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On Hilbert geometry

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ON HILBERT GEOMETRY

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Introduction. A geometry discovered by Hilbert arises from Klein's model of hyperbolic geometry through replacing the ellipse as absolute locus by a convex curve. Following H. Busemann [1][1], in Hilbert geometry a space R is a G-space[2] which is Desarguesian in his sense. In the paper we deal with some conditions for the space R to be non-Euclidean, i. e., hyperbolic.

In [1] we explain some preliminary concepts and show that the space is a Finsler space with convex indicatrices, if the underlying space is Euclidean. Further we show a relation between Hilbert geometry and non-Euclidean geometry. The main purpose of the paper is to prove Theorem (2. 1). Let $\mathfrak{F}$ be the system of all straight lines through a point p. In the theorem we show that, if the space admits a transitive group of motions Γ such that under an element Φ of Γ a straight line of $\mathfrak{F}$ is carried into a straight line of $\mathfrak{F}$, the space is then hyperbolic. We show this in § 2.

H. Busemann [2] proved that, if a compact G-space admits a transitive group of motions such that a geodesic is carried into a geodesic under a motion of the group, the space is spherical, elliptic, hermitian elliptic, quaternion elliptic or the cayley elliptic plane. But it seems difficult to guarantee the above even in Hilbert geometry. Hence our research is limited to the case of Theorem (2. 1). The G-space said in the above admits a pairwise transitive group of motions. H. C. Wang [7] and H. Busemann [1] proved that, if a compact G-space admits a pairwise transitive group of motions, the space has also the above property. A further extension was given by H. Busemann [1], i.e., if a G-space whose dimension is finite and odd (of two) and which possess a pairwise transitive group of motions is Euclidean, hyperbolic, spherical or elliptic. If in Hilbert geometry a space admits a pairwise transitive group of motions, the space is hyperbolic independently on dimension. This is a direct result of Theorem (2. 1).

In § 3 we define an L-space which arises from the Klein's model in hyperbolic geometry and prove that, if an L-space satisfies "l'axiom du libre mobilité" of E. Cartan [4], the space is locally Euclidean, hyperbolic or elliptic.

§ 1. Let $A^n (n \geqq 2)$ be an n-dimensional affine space. A point with coordinates $(x^1, \cdots, x^n)$ is denoted by x. Let K be a convex body with

1) Numbers in brackets refer to the references at the end of the paper.
2) See [1], p. 37.

interior points and an affine line $\mathfrak{x}$ intersect K at two distinct points p and q. Further let a and b be two points on the affine segment pq and represented as

$$a^i = (1-\tau)p^i + \tau q^i \text{ and } b^i = (1-\tau')p^i + \tau' q^i$$
$$(\tau \neq \tau',\ 0 < \tau,\ \tau' < 1,\ i = 1, \cdots, n).$$

Then the distancs $\rho(a, b)$ between these points is defined as

(1. 1). $$\rho(a, b) = \mathrm{k} \log \left| \frac{1-\tau}{1-\tau'} \frac{\tau'}{\tau} \right|,$$

where k is a positive number. Under such a metrization the segment which is cut off from an affine line by K is a geodesic. Such a geodesic is called a straight line. If K does not contain the pair of coplaner segments, for any two points x and y the segment from x to y (or from y to x) is a unique geodesic which connects these points. In the paper we assume conveniently the surface K is strictly convex, and the segment from p to q is simply denoted by pq. When we need the orientation of pq, we denote it by $\overrightarrow{pq}$ (or by $\overrightarrow{qp}$). Under the above assumption with respect to K a sphere $S(a, \tau)$ $(=\{x \mid \rho(a, x) < \tau\},\ \tau > 0)$ is strictly convex. The space defined above is called an n-dimensional H-space and will be denoted by R^n. The affine space A^n is considered as a Euclidean metrization. We assume that $(x^1, \cdots, x^n)$ are rectangular coordinates. Let x be a point of R^n and $\{a_\nu\}$ and $\{b_\nu\}$ be two sequences of points which con verges to the point x and such that the sequence of Euclidean unit vectors $\{\lambda_\nu\}$ converges to a Euclidean unit vector λ where λ_ν is the Euclidean unit vector of each segment $\overrightarrow{a_\nu b_\nu}$. Then $\lim_{\nu\to\infty} \mathrm{e}(a_\nu, b_\nu)/\rho(a_\nu, b_\nu)$ exists and is given as follows :

