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Abstract

In this paper, we shall consider Privalov space $N_{p,0}(D)$ ($p > 1$) which consists of holomorphic functions f on the upper half plane $D := \{z \in \mathbb{C} | \text{Im} z > 0\}$ such that $(\log^+ |f(z)|)^p$ has a harmonic majorant on D . We shall give some properties of $N_{p,0}(D)$.

KEYWORDS: Privalov space, Nevanlinna-type spaces, Hardy-Orlicz class

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PRIVALOV SPACE ON THE UPPER HALF PLANE

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ABSTRACT. In this paper, we shall consider Privalov space $N_0^p(D)$ ($p > 1$) which consists of holomorphic functions f on the upper half plane $D := \{z \in \mathbf{C} \mid \operatorname{Im} z > 0\}$ such that $(\log^+ |f(z)|)^p$ has a harmonic majorant on D . We shall give some properties of $N_0^p(D)$.

1. INTRODUCTION

Let U and T denote the unit disk and the unit circle in $\mathbf{C}$, respectively. For $p > 1$, Privalov space $N^p(U)$ is the class of all holomorphic functions f on U such that $(\log^+ |f(z)|)^p$ has a harmonic majorant on U . Letting $p = 1$, we have the Nevanlinna class $N(U)$.

As in [7], for each strongly convex function φ on $(-\infty, \infty)$ we define the Hardy-Orlicz class $H_\varphi(U)$ as the space of all holomorphic functions f on U such that $\varphi(\log^+ |f(z)|)$ has a harmonic majorant on U . Recall that a convex function φ is strongly convex if φ is non-negative, non-decreasing and $\varphi(t)/t \rightarrow \infty$ as $t \rightarrow \infty$. We define $N_*(U) = \bigcup \{H_\varphi(U) \mid \varphi : \text{strongly convex}\}$, which is called the Smirnov class.

For $0 < q < \infty$, the space $H_\varphi(U)$ with $\varphi(t) = e^{qt}$ coincides with the usual Hardy space $H^q(U)$. For each $p > 1$, if we define $\varphi_p(t)$ on $(-\infty, \infty)$ by $\varphi_p(t) = t^p$ for $t \geq 0$, and $\varphi_p(t) = 0$ for $t < 0$, we obtain $N^p(U)$ as a subspace of $N_*(U)$.

It is well-known that $H^q(U) \subset N^p(U) \subset N_*(U) \subset N(U)$ ($0 < q < \infty, p > 1$). These including relations are proper. $N^p(U)$ was treated by several authors ([2], [5], [7] and [8]). The spaces $N(U)$, $N_*(U)$, $N^p(U)$ and $H^q(U)$ are called *Nevanlinna-type spaces*.

Let $D := \{z \in \mathbf{C} \mid \operatorname{Im} z > 0\}$. We let the Nevanlinna class $N_0(D)$, as Krylov [4] introduced, consist of all holomorphic functions f on D such that $\log^+ |f(z)|$ has a harmonic majorant on D .

Rosenblum and Rovnyak [6] introduced the Hardy-Orlicz and Smirnov classes on D : for each strongly convex function ϕ on $(-\infty, \infty)$, $H_\phi(D)$ is the set of all holomorphic functions f on D such that $\phi(\log^+ |f(z)|)$ has a harmonic majorant on D . We define $N_0^*(D) = \bigcup \{H_\phi(D) \mid \phi : \text{strongly convex}\}$.

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Key words and phrases. Privalov space, Nevanlinna-type spaces, Hardy-Orlicz class.

In this paper, we shall define a new class $N_0^p(D)$, analogous to $N^p(U)$; i.e., we denote by $N_0^p(D)$ ($p > 1$) the set of all holomorphic functions f on D such that $(\log^+ |f(z)|)^p$ has a harmonic majorant on D . First we obtain a factorization theorem for the space $N_0^p(D)$. Moreover, some properties of $N_0^p(D)$ are also given.

