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Note on the Relation between S-reducibility, S-coreducibility and Stable Homotopy Types of Some Stunted Lens Spaces

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NOTE ON THE RELATION BETWEEN S-REDUCIBILITY, S-COREDUCIBILITY AND STABLE HOMOTOPY TYPES OF SOME STUNTED LENS SPACES

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1. Introduction

Let $q \geq 2$ and $n \geq 0$ be integers. Let $L_q^{2n+1} = S^{2n+1}/(\mathbb{Z}/q)$ be the $(2n+1)$-dimensional mod q lens space and L_q^{2n} the $2n$-skeleton of L_q^{2n+1} by the natural cell-decomposition. For $k \geq 0$, there is a natural inclusion $L_q^{n-1} \subset L_q^{n+k}$, and so we get the mod q stunted lens space $L_n^{n+k} = L_q^{n+k}/L_q^{n-1}$.

Two spaces X and Y are said to be of the same stable homotopy type if $\Sigma^u X$(the u-fold suspension of X) and $\Sigma^v Y$ are of the same homotopy type for some non-negative integers u and v.

A space X is reducible if there exists a map $f\colon S^n \to X$ that induces an isomorphism

$$f_*\colon \widetilde{H}_i(S^n;\mathbb{Z}) \xrightarrow{\cong} \widetilde{H}_i(X;\mathbb{Z}) \text{ for all } i \geq n,$$

and is S-reducible if $\Sigma^u X$ is reducible for some non-negative integer u.

Dually, a space X is coreducible if there exists a map $g\colon X \to S^n$ that induces an isomorphism

$$g^*\colon \widetilde{H}^i(S^n;\mathbb{Z}) \xrightarrow{\cong} \widetilde{H}^i(X;\mathbb{Z}) \text{ for all } i \leq n,$$

and is S-coreducible if $\Sigma^v X$ is coreducible for some non-negative integer v.

It is clear that S-reducibility and S-coreducibility are properties of the stable homotopy type, and that a space is S-reducible if and only if its S-dual, in the sense of Spanier and Whitehead [8], is S-coreducible.

By the integral homology and cohomology of mod q stunted lens spaces, it follows that $L_{2n+\delta}^{2n+2k+\varepsilon}$ is not S-reducible for $\varepsilon = 0$ and $\delta < 2k$, and not S-coreducible for $\delta = 1$ and $\varepsilon > 1 - 2k$.

The object of this paper is to determine a necessary and sufficient condition for two mod q stunted lens spaces to be of the same stable homotopy type in case either of two spaces is S-reducible or S-coreducible.

Let η be the canonical complex line bundle over L_q^{2k+1} and denote simply by η its restriction to L_q^{2k}. Let $\overline{\eta}$ be the realification of η. Let $J(\overline{\eta}-2)$ be the image of $\overline{\eta} - 2 \in \widetilde{KO}(L_q^l)$ by the J-homomorphism $J\colon \widetilde{KO}(L_q^l) \to \widetilde{J}(L_q^l)$.

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Then the order $h(l,q)$ of $J(\overline{\eta}-2) \in \widetilde{J}(L_q^l)$ is completely determined in [3, Theorem 2.1].

Our main theorem is as follows.

Theorem 1.1. (i) *Suppose that either the* mod q *stunted lens space* $L_{2n}^{2n+2k+\varepsilon}$ *or* $L_{2m}^{2m+2k+\varepsilon}$ *is* S*-coreducible for* $\varepsilon = 0$ *or* 1. *Then* $L_{2n}^{2n+2k+\varepsilon}$ *and* $L_{2m}^{2m+2k+\varepsilon}$ *are of the same stable homotopy type if and only if* $n \equiv m \bmod h(2k+\varepsilon, q)$.

(ii) *Suppose that either the* mod q *stunted lens space* $L_{2n+\delta}^{2n+2k+1}$ *or* $L_{2m+\delta}^{2m+2k+1}$ *is* S*-reducible for* $\delta = 0$ *or* 1. *Then* $L_{2n+\delta}^{2n+2k+1}$ *and* $L_{2m+\delta}^{2m+2k+1}$ *are of the same stable homotopy type if and only if* $n \equiv m \bmod h(2k+1-\delta, q)$.

Susumu Kôno stated the above results in [6, (2.10)(2), (4)] without proof and without mentioning the relation to S-coreducibility and S-reducibility. So we state the relation to the S-coreducibility and S-reducibility, and give the complete proof.

Stable homotopy types of mod q stunted lens spaces are completely determined by D. M. Davis and M. Mahowald [2] for the case $q = 2$, by H. Yang [9] for the case $q = 4$ and by J. González [4] for the case $q = p$ where p is an odd prime.

This paper is organized as follows. In Section 2, we recall the known results. A proof of the main theorem is given in Section 3. As a concluding remark, we consider the cases $q = 2^r (r \geq 1)$.

2. Preliminaries

Let α be a real vector bundle over a finite CW-complex X. Then the Thom complex X^α is defined to be the one-point compactification of α. Let $J(\alpha) \in \widetilde{J}(X)$ denote the stable fibre homotopy class of α.

