A \times B, X)$ in lemma 1.1 is given by

$$f = f|_{\Sigma A} \Sigma P_A + f|_{\Sigma B} \Sigma P_B + f \xi_{AB} \Sigma \pi,$$

where $f|_K$ denotes the restriction of f on K .

For example if h is a map $A \times B \rightarrow X$ then $\Sigma h \xi_{AB}$ is essentially the Hopf-construction of f , i.e. $C(h)$ (see [2]) and we have a representation:

$$\Sigma h = \Sigma h_A + \Sigma h_B + C(h) \Sigma \pi,$$

where h is a map of type (h_A, h_B) .

Secondly we define two maps $\varphi \in \pi(Y, \Sigma A \times B)$ and $\phi \in \pi(\Sigma A \times B, Y)$ for $Y = \Sigma A \vee \Sigma B \vee \Sigma A \wedge B$ by

$$\begin{aligned} \varphi &= \Sigma i_A \vee \Sigma i_B \vee \xi_{AB}, \\ \phi &= i_{\Sigma A} \Sigma P_A + i_{\Sigma B} \Sigma P_B + i_{\Sigma A \wedge B} \Sigma \pi. \end{aligned}$$

Lemma 2.2. φ is a homotopy equivalence with ϕ as its inverse.

Proof. Easy. \square

In the following of this section we assume that

- (1) A is a -connected and B is b -connected,
- (2) $a \leq b$,
- (3) $\dim X \leq 2a + b + 2$.

Then by Hilton-Milnor theorem, $f \in \pi(\Sigma X, \Sigma A \times B)$ can be represented as follows:

$$f = \Sigma i_A f_A + \Sigma i_B f_B + [\Sigma i_A, \Sigma i_B] f_{C1} + \xi_{AB} f_{C2},$$

where $f_* \in \pi(\Sigma X, \Sigma *)$ and $f_{C*} \in \pi(\Sigma X, \Sigma A \wedge B)$.

More precisely we have

Lemma 2.3. $f_A = \Sigma P_A f$, $f_B = \Sigma P_B f$, $f_{C1} = \Sigma(P_A \wedge P_B)H(f)$ and $f_{C2} = \Sigma \pi f$, where $H(f)$ denotes Hopf-invariant of f .

Proof. First we note that $[\gamma, \delta \Sigma \pi]H(f) = 0$ because this element is decomposed as follows:

$$\Sigma X \xrightarrow{H(f)} \Sigma(A \times B) \wedge (A \times B) \xrightarrow{\Sigma(1 \wedge \pi)} \Sigma(A \times B) \wedge (A \wedge B) \xrightarrow{[\gamma, \delta]} \Sigma A \times B.$$

Then the proof is deduced from our assumptions. Secondly apply f from the right to the equality. We obtain that

$$\begin{aligned} f &= (\Sigma i_A P_A + \Sigma i_B P_B + \xi_{AB} \Sigma \pi) f \\ &= (\Sigma i_A P_A + \Sigma i_B P_B) f + \xi_{AB} \Sigma \pi f \\ &= \Sigma i_A P_A f + \Sigma i_B P_B f + [\Sigma i_A, \Sigma i_B] (\Sigma P_A \wedge \Sigma P_B) H(f) + \xi_{AB} \Sigma \pi f. \end{aligned}$$

Thus the proof is completed. \square

3. ON THE GROUP $\pi(\Sigma A \times B, \Sigma A \times B)$

In this section we assume that A and B are both n -connected, $\dim A \leq \dim B$ and $\dim A + \dim B \leq 3n + 2$.

Lemma 3.1. Our **assumptions** contain

- (1) A , B and $A \wedge B$ are all suspensions, so $\pi(\Sigma K, X)$ is **abelian** for $K = A$, B and $A \wedge B$,
- (2) $\dim B \leq 2n + 1$. Hence $\pi(\Sigma *, \Sigma A \wedge B) = 0$ for $*$ = A or B ,
- (3) $\Sigma: \pi(X, Y) \rightarrow \pi(\Sigma X, \Sigma Y)$ is onto for any pair (X, Y) of $\{A, B\}$.

Proof. Easy. \square

In §1, $f \in \pi(\Sigma A \times B, \Sigma A \wedge B)$ has a representation:

$$f = f_A \Sigma P_A + f_B \Sigma P_B + f_C \Sigma \pi$$

for $f_* \in \pi(*, \Sigma A \times B)$ and $C = A \wedge B$.