2. PRELIMINARIES

Let ν be a real measure on T and $\Psi(z) = (z - i)/(z + i)$ ($z \in \overline{D}$). Then there corresponds a finite real measure μ on $\mathbf{R}$ such that

$$\int_{\mathbf{R}} h(t) d\mu(t) = \int_{T^*} (h \circ \Psi^{-1})(\eta) d\nu(\eta) \quad (h \in C_c(\mathbf{R})),$$

where $T^* = T \setminus \{1\}$. Let $H(w, \eta) = (\eta + w)/(\eta - w)$ ($(w, \eta) \in U \times T$). There holds

$$(1) \quad \frac{1}{i} \int_{\mathbf{R}} \frac{1 + tz}{t - z} d\mu(t) = \int_{T^*} H(\Psi(z), \eta) d\nu(\eta) \\ = \int_T H(\Psi(z), \eta) d\nu(\eta) - i\alpha z \quad (z \in D),$$

where $\alpha = -\nu(\{1\})$. We write the Poisson integrals of measures μ on $\mathbf{R}$ and ν on T as follows:

$$P[\mu](z) = \frac{1}{\pi} \int_{\mathbf{R}} \frac{y}{(x - t)^2 + y^2} d\mu(t) \quad (z = x + iy \in D), \\ Q[\nu](w) = \int_T \frac{1 - |w|^2}{|\eta - w|^2} d\nu(\eta) \quad (w \in U).$$

Taking the real parts in (1), we have

$$P[\pi(1 + t^2)d\mu(t)](z) = Q[\nu](\Psi(z)) + \alpha \cdot \operatorname{Im} z \quad (z \in D).$$

When $f \in L^1(\mathbf{R}, (1 + t^2)^{-1}dt)$ and $g \in L^1(T)$, we write $P[f]$ and $Q[g]$ instead of $P[f(t)dt]$ and $Q[g\sigma]$, respectively. If $g \in L^1(T)$, then we have $g \circ \Psi \in L^1(\mathbf{R}, (1 + t^2)^{-1}dt)$ and

$$(2) \quad P[g \circ \Psi](z) = Q[g](\Psi(z)).$$

3. SOME PROPERTIES OF $N_0(D)$, $N_0^*(D)$ AND $N^p(U)$

In this section, we shall summarize some properties of $N_0(D)$, $N_0^*(D)$ and $N^p(U)$ ($p > 1$). For the following results, the reader refers to [2], [3] and [6].

Recall that an *outer function* on D is of the form

$$d(z) = \exp \left(\frac{1}{\pi i} \int_{\mathbf{R}} \frac{1+tz}{t-z} \frac{1}{1+t^2} \log h(t) dt \right),$$

where $h(t) \geq 0$, $\log h \in L^1 \left(\mathbf{R}, \frac{dt}{1+t^2} \right)$.

Proposition 3.1. *Let $f \in N_0(D)$, $f \neq 0$. Then $f^*(x) = \lim_{z \rightarrow x} f(z)$ exists nontangentially a.e. for $x \in \mathbf{R}$.*

Proposition 3.2. *Let $H^\infty(D)$ be the class of all bounded holomorphic functions on D .*

- (i) $N_0(D) = \left\{ \frac{g}{h} : g, h \in H^\infty(D), h \neq 0 \right\}$.
- (ii) $N_0^*(D) = \left\{ \frac{g}{h} : g, h \in H^\infty(D), h \text{ is outer} \right\}$.

Proposition 3.3. *Let f be holomorphic on D .*

- (i) $f \in N_0(D)$ if and only if

$$\sup_{y>0} \int_{\mathbf{R}} \frac{\log^+ |f(x+iy)|}{x^2 + (y+1)^2} dx < \infty.$$

- (ii) If ϕ is a strongly convex function, then $f \in N_0^*(D)$ if and only if

$$\sup_{y>0} \int_{\mathbf{R}} \frac{\phi(\log^+ |f(x+iy)|)}{x^2 + (y+1)^2} dx < \infty.$$

Proposition 3.4. *Let $p > 1$ and f be holomorphic on U . Then the following are equivalent:*

- (i) $f \in N^p(U)$.
- (ii) $\sup_{0 < r < 1} \int_0^{2\pi} \left(\log^+ |f(re^{i\theta})| \right)^p d\theta < \infty$.
- (iii) $f \in N(U)$ and the condition

$$\int_0^{2\pi} \left(\log^+ |f^*(e^{i\theta})| \right)^p d\theta < \infty$$

holds, where $f^*(e^{i\theta}) = \lim_{r \rightarrow 1^-} f(re^{i\theta})$ (a.e. $e^{i\theta} \in T$).

Proposition 3.5. *Let $f \in N^p(U)$ ($p > 1$), $f \neq 0$. Then, $\log |f^*| \in L^1(T)$ and $\log(1 + |f^*|) \in L^p(T)$. Furthermore, f can be uniquely factored as follows,*

$$(3) \quad f(z) = aB(z)F(z)S(z) \quad (z \in U),$$

where the factors above have the following properties.