$$\lim_{\nu\to\infty} \frac{\mathrm{e}(a_\nu, b_\nu)}{\rho(a_\nu, b_\nu)} = \frac{\mathrm{k}\mathrm{e}(p, x)\mathrm{e}(x, q)}{\mathrm{e}(p, x) + \mathrm{e}(x, q)},$$

where p and q are the points of the intersection of K with a straight line through x with the direction λ, $\mathrm{e}(a_\nu, b_\nu)$ are the Euclidean distances between the points a_ν and b_ν and so on. We put

$$\mathrm{e}(x, q) = \mathrm{r}(x, \lambda),\ \mathrm{e}(x, p) = \mathrm{r}(x, -\lambda)$$

and if further we put

(1. 2). $$\mathrm{G}(x, \lambda) = \frac{\mathrm{k}\mathrm{r}(x, \lambda)\mathrm{r}(x, -\lambda)}{\mathrm{r}(x, \lambda) + \mathrm{r}(x, -\lambda)},$$

the equation $\mathrm{r} = \mathrm{G}(x, \lambda)$ is considered as a representation of the indicatrix γ_x at the point x by polar coordinates.

Let C be a curve of class D' and $x^i = x^i(\mathrm{t})$, $\alpha \leqq \mathrm{t} \leqq \beta$, $(i=1, \cdots, n)$ its parametrization. Then, by putting

(1. 3). $$F(x,\ \dot{x})=\frac{1}{k}\left\{\frac{1}{r(x,\ \dot{x}/|\dot{x}|)}+\frac{1}{r(x,\ -\dot{x}/|\dot{x}|)}\right\}|\dot{x}|$$

the length $l(C)$ of C is defined by integral as usual, where $\dot{x}=(dx^1/dt,\cdots,dx^n/dt)$ and $|\dot{x}|=\sqrt{\sum_{i=1}^n(\dot{x}^i)^2}$. The following is clear from the above.

(1. 4). *If the underlying spase A^n is Euclidean and the convex surface K is of class $C^r(r\geqq 0)$, the space R^n is a Finsler space of class C^r with convex indicatrices.*

If $n=2$ and K is an ellipse, K is transformed to the unit circle under a suitable affine transformation. Hence we assume K is the unit circle. Then by simple calculation the fundamental tensor g_{ij} is given as

$$g_{11}=\frac{4(1-y^2)}{k(1-x^2-y^2)^2},\quad g_{12}=g_{21}=\frac{8xy}{k(1-x^2-y^2)^2}$$
$$\text{and } g_{22}=\frac{4(1-x^2)}{k(1-x^2-y^2)^2}\ .$$

Such a space is hyperbolic with constant curvature $-1/4k^2$. The angle between two directions evaluated by use of the above tensor is identical with the pseudo-angle between these directions.

§ 2. In this paragraph we prove the following

Theorem (2. 1). *Let R^n be an n-dimensional* H*-space* $(n\geqq 2)$, *p a point of R^n and $\mathfrak{F}$ the system of all straight lines through p. If R^n admits a transitive group of motions Γ such that a straight line of $\mathfrak{F}$ is carried into a straight line of $\mathfrak{F}$ under an element Φ of Γ, then the space is hyperbolic.*

To prove the theorem let A^n be the underlying affine space of R^n and the space R^n defined by use of a strictly convex hypersurface K as absolute figure. We show some propositions and lemma firstly. The proof of the theorem will be given lastly.

(2. 2). *An r-flat B_r* $(0<r\leqq n-1)$ *is carried into an r-flat B'_r under a motion of the group Γ.*

Since the proposition is clear, the proof is omitted.

