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On Galois theory of division rings III

Nobuo Nobusawa* Hisao Tominaga†

*Osaka University

†Hokkaido University

ON GALOIS THEORY OF DIVISION RINGS III

NOBUO NOBUSAWA and HISAO TOMINAGA

Introduction. Throughout the present paper, we shall deal with a division ring K which is (left) locally finite over a division subring L, and use the following conventions: Let T be an arbitrary intermediate subring of K/L. Then, $\mathfrak{G}'(T/L)$ means the totality of L-(ring) isomorphisms of T into K. (In particular, if T/L is Galois, the Galois group of T/L will be denoted by $\mathfrak{G}(T/L)$ as in [3].) And, if σ is a map defined on a given subset U of K containing T, then $\sigma|T$ means the contraction of σ to T. Similarly, if $\mathfrak{S}$ is a set of maps defined on U, then $\mathfrak{S}|T=\{\sigma|T;\ \sigma\in\mathfrak{S}\}$. Finally, we set $H=V_K(_K(L))$, and for other notations and terminologies used here we follow [3].

Recently, in [3], the notion of locally Galois extensions was introduced: If, for any finite subset F of K, there exists a division subring N containing $L[F]$ such that N/L is Galois and $[N:L]_l<\infty$, then K/L is said to be *locally Galois*. In the present paper, we shall consider a class of locally finite division ring extensions which contains locally Galois extensions and also Galois extensions. More precisely, we shall consider the following conditions :

(I) For any subring L' of K such that $L'\supsetneqq L$ and $[L':L]_l<\infty$, $\mathfrak{G}'(L'/L)$ contains an isomorphism different from the identity map : $\mathfrak{G}'(L'/L)\neq 1$.

(I′) H/L is Galois.

(II) $\mathfrak{G}'(L_1/L)|L_2=\mathfrak{G}'(L_2/L)$ for any intermediate subrings L_1 and L_2 of K/L such that $L_1\supseteq L_2\supseteq L$ and $[L_1:L]_l<\infty$.

(II′) $\mathfrak{G}'(T/L)|H=\mathfrak{G}'(H/L)$ for any intermediate subring T of K/H with $[T:H]_l<\infty$.

We introduce here the notion of quasi-Galois extensions: If K/L is locally finite and satisfies the conditions (I) and (II), the extension K/L will be called *quasi-Galois*. Needless to say, if K/L is locally Galois, then it is quasi-Galois. It is also clear that, if K/L is Galois, then it is quasi-Galois. The notion of quasi-Galois coincides with that of Galois when K is finite over H as is seen later. In this sense, one may regard the notion of quasi-Galois as a natural generalization of that of Galois.

We shall start our study with §**1**, which contains preliminary lemmas that will play important roles in the subsequent sections. Secondly, in §**2**, we shall see that K/L is quasi-Galois if and only if K/L is locally finite and either the conditions (I′) and (II) or (I′) and (II′) are fulfilled, and prove our principal theorem, which claims that, if K/L is quasi-Galois, then so is K/T for any division subring T containing L such that $[V_K(L): V_K(T)]_r < \infty$. We shall conclude our study with §3, which contains some supplementary remarks to the results cited in [3].

1. Preliminaries. In this section, we always assume that K is a division ring which is locally finite over a division subring L.

Lemma 1. *Let T be an arbitrary subring of H containing L.*

(i) *If the condition* (II) *is satisfied, then $T\mathfrak{G}'(T/L) \subseteq H$.*

(ii) *If the condition*(I′) *is satisfied, then also $T\mathfrak{G}'(T/L) \subseteq H$. In particular, $\mathfrak{G}'(H/L) = \mathfrak{G}(H/L)$.*

Proof. It suffices to prove our lemma for the case $[T: L]_l < \infty$.

(i) Let σ be in $\mathfrak{G}'(T/L)$. If $T\sigma \not\subseteq H$, then there exist a non-zero element v in $V_K(L)$ and an element t in T such that $v^{-1}(t\sigma)v \neq t\sigma$. Since $\sigma^{-1} \in \mathfrak{G}'(T\sigma/L) = \mathfrak{G}'((T\sigma)[v]/L)|T\sigma$ by (II), there holds $\sigma^{-1} = \tau|T\sigma$ for some $\tau \in \mathfrak{G}'((T\sigma)[v]/L)$. Then, $v\tau$ being evidently contained in $V_K(L)$, we have $t = (v\tau)^{-1}t(v\tau) = (v^{-1}(t\sigma)v)\tau \neq t\sigma\tau = t$, which is a contradiction.