(i) $a \in T$ is a constant.

(ii) $B(z) = z^m \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n}z}$ ($z \in U$) is a Blaschke product with respect to the zeros of f .

(iii) $F(z) = \exp \left(\int_T \frac{\zeta + z}{\zeta - z} \log |f^*(\zeta)| d\sigma(\zeta) \right)$, where σ denotes normalized Lebesgue measure on T .

(iv) $S(z) = \exp \left(- \int_T \frac{\zeta + z}{\zeta - z} d\nu(\zeta) \right)$, where ν is a positive singular measure on T .

If f is expressed in the form (3), then $f \in N^p(U)$.

Proposition 3.6. *Let $f \in N^p(U)$, $p > 1$. Then $(\log^+ |f|)^p$ has the least harmonic majorant $Q[(\log^+ |f^*|)^p]$.*

4. A FACTORIZATION THEOREM FOR THE SPACE $N_0^p(D)$

Theorem 4.1. *Let $p > 1$. $f \in N_0^p(D)$, $f \neq 0$, is factorized in the form*

$$(4) \quad f(z) = ae^{i\alpha z} b(z)d(z)g(z) \quad (z \in D),$$

with the following properties.

(i) $a \in T$, $\alpha \geq 0$.

(ii) $b(z)$ is the Blaschke product with respect to the zeros of f .

(iii) $d(z) = \exp \left(\frac{1}{\pi i} \int_{\mathbf{R}} \frac{1+tz}{t-z} \frac{1}{1+t^2} \log h(t) dt \right)$, where $h(t) \geq 0$, $\log h \in L^1(\mathbf{R}, (1+t^2)^{-1}dt)$ and $\log^+ h \in L^p(\mathbf{R}, (1+t^2)^{-1}dt)$.

(iv) $g(z) = \exp \left(-\frac{1}{i} \int_{\mathbf{R}} \frac{1+tz}{t-z} d\mu(t) \right)$, where μ is a finite real measure

on $\mathbf{R}$, singular with respect to Lebesgue measure.

If f is expressed in the form (4), then $f \in N_0^p(D)$.

Proof. Suppose $f \in N_0^p(D)$, $f \neq 0$. Then $f \circ \Psi^{-1} \in N^p(U)$, and Proposition 3.5 implies $(f \circ \Psi^{-1})(w) = aB(w)F(w)S(w)$ ($w \in U$). In the factorization $f(z) = aB(\Psi(z))F(\Psi(z))S(\Psi(z))$ ($z \in D$), $b(z) := B(\Psi(z))$ is the Blaschke product formed from the zeros of f , and the change of the variables $\eta = \Psi(t)$ ($t \in \mathbf{R}$) shows that

$$d(z) := F(\Psi(z)) = \exp \left(\frac{1}{\pi i} \int_{\mathbf{R}} \frac{1+tz}{t-z} \frac{1}{1+t^2} \log |f^*(t)| dt \right).$$

This is of the form (iii). Since $\log |(f \circ \Psi^{-1})^*| \in L^1(T)$, we have $\log |f^*| \in L^1(\mathbf{R}, (1+t^2)^{-1}dt)$ by (2). Next $\log^+ |f(\Psi^{-1}(\eta))| \in L^p(T)$ implies $\log^+ |f^*| \in L^p(\mathbf{R}, (1+t^2)^{-1}dt)$. Setting $\alpha = \nu\{1\}$, we have $S(\Psi(z)) = g(z)e^{i\alpha z}$, where g is of the form (iv).

Suppose, conversely, that f is of the form (4). Then

$$|f(z)| = |e^{i\alpha z}| |b(z)| \exp(P[\log h - \pi(1+t^2)d\mu(t)](z)) \leq \exp(P[\log h](z)).$$

Since $\log^+ |(f \circ \Psi^{-1})(w)| \leq Q[\log^+ |(h \circ \Psi^{-1})|](w)$, we have $f \circ \Psi^{-1} \in N^p(U)$. Letting $y \rightarrow 0^+$ in $|f(x+iy)|$, we have $|f^*(x)| = h(x)$ a.e. for $x \in \mathbf{R}$. Furthermore, $(\log^+ |f \circ \Psi^{-1}|)^p$ has the least harmonic majorant $v' = Q[(\log^+ |(f \circ \Psi^{-1})^*|)^p]$ by Proposition 3.6, hence $v := v' \circ \Psi$ is the least harmonic majorant of $(\log^+ |f|)^p$; i.e., $(\log^+ |f(z)|)^p \leq P[(\log^+ |f^*|)^p](z)$. Integrating the both sides, we have $f \in N_0^p(D)$. $\square$

5. SOME THEOREMS FOR THE SPACE $N_0^p(D)$

In this section, we prove some theorems for the space $N_0^p(D)$.