In the affine space A^n every point x has neighborhood

$$V:\ |x^i-y^i|<\varepsilon\ \ (i=1,\cdots,n),$$

where ε is a positive number. The topology of the space R^n coincides with that of the original affine space. Let ab be a straight line of R^n where a and b are the points of the intersection of K with the affine line ab and Φ an element of Γ. If ab is carried into a straight line cd under Φ, by extending the definite range of Φ to the convex hypersurface K, we can put $a\Phi=c$ (or d) and $b\Phi=d$ (or c) in such a way that for two distinct points

x and y on ab the order of thc four points a, b, x, y coincides with that of c, d, x, y (or d, c, x, y). Under the motion Φ the double ratio of these points is of course invariant. Hence every element of Γ is represented as a projective transformation by use of plane at infinity $x_{n+1}=0$.

It is also easy to show that, if $\overrightarrow{ab}$ is a straight line of R^n where $a, b \in K$, then there exists a motion Φ of Γ under which the straight line $\overrightarrow{ab}$ is carried into $\overrightarrow{ba}$.

(2. 3). *The convex hypersurface K is differentiable.*

Proof. Since K is convex, K is almost everywhere differentiable. If K is not differentiable at a point a, then the tangent cone at a is not a hyperplane. Every element of Γ carries a supporting plane into a supporting plane and a tangent line also into a tangent line. Since the group is transitive, K has the tangent cone at every point of K which is not a hyperplane. But this contradicts the above, from which the proposition follows.

Next we prove the following

Lemma (2. 4). *Let the space be a 2-dimensional* H-*space and aa' the straight line of $\mathfrak{R}$ where $a, a' \in K$. Further let A and A' be the tangent lines of K at the points a and a' respectively and p' the point which lies on the affine line through a and a' and such that $(aa', pp') = -1$. Then, by choosing a coordinate system such that the affine line through the points $A \cap A'$ and p' coincides with line at infinity, every element of Γ is represented as an affine transformation.*

Proof. Let P be an affine line which passes through $A \cap A'$ and the point p and P' the line at infinity. Since $(aa', pp') = -1$, the point p is the affine center of a and a'. Since the group Γ is transitive, there exists a motion Φ of Γ under which the straight line $\overrightarrow{aa'}$ is carried into $\overrightarrow{a'a}$. In the case we can put $a\Phi = a'$ and $a'\Phi = a$. Then the straight lines A and A' are carried into A' and A respectively. A and A' are parallel to each other and the point p is fixed under Φ. Hence the motion Φ is represented as an affine transformation. Since a point of P is not necessarily invariant under Φ, we can consider the following two cases.

i) $\Phi = E$ (identity) on P.

Let $\mathfrak{x}$ be an affine line through the points a and a' and let x and x' be two points on $\mathfrak{x}$ with p as their affine center. Then it is easy to see $x\Phi = x'$ and $x'\Phi = x$. Let P_x and $P_{x'}$ be the affine lines through x, $A \cap A'$ and x', $A \cap A'$ respectively. The affine lines P_x and $P_{x'}$ are carried into $P_{x'}$ and P_x under Φ respectively. Let y be a point of P_x and y' the point of the intersection of $P_{x'}$ with the affine line parallel to $\mathfrak{x}$ through the point y.

$y'\Phi = y$ and $y\Phi = y'$. It follows from this that the motion Φ is an involution.

ii) A point on P is not necessarily invariant under Φ.

Let b and b' be the points of $P \cap K$. If the points b and b' are fixed under Φ, then every point on P is also fixed under Φ and hence the case i) holds. We see from this $b\Phi = b'$ and $b'\Phi = b$. Let z and z' be two points on P with p as their affine center, then $z\Phi = z'$ and $z'\Phi = z$. Let y be a point on P_x and z the point at which the affine line parallel to $\mathfrak{x}$ through y intersects P. Then the point $y\Phi$ coincides with the point y' at which the affine line parallel to $\mathfrak{x}$ through the point z' intersects $P_{x'}$. Further it is also easy to see $y'\Phi = y$. Thus we see that Φ is also an involution.

From the above we see that, if the case ii) holde, K has p as its affine center. Therefore the tangent lines of K at points which are antipodal with respect to p are parallel to each other. Thus every element of Γ is represented as an affine transformation under which the point p is fixed. Hence the convex curve K is an ellipse [1], [5]. It remains to show that, if the case i) holds, the convex curve K has p as its affine center. Next we show this.