(ii) Let N be an intermediate subring of H/T such that N/L is Galois and $[N: L]_l < \infty$ [2, Theorem 3]. Since $\mathrm{Hom}_{L_l}(N, K) = \mathfrak{G}(N/L)K_r$, we obtain $\mathrm{Hom}_{L_l}(T, K) = (\mathfrak{G}(N/L)|T)K_r = \mathfrak{G}'(T/L)K_r$. Accordingly, for each $\sigma \in \mathfrak{G}'(T/L)$, we can find some $\tau \in \mathfrak{G}(N/L)|T$ such that σK_r is $T_r - K_r$-isomorphic to τK_r. And then, one will easily see that $\sigma = \tau\tilde{v}$ with some $v \in V_K(L)$. Recalling here that $T\tau \subseteq N \subseteq H$, we obtain $\sigma = \tau$. Consequently, it follows $T\sigma \subseteq H$. The latter part is a conseduence of [2, Theorem 11].

Corollary 1. *If K/L is quasi-Galois, then the condition* (I′) *is satisfied.*

Proof. Let L' be an arbitrary subring of H containing L such that $[L': L]_l < \infty$. Since $[\mathfrak{G}'(L'/L)K_r: K_r]_r < \infty$, we obtain $\mathfrak{G}'(L'/L)K_r = \sum_{i=1}^{n} \oplus \sigma_i K_r$ with some σ_i′ s in $\mathfrak{G}'(L'/L)$. (One may remark here that each $\sigma_i K_r$ is $L'_r - K_r$-irreducible.) Now, for any $\sigma \in \mathfrak{G}'(L'/L)$, there exists some σ_i such that σK_r is $L'_r - K_r$-isomorphic to $\sigma_i K_r$. And then, we see that $\sigma = \sigma_i \tilde{v}$ with some $v \in V_K(L)$. Since $L'\sigma_i \subseteq H$ by Lemma 1 (i), it follows $\sigma = \sigma_i$. Thus we have proved $\mathfrak{G}'(L'/L)$ is finite (and so is compact). Now let $\mathfrak{G}'(\ni 1)$ be the inverse limit of the system $\{\mathfrak{G}'(L_\alpha/L);\ L_\alpha$ run over all the intermediate subrings of H/L such that $[L_\alpha: L]_l < \infty\}$, which may be regarded as a set of L-(ring) isomorphisms of H into H. Then, since each $\mathfrak{G}'(L_\alpha/L)$ is finite and, by (II), $\mathfrak{G}'(L_\alpha/L)|L_\beta = G'(L_\beta/L)$ for $L_\alpha \supseteq L_\beta$, there holds $\mathfrak{G}'|L_\alpha = \mathfrak{G}'(L_\alpha/L)$ by [1, Corollary 3.9]. In virtue of this fact, we readily see that, for arbitrary $\sigma \in \mathfrak{G}'$ and $x \in H$, there exists some positive integer n such that $x\sigma^n = x$, whence it follows that σ maps H onto H, that is, σ is an automorphism of H. Finally, if H contains an element a not contained in L such that $a\tau = a$ for all $\tau \in \mathfrak{G}'$, then $\mathfrak{G}'(L[a]/L) = \mathfrak{G}' | L[a] = 1$, which contradicts (I). Thus we have proved that H is Galois over L.

Corollary 2. *If K/L is locally Galois, then H/L is Galois.*

Proof. Since K/L is quasi-Galois as is noted in the introduction, our assertion is a direct consequence of Corollary 1.

Lemma 2. *Let L' be an intermediate subring of K/L such that $[L': L]_l < \infty$.*

(i) *If* (I′) *and* (II) *are satisfied, then $Hom_{L_l}(L', K) = \mathfrak{G}'(L'/L)K_r$.*

(ii) *If* (I′) *and* (II′) *are satisfied, then also $Hom_{L_l}(L', K) = \mathfrak{G}'(L'/L)K_r$.*