And moreover in §2, f_* has a representation:

$$f_* = \Sigma i_A f_{*1} + \Sigma i_B f_{*2} + \xi_{AB} f_{*3} + [\Sigma i_A, \Sigma i_B] f_{*4}$$

for $f_{*1} \in \pi(\Sigma^*, \Sigma A)$, $f_{*2} \in \pi(\Sigma^*, \Sigma B)$ and $f_{*3}, f_{*4} \in \pi(\Sigma^*, \Sigma A \wedge B)$.

Thus $f \in \pi(\Sigma A \times B, \Sigma A \times B)$ has a form of a (3×4) -matrix (note lemma 3.1):

$$(f_{*k}) = \begin{pmatrix} f_{A1} & f_{A2} & 0 & 0 \\ f_{B1} & f_{B2} & 0 & 0 \\ f_{C1} & f_{C2} & f_{C3} & f_{C4} \end{pmatrix}.$$

We want to compute the composition gf for $f, g \in \pi(\Sigma A \times B, \Sigma A \times B)$. Since we have that $gf = (gf_A)\Sigma P_A + (gf_B)\Sigma P_B + (gf_C)\Sigma \pi$ it is sufficient for our purpose to compute $gf_* \in \pi(\Sigma^*, \Sigma A \times B)$.

Lemma 3.2. $g_C = g\xi_{AB}$.

Proof. First we have

$$\begin{aligned} g &= g1_{\Sigma A \times B} \\ &= g(\Sigma i_A \Sigma P_A + g \Sigma i_B \Sigma P_B + g\xi_{AB} \Sigma \pi) \\ &= g|_{\Sigma A} \Sigma P_A + g|_{\Sigma B} \Sigma P_B + g\xi_{AB} \Sigma \pi. \end{aligned}$$

On the other hand $g = g_A \Sigma P_A + g_B \Sigma P_B + g_C \Sigma \pi$. Hence we have $g_C = g\xi_{AB}$. \square

Now we proceed to gf_* :

$$\begin{aligned} gf_* &= g(\Sigma i_A f_{*1} + \Sigma i_B f_{*2} + \xi_{AB} f_{*3} + [\Sigma i_A, \Sigma i_B] f_{*4}) \\ &= g|_{\Sigma A} f_{*1} + g|_{\Sigma B} f_{*2} + g\xi_{AB} f_{*3} + [g|_{\Sigma A}, g|_{\Sigma B}] f_{*4} \\ &= g_A f_{*1} + g_B f_{*2} + g_C f_{*3} + [g_A, g_B] f_{*4}. \end{aligned}$$

Lemma 3.3.

$$\begin{aligned} [g_A, g_B] &= \Sigma i_A [g_{A1}, g_{B1}] + \Sigma i_B [g_{A2}, g_{B2}] \\ &\quad + [\Sigma i_A, \Sigma i_B] (\Sigma g'_{A1} \wedge \Sigma g'_{B2} - \tau \Sigma g'_{A2} \wedge g'_{B1}), \end{aligned}$$

where $g_{*k} = \Sigma g'_{*k}$ for $* \in \{A, B\}$ and $k = 1, 2$.

Proof. Apply lemma 1.2 and Propositions 3.3, 3.4 of [1] to the equality:

$$[g_A, g_B] = [\Sigma i_A g_{A1} + \Sigma i_B g_{A2}, \Sigma i_A g_{B1} + \Sigma i_B g_{B2}],$$

then the proof is completed. \square

Lemma 3.4. We have

$$g_A f_{C1} = \Sigma i_A (g_{A1} f_{C1}) + \Sigma i_B (g_{A2} f_{C1}) + [\Sigma i_A, \Sigma i_B] \Sigma g'_{A1} \wedge g'_{A2} H(f_{C,1}).$$

Proof. Apply the distributive law to the equality:

$$g_A f_{C1} = (\Sigma i_A g_{A1} + \Sigma i_B g_{A2}) f_{C1},$$

then the proof is completed. \square

From these lemmas we have

Theorem 3.5. *If $f = (f_{*k})$ and $g = (g_{*k})$ then $gf = h = (h_{*k})$ is given by*