Theorem 5.1. *Let f be holomorphic on D . Then, for $p > 1$,*

$$N_0^p(D) = \left\{ \frac{k_1}{k_2} : k_1, k_2 \in H^\infty(D), k_2 \text{ is invertible in } N_0^p(D) \right\}.$$

Proof. Let $f \in N_0^p(D)$. Then $f(z) = ae^{i\alpha z}b(z)d(z)g(z)$ ($z \in D$) by Theorem 4.1. Now d takes the form

$$d(z) = \exp \left(\frac{1}{\pi i} \int_{\mathbf{R}} \frac{1+tz}{t-z} \frac{1}{1+t^2} \log |f^*(t)| dt \right) = \frac{d_1(z)}{d_2(z)},$$

where

$$d_1(z) = \exp \left(-\frac{1}{\pi i} \int_{\mathbf{R}} \frac{1+tz}{t-z} \frac{1}{1+t^2} \log^- |f^*(t)| dt \right)$$

and

$$d_2(z) = \exp \left(-\frac{1}{\pi i} \int_{\mathbf{R}} \frac{1+tz}{t-z} \frac{1}{1+t^2} \log^+ |f^*(t)| dt \right).$$

We note that d_1 and d_2 both belong to $H^\infty(D)$ and are outer functions on D . Moreover, we find $d_2, 1/d_2 \in N_0^p(D)$. Therefore we have $f = ae^{i\alpha z}bdg = ae^{i\alpha z}bd_1g/d_2$, where $k_1 := ae^{i\alpha z}bd_1g$ and $k_2 := d_2$ are both in $H^\infty(D)$ and k_2 is invertible in $N_0^p(D)$.

On the other hand, let $f = k_1/k_2$, where $k_1, k_2 \in H^\infty(D)$ and k_2 is invertible in $N_0^p(D)$. Since $N_0^p(D)$ is an algebra, it follows that $f \in N_0^p(D)$. $\square$

Theorem 5.2. *Let f be holomorphic on D . Then, for $p > 1$, $f \in N_0^p(D)$ if and only if*

$$\sup_{y>0} \int_{\mathbf{R}} \frac{(\log^+ |f(x+iy)|)^p}{x^2 + (y+1)^2} dx < \infty.$$

Proof. $(\log^+ |f(x+iy)|)^p$ is non-negative and subharmonic on D . Therefore we prove the result by the theorem of Flett and Kuran [6, p.89]. $\square$

Theorem 5.3. *Let $f(z) \in N_0(D)$ and $p > 1$. Then f belongs to $N_0^p(D)$ if and only if*

$$(5) \quad \frac{1}{\pi} \int_{\mathbf{R}} \frac{(\log^+ |f^*(x)|)^p}{1+x^2} dx < \infty.$$

Proof. The function $f(z)$ is in $N_0^p(D)$ if and only if $F(z) = f(\Psi^{-1}(z))$ is in $N^p(U)$. By Proposition 3.4, this is the case if and only if

$$\int_0^{2\pi} (\log^+ |f^*(e^{i\theta})|)^p d\theta < \infty.$$

This is the same as condition (5). $\square$

Theorem 5.4. *Let $p > 1$. If $f \in N_0^p(D)$, then*

$$\lim_{y \rightarrow 0^+} \int_{\mathbf{R}} |\log^+ |f(x+iy)| - \log^+ |f^*(x)||^p dx = 0.$$

Proof. Let $f \in N_0^p(D)$. Then $F(z) = f(\Psi^{-1}(z))$ belongs to $N^p(U)$. By [7, Proposition 4.1], we have $(\log^+ |f(z)|)^p \geq P[(\log^+ |f^*|)^p](z)$. Integrating the both sides, it follows that

$$\int_{\mathbf{R}} (\log^+ |f(x+iy)|)^p dx \geq \int_{\mathbf{R}} (\log^+ |f^*(x)|)^p dx.$$

Using Fatou's lemma, we obtain

$$\lim_{y \rightarrow 0^+} \int_{\mathbf{R}} (\log^+ |f(x+iy)|)^p dx = \int_{\mathbf{R}} (\log^+ |f^*(x)|)^p dx.$$

Applying [1, Lemma 1, p.21], we have the desired result. $\square$

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