Proof. Let $U = \{u_1, \cdots, u_n\}$ be a maximal subset of L' which is linearly left-independent over H. Then we can find an intermediate subring N of H/L such that $[N: L]_l < \infty$, N/L is Galois, and $M = \sum_{j=1}^{n} Nu_j \supseteq L'$. Since K/H is Galois, by Jacobson's theorem, there exist some $\xi_k \in \mathfrak{G}(K/H)K_r (k = 1, \cdots, n)$ such that $u_j \xi_k = \delta_{jk}$(Kronecker's δ). Here, we set $\xi_k = \sum_p \sigma_{kp}(a_{kp})_r$ where $\sigma_{kp} \in \mathfrak{G}(K/H)$ and $a_{kp} \in K$. On the other hand, if $\{x_1, \cdots, x_m\}$ is a linearly left-independent L-basis of N, then there exist some $\rho_h \in \mathfrak{G}'(N/L)K_r (h = 1, \cdots, m)$ such that $x_i \rho_h = \delta_{ih}$. We set $T = N[U]$ and $T' = T[(\bigcup_{k,p} T\sigma_{kp}) \cup \{a_{kp}'s\}]$, which are evidently left finite over L. Further, we set $T_0 = H[T]$ and $T^* = H[T']$, which are left finite over H by [4, Theorem 1].

(i) By (II), $\rho_h = \rho'_h | N$ for some $\rho'_h = \sum_q \tau_{hq}(b_{hq})_r \in \mathfrak{G}'(T'/L)K_r$ where $\tau_{hq} \in \mathfrak{G}'(T'/L)$ and $b_{hq} \in K$. Setting here $\gamma_{hk} = (\xi_k | T)\rho'_h$, one will readily see that $\gamma_{kh} = \sum_{p,q} (\sigma_{kp} | T)\tau_{hq}(a_{kp}\tau_{hq})_r(b_{hq})_r$ is contained in $\mathfrak{G}'(T/L)K_r$ and

$$x_i u_j \gamma_{kh} = \begin{cases} 1 \text{ if } j = k \text{ and } i = h, \\ 0 \text{ otherwise.} \end{cases}$$

We obtain therefore $\mathrm{Hom}_{L_l}(M, K) = \mathfrak{G}'(T/L)K_r | M$, whence $\mathrm{Hom}_{L_l}(L', K) = \mathrm{Hom}_{L_l}(M, K) | L' = (\mathfrak{G}'(T/L)K_r) | L' \subseteq \mathfrak{G}'(L'/L)K_r$. Since the converse inclusion $\mathrm{Hom}_{L_l}(L', K) \supseteq \mathfrak{G}'(L'/L)K_r$ is clear, we obtain our assertion.

(ii) By (II′) and Lemma 1 (ii), we can easily see that $\mathfrak{G}'(N/L) = \mathfrak{G}'(T^*/L) | N$. Consequently, $\rho_h = \rho^*_h | N$ for some $\rho^*_h \in \mathfrak{G}'(T^*/L)$. Then, as in the proof of (i), we can see that $\gamma^*_{kh} = (\xi_k | T_0)\rho^*_h$ is contained in $\mathfrak{G}'(T_0/L)K_r$ and

$$x_i u_j \eta^*_{kh} = \begin{cases} 1 & \text{if } j = k \text{ and } i = h, \\ 0 & \text{otherwilse.} \end{cases}$$

Therefore we have eventually $\mathrm{Hom}_{L_l}(L', K) = \mathfrak{G}'(L'/L)K_r$.

2. Quasi-Galois extensions.

In this section, too, we shall assume that K/L is a locally finite division ring extension. To prove our first theorem, further several lemmas will be needed.

Lemma 3. *Let the condition* (I′) *be satisfied, and L_1 and L_2 division subrings of K such that $L_1 \supseteq L_2 \supseteq L$ and $[L_1 : L]_l < \infty$.*

(i) *If* (II) *is satisfied, then* $\mathfrak{G}'(L_1/L_2) \neq 1$.

(ii) *If* (II′) *is satisfied, then also* $\mathfrak{G}'(L_1/L_2) \neq 1$.

Proof. (i) Since $\mathrm{Hom}_{L_l}(L_1, K) = \mathfrak{G}'(L_1/L)K_r$ by Lemma 2 (i), a similar argument as in the proof of [4, Theorem 2] will apply to see that $\mathrm{Hom}_{L_{2l}}(L_1, K) = \mathfrak{G}'(L_1/L) \cap \mathrm{Hom}_{L_{2l}}(L_1, K))\, K_r = \mathfrak{G}'(L_1/L_2) K_r$. Recalling that $[\mathrm{Hom}_{L_{2l}}(L_1, K) : K_r]_r = [L_1 : L_2]_l > 1$, we obtain $\mathfrak{G}'(L_1/L_2) \neq 1$.

(ii) Since $\mathrm{Hom}_{L_l}(L_1, K) = \mathfrak{G}'(L_1/L)K_r$ by Lemma 2 (ii), the proof will proceed just as in (i).