Now let us recall that the relations given in [1] between stable fibre homotopy classes and stable homotopy types or S-coreducibility of Thom complexes.

Proposition 2.1 ([1, Proposition (2.6)]). *Let* α *and* β *be real vector bundles over* X. *Then* X^α *and* X^β *are of the same stable homotopy type if* $J(\alpha) = J(\beta)$.

Proposition 2.2 ([1, Proposition (2.8)]). *Let* α *be a real vector bundle over a connected space* X. *Then* X^α *is* S*-coreducible if and only if* $J(\alpha) = 0$.

Let η be the canonical complex line bundle over L_q^{2k+1} and denote simply by η its restriction to L_q^{2k}. Let $\overline{\eta}$ be the realification of η. Then there are natural homeomorphisms given in [5, Theorem 4.7, Corollary 4.8]:

$$L_{2n}^{2n+2k+\varepsilon} = (L_q^{2k+\varepsilon})^{n\overline{\eta}},$$
$$L_{2n+1}^{2n+2k+\varepsilon} = (L_q^{2k+\varepsilon})^{n\overline{\eta}}/S^{2n}$$

for $\varepsilon = 0$ or 1.

S-duality of mod q stunted lens spaces is given in [6].

Lemma 2.3 ([6, Lemma 2.9] and [7, Proposition 5]). *Let* $\delta, \varepsilon \in \{0, 1\}$. *Suppose that*

$$N \equiv 0 \bmod \begin{cases} h(2k+1-\delta, q) & \text{if } \varepsilon = 1, \\ h(2k, q) & \text{if } \varepsilon = 0, \end{cases}$$

and $2N > 2n + 2k + \varepsilon + 1$. *Then an* S*-dual of the* mod q *stunted lens space* $L_{2n+\delta}^{2n+2k+\varepsilon}$ *is* $L_{2N-2n-2k-\varepsilon-1}^{2N-2n-\delta-1}$.

3. Proof of Main Theorem

K. Fujii, T. Kobayashi and M. Sugawara [3] proved the following.

Theorem 3.1 ([3, Theorem 1.5]). *Two* mod q *stunted lens spaces* $L_{2n+\delta}^{2n+2k+\varepsilon}$ *and* $L_{2m+\delta}^{2m+2k+\varepsilon}$ *for* $\delta, \varepsilon \in \{0, 1\}$ *are of the same stable homotopy type if* $n \equiv m \bmod h(2k + \varepsilon, q)$.

However, by making use of S-duality, we have

Theorem 3.2. *Two* mod q *stunted lens spaces* $L_{2n+\delta}^{2n+2k+1}$ *and* $L_{2m+\delta}^{2m+2k+1}$ *for* $\delta = 0$ *or* 1 *are of the same stable homotopy type if* $n \equiv m \bmod h(2k+1-\delta, q)$.

Theorem 3.2 is better than Theorem 3.1 for $\varepsilon = 1$ since in general

$$h(2k + 1 - \delta, q) \leq h(2k + 1, q).$$

Proof of Theorem 3.2. Since $n \equiv m \bmod h(2k + 1 - \delta, q)$, we have

$$(N - n - k - 1)J(\overline{\eta} - 2) = (M - m - k - 1)J(\overline{\eta} - 2) \in \widetilde{J}(L_q^{2k+1-\delta})$$

for some integers N and M such that $N, M \equiv 0 \bmod h(2k + 1 - \delta, q)$ and $N > n + k + 1$, $M > m + k + 1$. Then, by Proposition 2.1,

$$(L_q^{2k+1-\delta})^{(N-n-k-1)\overline{\eta}} = L_{2N-2n-2k-2}^{2N-2n-\delta-1}$$

and

$$(L_q^{2k+1-\delta})^{(M-m-k-1)\overline{\eta}} = L_{2M-2m-2k-2}^{2M-2m-\delta-1}$$

are of the same stable homotopy type.

On the other hand, by Lemma 2.3, $L_{2N-2n-2k-2}^{2N-2n-\delta-1}$ is an S-dual of $L_{2n+\delta}^{2n+2k+1}$. Thus, by the property of S-duality, $L_{2n+\delta}^{2n+2k+1}$ and $L_{2m+\delta}^{2m+2k+1}$ are of the same stable homotopy type. □

S-coreducibility and S-reducibility of mod q stunted lens spaces are stated as follows.

Proposition 3.3. (i) *The* mod q *stunted lens space* $L_{2n}^{2n+2k+\varepsilon}$ *is* S*-coreducible for* $\varepsilon = 0$ *or* 1 *if and only if* $n \equiv 0 \bmod h(2k+\varepsilon, q)$.

(ii) *The* mod q *stunted lens space* $L_{2n+\delta}^{2n+2k+1}$ *is* S*-reducible for* $\delta = 0$ *or* 1 *if and only if* $n+k+1 \equiv 0 \bmod h(2k+1-\delta, q)$.