Under the assumption let u and u' be the points of $P_x \cap K$. Then $u\Phi$ and $u'\Phi$ coincide with the points of $P_{x'} \cap K$. We denote these points by v and v' respectively. Then we have $u\Phi = v$, $v\Phi = u$, $u'\Phi = v'$ and $v'\Phi = u'$. The points u, u', v' and v are the vertices of a parallel quadrangle. Let w and w' be the points at which the affine lines uv and $u'v'$ intersect P respectively. It is easy to show that the tangent lines of K at b and b' are parallel to each other and also to the affine lines uv and $u'v'$. Let Ψ be a motion of Γ under which the straight line aa' is carried into bb'. In the case we put $a\Psi = b$ and $a'\Psi = b'$ as before. Then the tangent lines A and A' are carried into the tangent lines B and B' of K at the points b and b' and the affine line P into $\mathfrak{x}$. We see from this that the motion Ψ is also represented as an affine transformation.

From the above there exists a motion Φ' under which the straight line $\overrightarrow{bb'}$ is carried into $\overrightarrow{b'b}$ and such that $b\Phi' = b'$ and $b'\Phi' = b$. Obviously the motion Φ' can clearly be chosen so as to be the reflexion of the space with respect to the affine line $\mathfrak{x}$ by use of the direction parallel to P. Then the affine lines uv and $u'v'$ are carried into $u'v'$ and uv respectively. Hence p is the affine center of the points w and w' and further the points u and v' has also p as their affine center. The same holds for the points u' and v. From this we see again that, if the case i) holds, p is the affine center of K. Thus the proposition is proved.

Let R be a G-space in H. Busemann's sense. In R "l'axiom du libre mobilité" of E. Cartan [4] is stated as follows:

Every point p has a neighborhood $S(p, \rho_p)$ ($\rho_p > 0$) such that, if for four points x, y, x' and y' of $S(p, \rho_p)$, $\rho(p, x) = \rho(p, x')$, $\rho(p, y) = \rho(p, y')$ and $\rho(x, y) = \rho(x', y')$ hold good, then there exists a motion of $S(p, \rho_p)$ under which the points x, p ahd y are carried into x', p and y' respectively.

If the space satisfies the above axiom, then the space is a Riemann space of constant curvature. Since a Riemann space or a Finsler space is a G-space under a suitable assumption of differentiabity, the same holds for a Rinmann space or a Finsler space. But in Hilbert geometry the space R^n is hyperbolic under more weak condition than the above, i. e.,

Theorem (2. 5). *Let R^n be an n-dimensional* H*-space and p a point of R^n. If the space R^n permits rotation about p, i.e., if for four points x, y, x' and y', $\rho(p, x) = \rho(p, x')$, $\rho(p, y) = \rho(p, y')$ and $\rho(x, y) = \rho(x', y')$ hold, then there exists a motion under which the points x, p and y are carried into the points x', p and y' respectively, the space is hyperbolic.*

Proof. We use the same notation as before. Let H be a 2-flat through p. Then it is easy to see that there exists a group of motions with p as fixed point under which $H \cap K$ is carried into itself and which is transitive on $H \cap K$. In virtue of Lemma (2. 4) $H \cap K$ has p as its center. As we said in the proof of Lemma (2. 4) $H \cap K$ is an ellipse from which the theorem follows.

To prove Theorem (2. 1) in the n-dimensional case we use again the same notations as in the proof of Lemma (2. 4). In the case, A and A' are hyperplanes tangent to K at the points a and a' respectively and P is hyperplane through p parallel to A and A'. The hyperplane P' coincides with the plane at infinity. The motion Φ which carries the straight line $\overrightarrow{aa'}$ into $\overrightarrow{a'a}$ keeps the hyperplane P' fixed. We can consider the following three cases, since the motion Φ carries the hyperplane P into itself.

i) On P the motion Φ is identity.

ii) $\{\Phi^k\}$ $(k = 0, \pm 1, \pm 2, \cdots)$ is an infinite group on P.

iii) $\{\Phi^k\}$ $(k = 0, \pm 1, \pm 2, \cdots)$ is a cyclic group of finite order on P.