As a direct consequence of Corollary 1 and Lemma 3 (i), it will be easily seen that K/L is quasi-Galois if and only if K/L is locally finite and the conditions (I′) and (II) are satisfied. Moreover, Corollary 1 and Lemma 3 (i) yield at once the first and second assertion of the next

Corollary 3. *If K/L is quasi-Galois and L' a subring of K containing L with $[L' : L]_l < \infty$, then K/L' is quasi-Galois, $H' = V_K(V_K(L'))$ is Galois over L', and $\mathfrak{G}(H'/L')$ is (topologically) isomorphic to $\mathfrak{G}(H/H \cap L')$ by the restriction map.*

Proof. Recalling that $\mathfrak{G}(H'/L')$ is compact and $\mathfrak{G}'(H/L) = \mathfrak{G}(H/L)$, our last assertion will be easily obtained. (Cf. the proof of [2, Lemm 9 (iii)].)

Lemma 4. *Let K/L be quasi-Galois, and L' a division subring containing L with $[L' : L]_l < \infty$. If $H' = V_K(V_K(L'))$, then $\mathfrak{G}'(L'/L) = \mathfrak{G}'(H'/L)|L'$.*

Proof. Since H'/L' is outer Galois by Corollary 3, $H' = \bigcup L'_\alpha$ where L'_α run over all the $G(H'/L')$-normal intermediate subrings of H'/L' such that $[L'_\alpha : L']_l < \infty$ (Cf. [2]). Now let ρ be an arbitrary element of $\mathfrak{G}'(L'/L)$, and denote by $\mathfrak{E}_\alpha$ the set of all the extensions of ρ to L-isomorphisms of L'_α, which is evidently non-empty by (II). And then, we denote by $\mathfrak{E}$ the inverse limit of the system $\{\mathfrak{E}_\alpha\}$. If σ and τ are in $\mathfrak{E}_\alpha$, then $\sigma =$

$\tau\varepsilon$ with some $\varepsilon\in\mathfrak{G}'(L'_{\alpha}\tau/L'\rho)$. But $\mathfrak{G}'(L'_{\alpha}\tau/L'\rho)(\cong\mathfrak{G}'(L'_{\alpha}/L')=\mathfrak{G}(L'_{\alpha}/L'))$ is finite, that is, $\mathfrak{F}_{\alpha}$ is finite. Consequently, the inverse limit $\mathfrak{F}$ is non-empty by [1, Theorem 3.6], and is a subset of $\mathfrak{G}'(H'/L)$.

Now we are at the position to present the following theorem.

Theorem 1. *For a locally finite division ring extension K/L, the following three conditions are equivalent* :

(1) *K/L is quasi-Galois.*

(2) (I′) *and* (II) *are satisfied.*

(3) (I′) *and* (II′) *are satisfied.*

Proof. The equivalence of (1) and (2) was already remarked immediately afer the proof of Lemma 3.

(3) *implies* (2). If L_1 and L_2 are division subrings such that $L_1\supseteq L_2\supseteq L$ and $[L_1:L]_l<\infty$, then $\mathfrak{G}'(L_2/L)K_r=\mathrm{Hom}_{L_l}(L_2,K)=\mathrm{Hom}_{L_l}(L_1,K)|L_2=(\mathfrak{G}'(L_1/L)|L_2)K_r$ by Lemma 2 (ii). In virtue of this fact, it will be easy to see that $\mathfrak{G}'(L_2/L)=\mathfrak{G}'(L_1/L)|L_2$. (Cf. the proof of Lemma 1 (ii).)

(2) *implies* (3). If T is a division subring containing H such that $[T:H]_l<\infty$, then by [4, Lemma 2], $T=H[L']=V_K(V_K(L'))$ with some intermediate subring L' of T/L such that $[L':L]_l<\infty$. Hence, by Lemma 1 (ii) and Corollary 3, we obtain $\mathfrak{G}'(T/L')|H=\mathfrak{G}(T/L')|H=\mathfrak{G}(H/H\cap L')=\mathfrak{G}'(H/H\cap L')$. On the other hand, since $\mathfrak{G}'(T/L)|L'=\mathfrak{G}'(L'/L)$ by Lemma 4, (II) yields $\mathfrak{G}'(T/L)|H\cap L'=\mathfrak{G}'(H\cap L'/L)$. Now, combining these with Lemma 1(ii), one can easily obtain $\mathfrak{G}'(T/L)|H=\mathfrak{G}'(H/L)(=\mathfrak{G}(H/L))$.