Proof. (i) By Proposition 2.2, $L_{2n}^{2n+2k+\varepsilon} = (L_q^{2k+\varepsilon})^{n\overline{\eta}}$ is S-coreducible if and only if

$$J(n\overline{\eta}) = nJ(\overline{\eta}-2) = 0.$$

Hence we get $n \equiv 0 \bmod h(2k+\varepsilon, q)$.

(ii) By Lemma 2.3, an S-dual of $L_{2n+\delta}^{2n+2k+1}$ is $L_{2N-2n-2k-2}^{2N-2n-\delta-1}$ for some integer N such that $N \equiv 0 \bmod h(2k+1-\delta, q)$ and $N > n+k+1$. Since a space is S-reducible if and only if its S-dual is S-coreducible, by (i), $L_{2n+\delta}^{2n+2k+1}$ is S-reducible if and only if

$$N-n-k-1 \equiv 0 \bmod h(2k+1-\delta, q)$$

for some integer N such that $N \equiv 0 \bmod h(2k+1-\delta, q)$ and $N > n+k+1$. Hence we get $n+k+1 \equiv 0 \bmod h(2k+1-\delta, q)$. □

Now we can prove main theorem.

Proof of Theorem 1.1. The sufficient condition follows immediately from Theorems 3.1 and 3.2.

Next, we consider the necessary condition. Since two spaces are of the same stable homotopy type, $L_{2n}^{2n+2k+\varepsilon}$ is S-coreducible if and only if $L_{2m}^{2m+2k+\varepsilon}$ is S-coreducible. That is, by Proposition 3.3(i), we see that

$$n \equiv 0 \bmod h(2k+\varepsilon, q) \iff m \equiv 0 \bmod h(2k+\varepsilon, q).$$

Hence we get $n \equiv m \bmod h(2k+\varepsilon, q)$.

Similarly, $L_{2n+\delta}^{2n+2k+1}$ is S-reducible if and only if $L_{2m+\delta}^{2m+2k+1}$ is S-reducible. That is, by Proposition 3.3(ii), we see that

$$n+k+1 \equiv 0 \bmod h(2k+1-\delta, q) \iff m+k+1 \equiv 0 \bmod h(2k+1-\delta, q).$$

Hence we get $n \equiv m \bmod h(2k+1-\delta, q)$. □

Concluding remarks

For the cases $q = 2^r (r \geq 1)$, Theorem 1.1 is restated as follows.

Theorem 1.1'. (i) *Suppose that* $n \equiv 0 \bmod h(2k+\varepsilon, 2^r)$. *Then two* mod 2^r *stunted lens spaces* $L_{2n}^{2n+2k+\varepsilon}$ *and* $L_{2m}^{2m+2k+\varepsilon}$ *for* $\varepsilon = 0$ *or* 1 *are of the same stable homotopy type if and only if* $n \equiv m \bmod h(2k+\varepsilon, 2^r)$.

(ii) *Suppose that* $n+k+1 \equiv 0 \bmod h(2k+1-\delta, 2^r)$. *Then two* mod 2^r *stunted lens spaces* $L_{2n+\delta}^{2n+2k+1}$ *and* $L_{2m+\delta}^{2m+2k+1}$ *for* $\delta = 0$ *or* 1 *are of the same stable homotopy type if and only if* $n \equiv m \bmod h(2k+1-\delta, 2^r)$.

However, it is known that the above result holds under the weaker assumptions

(i) $2n \equiv 0 \bmod h(2k+\varepsilon, q)$,

(ii) $2(n+k+1) \equiv 0 \bmod h(2k+1-\delta, q)$,

for the cases $q = 2$ ([2]), $q = 4$ ([9]) and $q = 8$ ([6]).

Susumu Kôno claims that for all the cases $q = 2^r (r \geq 1)$, above result holds under the weaker assumptions.

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References

[1] M. F. Atiyah, *Thom complexes*, Proc. London. Math. Soc. **11**(1961), 291–310.
[2] D. M. Davis and M. Mahowald, *Classification of the stable homotopy types of stunted real projective spaces*, Pacific J. Math. **125**(1986), 335–345.
[3] K. Fujii, T. Kobayashi and M. Sugawara, *Stable homotopy types of stunted lens spaces*, Mem. Fac. Sci. Kochi Univ. Ser. A Math. **3**(1982), 21–27.
[4] J. González, *Classification of the stable homotopy types of stunted lens spaces for an odd prime*, Pacific. J. Math. **176**(1996), 325–343.
[5] T. Kobayashi and M. Sugawara, *On stable homotopy types of stunted lens spaces*, Hiroshima Math. J. **1**(1971), 287–304.
[6] S. Kôno, *Stable homotopy types of stunted lens spaces* mod 4, Osaka J. Math. **29**(1992), 697–717.
[7] M. Mimura, J. Mukai and G. Nishida, *Representing elements of stable homotopy groups by symmetric maps*, Osaka J. Math. **11**(1974), 105–111.
[8] E. H. Spanier and J. H. C. Whitehead, *Duality in homotopy theory*, Mathematika **2**(1955), 56–80.
[9] H. Yang, *The stable homotopy types of stunted lens spaces* mod 4, Trans. Amer. Math. Soc. **350**(1998), 4775–4798.

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