Next we show that the case ii) is reduced to the case i) and the case iii) to the case i) or the case where Φ is an involution, i. e., $\Phi^2 = E$ (identity).

Let y be a point of $K \cap P_x$ and $\mathfrak{l}$ the affine line through y parallel to the affine line $\mathfrak{x}$. Let z be the point of the intersection of $\mathfrak{l}$ with P, y' the point on $\mathfrak{l}$ such that z is the affine center of y and y' and y'' the point of the intersection of $P_{x'}$ with the affine line $\mathfrak{l}'$ through the point $z\Phi (\in P)$ parallel to the affine line $\mathfrak{x}$. Then $y\Phi = y''$ and $y'' \in K \cap P_{x'}$. To prove the above we prove firstly the following

(2. 6). *If* $\{\Phi^k\}$ $(k=0, \pm 1, \pm 2, \cdots)$ *is an infinite cyclic group, there exists an element* Ψ *of the group* Γ *such that* $a\Psi = a'$, $a'\Psi = a$ *and* $\Psi = E$ *on* P.

Proof. Let V be a relatively open set of y on $K \cap P_x$. Then the set V is carried into a subset V' of $P_{x'}$ under the reflexion with respect to P and of the direction parallel to $\mathfrak{x}$ and $V\Phi$ also a relatively open set of y'' on $K \cap P_{x'}$. If the set V' is also such an open set on $K \cap P_{x'}$, the proposition is proved. Suppose that this doss not hold and V is such a maximal open set. Then the open set V does not coincide with $K \cap P_x$. Since each of the motions $\Phi^{2k}(k=0, 1, 2, \cdots)$ carries $K \cap P_x$ into itself, the sets $V\Phi^{2k}(k=0, 1, 2, \cdots)$ are open sets on $K \cap P_x$. The set $K \cap P_x$ is compact and hence there exist two open sets $V\Phi^{2k_1}$ and $V\Phi^{2k_2}$ $(k_1 < k_2)$ which have a common point. Then the sets V and $V\Phi^{2(k_2-k_1)}$ have also a common point which contradicts that V is maximal, since $V \subsetneqq V \cap V\Phi^{2(k_2-k_1)}$. Let Ψ be the reflexion with respect to the hyperplane P and of the direction parallel fo $\mathfrak{x}$. It follows from the above that the set V is carried into a set of $K \cap P'_x$ under Ψ. Obviously the reflexion Ψ is a motion and may be supposed to be an element of Γ. If we replace the motion Φ by Ψ, then Ψ is a desired motion in the proposition.

The case iii) is divided into the following three cases.

a) $m \equiv 1$ or -1 mod 4, b) $m \equiv 2$ mod 4, c) $m \equiv 0$ mod 4,
where m is the order of the group $\{\Phi^k\}$ $(k=0, \pm 1, \pm 2, \cdots)$. As can easily be seen from the fact mentioned above, the case a) is reduced to the case i). If the case c) holds, we put $\Psi = \Phi^k$ $(m=2k)$. Then Ψ is also an involution which leaves the points a and a' fixed. But, as shown in the latter, the theorem is proved in the same way as in the case b). Thus we see that, if the theorem is proved in the two cases : $\Phi = E$ on P and $\Phi^2 = E$ on P, the proof is complete. Next we prove the following

(2. 7). *Let* Z *be the* $(n-1)$*-dimensional tangential cylinder of* K *whose generating lines are parallel to* $\mathfrak{x}$. *Then* $K \cap Z$ *is identical with* $K \cap P$.

Proof. If $\Phi = E$ on P, the tangent line at each point of $K \cap P$ is clearly parallel to $\mathfrak{x}$. Since the convex surface K is strictly convex, the set $K \cap Z$ is identical with $K \cap P$. Next we consider the case where $\Phi^2 = E$ on P.

Since Φ is an involution, it is easy to see that there exist on P linearly independent $(n-1)$ affine lines such that any of these lines is carried into itself under Φ. We denote these by $\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_r ; \mathfrak{x}_{r+1}, \cdots, \mathfrak{x}_{n-1}$ where Φ coincides on each $\mathfrak{x}_i$ $(i=1, \cdots, r)$ with the reflexion with respect to the point p and on each $\mathfrak{x}_i$ $(i=r+1, \cdots, n-1)$ with the identity. Now we consider the case where $a\Phi = a'$ and $a'\Phi = a$.