From (3) of this theorem, we can conclude that, if K/L is quasi-Galois such that $[K:H]_l<\infty$, then K/L is Galois as was declared in the introduction.

In virtue of the validity of Corollary 3, a similar argument as in the proof of [5, Proposition 2] will apply to obtain the following lemma, whose proof may be left to readers.

Lemma 5. *If K/L is quasi-Galois and T an intermediate subring of H/L, then K/T is locally finite.*

We shall conclude this section with the principal theorem.

Theorem 2. *If K/L is quasi-Galois, then so is K/T for any division subring T containing L such that $[V_K(L):V_K(T)]_r<\infty$.*

Proof. Let L' be an intermediate division subring of T/L such that $[L':L]_l<\infty$ and $V_K(T)=V_K(L')$. Then K/L' is quasi-Galois by Corollary 3. Since $H'=V_K(V_K(L'))\supseteq T\supseteq L'$, K/T is locally finite by Lemma 5. Moreover, noting that $V_K(V_K(T))=H'$, K/T satisfies (II′) by Theorem 1, and H'/T is Galois by [2, Theorem 9]. It follows, again by Theorem 1, K/T is quasi-Galois.

3. Locally Galois extensions.

In this section, we shall consider a locally Galois division ring extension K/L. Our first theorem of this section is the next one, which asserts [3, Theorem 2] is still valid without assuming that K/L is Galois.

Theorem 3. *If K/L is locally Galois, and T a division subring containing L such that $[V_K(L): V_K(T)]_r < \infty$, then K/T is locally Galois.*

Proof. Let L' be an intermediate subring of T/L such that $V_K(L') = V_K(T)$ and $[L': L]_l < \infty$. Then, for an arbitrary finite subset F of K, there exists a division subring L^* containing $L'[F]$ such that L^*/L is Galois and $[L^*: L]_l < \infty$. And then, by [4, Lemma 2], $H^* = V_K(V_K(L^*))$ coincides with $H[L^*]$(which is left finite over H by [4, Theorem 1]) and contains $T[F]$. Since K/L is evidently quasi-Galois, there holds $\mathfrak{G}'(L^*/L) = \mathfrak{G}'(H^*/L)\,|\,L^*$ by Lemma 4. Hence each L-automorphism of L^* can be extended to an isomorphism of H^*. Further, recalling that $\mathfrak{G}'(H/L) = \mathfrak{G}(H/L)$ (Lemma 1 (ii)), one will easily see that the extended isomorphism is in fact an automorphism of $H^* = H[L^*]$. Combining this with the fact that H^*/L^* is Galois by Corollary 3, it will be easily seen that H^*/L is Galois. Moreover, by [3, Lemma 1 (i)], there holds $[V_{H^*}(L): V_{H^*}) H^*)]_r < \infty$, whence it follows that H^*/T is locally Galois by [3, Theorem 1]. Since $H^* \supseteq T[F]$, we have proved our assertion.

Now from the proof of Theorem 3, one will penetrate the necessary part of the following

Corollary 4. *K/L is locally Galois, if and only if K/L is locally finite and, for each finite subset F of K, there exists a division subring T containing $H[F]$ such that $[T: H]_l < \infty$ and T/L is Galois.*

Proof. We shall prove here the sufficiency only. Since $\infty > [T: H]_l \geq [V_T(H): V_T(T)]_r = [V_T(L): V_T(T)]_r$ by [3, Lemma 1], T/L is locally Galois again by [3, Theorem 1].

In [3], Galois theory of division rings was undertaken under the assumpion that the division ring extension K/L considered is Galois and locally Galois and $[K: H]_l \leq \aleph_0$. Our next theorem contains Corollary 1, and enables us to exclude the assumtion cited just now that K/L is Galois.

Theorem 4. *If K/L is locally Galois and $[K: H]_l \leq \aleph_0$, then K/L is Galois.*

Proof. By the validity of Corollary 4, the same argument as in the proof of [3, Lemma 4] will apply to see that $\mathfrak{G}(H/L) = \mathfrak{G}^* | H$, where $\mathfrak{G}^*$ is the group of all L-automorphisms of K. Since $\mathfrak{G}^* \supseteqq V_K(L)$, we obtain eventually $J(\mathfrak{G}^*, K) = L$.

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DEPARTMENTS OF MATHEMATICS,
OSAKA UNIVERSITY,
HOKKAIDO UNIVERSITY

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