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SELF MAPS OF SUSPENSION OF SPHERE BUNDLES OVER SPHERES

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1. Introduction. Since $\Sigma(S^m \times S^n)$ is homotopy equivalent to $\Sigma S^m \vee \Sigma S^n \vee \Sigma S^{m+n}$, there exists a bijection

$$[\Sigma(S^m \vee S^n), Y] \rightarrow [\Sigma S^m \vee \Sigma S^n \vee \Sigma S^{m+n}, Y].$$

However this bijection is not always homomorphic with respect to the natural multiplication of two sets. Of course the latter is necessarily abelian for any Y . However, the first group is not always abelian. For example, let $p_n : S^m \times S^n \rightarrow S^n$ and $p_m : S^m \times S^n \rightarrow S^m$ be the projections onto each factor and let $q : S^m \times S^n \rightarrow S^{m+n}$ be the projection. For brevity, suppose i denotes canonical inclusions $\Sigma S^m \rightarrow \Sigma(S^m \times S^n), \Sigma S^n \rightarrow \Sigma(S^m \times S^n), \Sigma S^m \vee \Sigma S^n \rightarrow \Sigma(S^m \times S^m)$ by the same symbol. Then the commutator $\langle i \circ \Sigma p_n, i \circ \Sigma p_m \rangle = \pm i_*[\iota_{n+1}, \iota_{m+1}] \circ \Sigma q \in [\Sigma(S^m \times S^n), \Sigma(S^m \times S^n)]$ is a non-trivial element (Corollary 2.2) and so $[\Sigma(S^m \times S^n), \Sigma(S^m \times S^n)]$ is non-abelian. More generally we show,

Theorem. *Let $E(\xi)$ be an S^m -bundle over S^n with its characteristic class $\xi \in \pi_{n-1}(SO(m+1))$. If $2 < m+1 < n$, then the group $[\Sigma E(\xi), \Sigma E(\xi)]$ is not abelian.*

For example the cases of $S^3 \rightarrow Sp(2) \rightarrow S^7$ and $S^3 \rightarrow SU(3) \rightarrow S^5$ are known by Ohshima.

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2.A commutator in $[\Sigma E(\xi), \Sigma E(\xi)]$ We use the notations:

$p_n : S^m \times S^n \rightarrow S^n$ and $p_m : S^m \times S^n \rightarrow S^m$ be the projections on the each of factors, respectively,

$q : S^m \times S^n \rightarrow S^{m+n} = S^m \times S^n / S^m \vee S^n$ be the projection,

$i_n : S^n \subset \Omega \Sigma(S^m \vee S^n)$ and $i_m : S^m \subset \Omega \Sigma(S^m \vee S^n)$ be the canonical inclusions respectively,

i denotes the adjoints of i_n and i_m by the same symbol,

$+, -$ denotes the loop sum operation and its inverse,

and assume $2 < m+1 < n$.

Lemma 2.1. *In $[\Sigma(S^m \times S^n), \Sigma(S^m \vee S^n)]$, we have the following*

relation

$$\{i \circ \Sigma p_n, i \circ \Sigma p_m\} = [\iota_{n+1}, \iota_{m+1}] \circ \Sigma q.$$

where $\{a, b\}$ denotes the commutator $(-a - b) + (a + b)$.

Proof. The adjoint of $\{i \circ \Sigma p_n, i \circ \Sigma p_m\}$ is represented $\langle i_n, i_m \rangle \circ q$ using the Samelson product $\langle i_n, i_m \rangle$ of i_n and i_m . Considering the adjoint by [3], this turns to the identity

$$\{i \circ \Sigma p_n, i \circ \Sigma p_m\} = \pm [\iota_{n+1}, \iota_{m+1}] \circ \Sigma q.$$

Applying the inclusion $\Sigma(S^m \vee S^n) \subset \Sigma(S^m \times S^n)$, the following is easy;

Corollary 2.2. $[\Sigma(S^m \times S^n), \Sigma(S^m \times S^n)]$ is non-abelian.

Next we consider more generally the case $[\Sigma E(\xi), \Sigma E(\xi)]$ where $p : E(\xi) \rightarrow S^n$, an S^m -bundle over S^n with its characteristic class $\xi \in \pi_{n-1}(SO(m+1))$,
 $q : E(\xi) \rightarrow E(\xi)/S^m \cup e^n = S^{m+n}$.

We assume $2 < m+1 < n$. The following lemma is an extension of [2].

Lemma 2.3. Let X be a connected finite CW complex of $\dim X = n$ and let $P : X \rightarrow X/X_{n-1} = \vee_k S^n$ be the projection where X_{n-1} denotes the $(n-1)$ -skeleton of X . Then for $f \in [\Sigma X, Y]$ and $\{\alpha_k\} \in \pi_{n+1}(Y)$, we have the equality,

$$f + (\vee_k \alpha_k) \circ \Sigma P = (\vee_k \alpha_k) \circ \Sigma P + f$$

in the group $[\Sigma X, Y]$.

Proof. We put $S = \Sigma(X/X_{n-1}) = \vee_k S^{n+1}$ and consider the map $\Sigma P + \Sigma id : \Sigma X \rightarrow S \vee \Sigma X$. Since the inclusion $i : S \vee \Sigma X \rightarrow S \times \Sigma X$ induces a bijection $[\Sigma X, S \vee \Sigma X] \rightarrow [\Sigma X, S \times \Sigma X]$ because of $(S \times \Sigma X)_{n+2} = S \vee \Sigma X$ we obtain $\Sigma P + \Sigma id = \Sigma id + \Sigma P$. Then the proof is completed by applying the map $(\vee_k \alpha_k) \vee f : S \vee \Sigma X \rightarrow Y$ to both sides of this equality.