(1) *the case of $*$ = A, or B,*

$$h_{*1} = g_{A1} f_{*1} + g_{B1} f_{*2},$$

$$h_{*2} = g_{A2} f_{*1} + g_{B2} f_{*2},$$

(2) *the case of $*$ = C = A \wedge B,*

$$h_{C1} = g_{A1} f_{C1} + g_{B1} f_{C2} + [g_{A1}, g_{B1}] f_{C4} + g_{C1} f_{C3},$$

$$h_{C2} = g_{A2} f_{C1} + g_{B2} f_{C2} + [g_{A2}, g_{B2}] f_{C4} + g_{C2} f_{C3},$$

$$h_{C3} = g_{C3} f_{C3},$$

$$h_{C4} = \Sigma g'_{A1} \wedge g'_{A2} H(f_{C1}) + \Sigma g'_{B1} \wedge g'_{B2} H(f_{C2}) \\ + (\Sigma g'_{A1} \wedge g'_{B2} - \Sigma \tau \Sigma g'_{A2} \wedge g'_{B1}) f_{C4} + g_{C4} f_{C3}.$$

As an example we take $A=B=S^n$. Let us be $f \in \pi(\Sigma S^n \times S^n, \Sigma S^n \times S^n)$ with its matrix:

$$\begin{pmatrix} f_{11} & f_{12} & 0 & 0 \\ f_{21} & f_{22} & 0 & 0 \\ f_{31} & f_{32} & f_{33} & f_{34} \end{pmatrix},$$

where $f_{ij}(\{i, j\} = \{1, 2\})$, $f_{33}, f_{34} \in \mathbf{Z}$ and $f_{31}, f_{32} \in \pi_{2n+1}(S^{n+1})$.

If $h = gf$ then h_{ij} is given by

$$h_{11} = g_{11} f_{11} + g_{21} f_{12},$$

$$h_{12} = g_{12} f_{11} + g_{22} f_{12},$$

$$h_{21} = g_{11} f_{21} + g_{21} f_{22},$$

$$h_{22} = g_{12} f_{21} + g_{22} f_{22},$$

$$h_{31} = g_{11} \circ f_{31} + g_{21} \circ f_{32} + f_{33} g_{31} + f_{34} g_{12} g_{22} [\iota_{n+1}, \iota_{n+1}],$$

$$h_{32} = g_{12} \circ f_{31} + g_{22} \circ f_{32} + f_{33} g_{32} + f_{34} g_{12} g_{22} [\iota_{n+1}, \iota_{n+1}],$$

$$h_{33} = g_{33} f_{33},$$

$$h_{34} = (g_{11} g_{22} - (-1)^n g_{12} g_{21}) f_{34} + g_{34} f_{33} + g_{11} g_{12} H(f_{31}) + g_{21} g_{22} H(f_{32}),$$

where $g_{**} \circ f_{**}$ denotes $(g_{**} \iota_{n+1}) f_{**}$.

Here we describe some results obtained from the above table of the composition.

1. f is a homotopy equivalence if and only if $f_{33} = \pm 1$ and

$$\begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = \pm 1.$$

2. We denote the group of self-homotopy equivalences of $\Sigma S^n \times S^n$ with $\varepsilon(G_n)$ and orientation-preserving self-homotopy equivalences with $\varepsilon_0(G_n)$, where **orientation-preserving** means that the above determinant is 1 and $f_{33} = 1$, then $\varepsilon_0(G_n)$ is a normal subgroup and the quotient is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2$.
3. $\varepsilon_0(G_n)$ contains a subgroup:

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a \end{pmatrix} \right\} \cong \mathbf{Z}.$$

4. $\varepsilon_0(G_n)$ contains a subgroup:

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \alpha & \beta & 1 & 0 \end{pmatrix} \right\} \cong \pi_{2n+1}(S^{n+1}) \oplus \pi_{2n+1}(S^{n+1}).$$

REFERENCES

- [1] M. ARKOWITZ, *The generalized Whitehead product*, Pacific J. Math. **12**(1962), 7–23.
- [2] K. MORISUGI, *Hopf construction, Samelson products and suspension maps*, Contemporary Math. **239**(1999), 225–238.
- [3] G. W. WHITEHEAD, *Elements of homotopy theory*, Graduate texts in Math. **61**, Springer-Verlag, 1978.

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