Lemma 2.4. Let (X, μ) be a connected CW Hopf space, $n > m+1 > 2$, $\alpha \in \pi_n(X)$ and $g : E(\xi) \rightarrow X$. Then

$$\{\alpha \circ p, g\}_\mu = \pm \langle \alpha, g \circ i \rangle_\mu \circ q,$$

where $\{\alpha, \beta\}_\mu = (-\alpha - \beta) + (\alpha + \beta)$ is the commutator in the algebraic loop $[E(\xi), X]$ with respect to μ , $\langle -, - \rangle_\mu$ is the Samelson product with

respect to μ .

Proof. It follows from the commutativity of the following diagram.

$$\begin{array}{ccccc}
 E(\xi) & \xrightarrow{q} & S^n \wedge S^m & \xrightarrow{<\alpha, g \circ i>_\mu} & X \\
 \downarrow d & & \downarrow id \wedge i & & \uparrow \{\cdot\}_\mu \\
 E(\xi) \wedge E(\xi) & \xrightarrow{(\epsilon \iota_n \circ p) \wedge id} & S^n \wedge E(\xi) & \xrightarrow{\alpha \wedge g} & X \wedge X
 \end{array}$$

where d denotes the diagonal map, $\{\cdot\}_\mu$ denotes the commutator map with respect to μ and ϵ is ± 1 . The commutativity of the first square follows from the facts that $(id \wedge i) \circ q$ and $\{(\epsilon \iota_n \circ p) \wedge id\} \circ d$ have the same induced homomorphism $H^{m+n}(S^n \wedge E(\xi)) \rightarrow H^{m+n}(E(\xi))$ and the natural transformation

$$[E(\xi), S^n \wedge E(\xi)] \rightarrow Hom(H^{m+n}(S^n \wedge E(\xi)), H^{m+n}(E(\xi))).$$

is isomorphic. The commutativity of the second square follows from the definition. Thus the proof is completed.

Proposition 2.5. *If $\alpha \in [\Sigma S^n, \Sigma E(\xi)]$ and $g \in [\Sigma E(\xi), \Sigma E(\xi)]$, then we have*

$$\{\alpha \circ \Sigma p, g\}_\mu = (-1)^n \epsilon [\alpha, g \circ \Sigma i] \circ \Sigma q.$$

Proof. It follows from the adjoint isomorphism

$$\begin{array}{ccc}
 [\Sigma E(\xi), \Sigma E(\xi)] & \simeq & [E(\xi), \Omega \Sigma E(\xi)] \\
 (\Sigma q)^* \uparrow & & \uparrow q^* \\
 \pi_{m+n+1}(\Sigma E(\xi)) & \simeq & \pi_{m+n}(\Omega \Sigma E(\xi))
 \end{array}$$

and Lemma 2.4.

Proof of Theorem. From the exact sequence of homotopy groups of the fiber bundle $p : E(\xi) \rightarrow S^n$, there exists an element $\alpha \in \pi_{n+1}(\Sigma E(\xi))$ such that $\deg(\Sigma p \circ \alpha)$ is non-zero, because $\pi_{n-1}(S^m)$ is finite. By [1], $\Sigma E(\xi)$ has the homotopy type of the mapping cone of $\Sigma \Delta \iota_n \vee J(\xi) : \Sigma S^{n-1} \vee \Sigma S^{m+n-1} \rightarrow \Sigma S^m$ where Δ denotes the boundary homomorphism $\pi_n(S^n) \rightarrow \pi_{n-1}(S^m)$ of the exact sequence of homotopy groups of the fiber bundle and J is the Hopf-Whitehead J -homomorphism. By our assumption $n > m + 1$, $J(\xi) \in \pi_{n+m}(\Sigma S^m)$ has the finite order. For such an element α , it follows that $[\alpha, \Sigma i]$ has infinite order because there

exists no map $S^{n+1} \times S^{m+1} \rightarrow \Sigma E(\xi)$ of type $(k\alpha, \Sigma i)$ for any integer $k \neq 0$.
From the Puppe sequence

$$[\Sigma^2(S^m \cup e^n), \Sigma E(\xi)] \xrightarrow{\Sigma(\Sigma i \circ J(\xi))^*} \pi_{m+n+1}(\Sigma E(\xi)) \xrightarrow{\Sigma q^*} [\Sigma E(\xi), \Sigma E(\xi)]$$

it follows that the order of the kernel of $\Sigma q^* : \pi_{m+n+1}(\Sigma E(\xi)) \rightarrow [\Sigma E(\xi), \Sigma E(\xi)]$ is finite and so we have that the commutator $\langle \alpha \circ \Sigma p, id_{\Sigma E} \rangle$ is also non-trivial by Proposition 2.5. Thus the proof is completed.

Remark. (1) Any maps in $[\Sigma E(\xi), \Sigma E(\xi)]$ can be represented as the formula

$$s id_{\Sigma E} + \alpha \circ \Sigma p + \beta \circ \Sigma q$$

for $s \in \mathbb{Z}$ (integers), $\alpha \in \pi_{n+1}(\Sigma E(\xi))$ and $\beta \in \pi_{m+n}(\Sigma E(\xi))$.

(2) On the set $[\Sigma E(\xi), \Sigma E(\xi)]$, the iterated commutators are trivial.

(3) For the case $n=m+1$, we have a counterexample for the Theorem, that is, the Hopf fibration $S^3 \rightarrow S^7 \rightarrow S^4$ or $S^7 \rightarrow S^{15} \rightarrow S^8$.

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