If $r = n-1$, the motion Φ coincides with the reflexion with respect to the point p. Hence the surface K has p as its affine center. The supporting plane of K at the end points of a chord through p are parallel to each other. It follows from this that each motion of the group Γ is an affine transformation. Under the assumption the group Γ is transitive and hence the surface K is an ellipsoid [1]. The proposition holds in the case and the theorem is also proved.

Suppose now $r=n-2$. Let b and b' be the points at which the affine line $\mathfrak{x}_{n-1}$ intersects K. Then the supporting planes B and B' of K at b and b' are parallel to each other and each of these supporting planes is carried into itself under Φ. Hence the hyperplane Q through p parallel to these supporting planes is also carried into itself. Obviously the hyperplane Q contains the affine lines $\mathfrak{x}$, $\mathfrak{x}_1$, $\cdots$ and $\mathfrak{x}_{n-2}$. If a point z is on $Q\cap K$, then the points z and $z\Phi$ have p as their affine center.

Let C and C' be the supporting planes of K at z and z' $(=z\Phi)$ respectively and Q' and Q'' the hyperplanes through z and z' parallel to P respectively. Further let x and x' be the points at which the affine line $\mathfrak{x}$ intersects Q' and Q'' respectively. Then p coincides with the affine center of the points x and x'. Hence we can put $Q'=P_x$ and $Q''=P_{x'}$. It is easy to see that under Φ the set $K\cap P_x$ is carried into $K\cap P_{x'}$. Since $K\cap Q$ has p as its affine center, the hyperplanes C and C' are parallel to each other.

If the affine line $\mathfrak{y}$ through the points b and b' intersects the hyperplane C at a point c, then the hyperplane C' also intersects $\mathfrak{y}$ at c, since one of the hyperplanes C and C' is carried into the other and each point of $\mathfrak{y}$ is fixed under Φ. It follows from this that the affine line $\mathfrak{y}$ is parallel to C and C'. Thus we see that the intersection of the surface K with the tangential $(n-1)$-cylinder whose generating lines are parallel to $\mathfrak{y}$ coincides with $K\cap Q$. Since there exists a motion Ψ of Γ under which b and b' are carried into a and a' respectively and which is an affine transformation, the points b and b' have p as their affine center and $K\cap P$ coincides with the intersection of K with the cylinder Z. For the hyperplane Q is carried into P and the affine line $\mathfrak{y}$ into $\mathfrak{x}$ under Φ.

Suppose finally $r\leqq n-3$. To prove the proposition suppose further that $K\cap P$ does not coincide with $K\cap Z$. Let x be a point of $K\cap Z$ but $x\notin K\cap P$. The system of half affine lines joining p to the points of $K\cap Z$ forms a cone which is not a hyperplane. Let z be the point at which the 2-flat determined by x and $\mathfrak{x}$ intersects $P\cap K$. Then the point z has a neighborhood

$$V: |z^i-y^i|<\varepsilon\,(i=1, \cdots, n)$$

which is disjoint from the cone.

The neighborhood V is divided into two open sets by a hyperplane D through x which contains the affine line $\mathfrak{x}$. Let H be the 2-flat spanned by the affine lines $\mathfrak{x}_{n-2}$ and $\mathfrak{x}_{n-1}$. Further let y be the point of V such that $y \in K \cap H \cap D$. Since each point of H is fixed under Φ, each point of the affine line through p and y is fixed under Φ. The surface K has the tangent line at y parallel to $\mathfrak{x}$. But the point y does not belong to $K \cap Z$ which is a contradiction.

If $\Phi^2 = E$ on P, $a\Phi = a$ and $a'\Phi = a'$, then there exists on P an affine line $\mathfrak{x}'$ which is carried into itself and such that the affine lines $\mathfrak{x}'$ and $\mathfrak{x}'\Phi$ have opposite orientations. Let b and b' be the points at which the affine line $\mathfrak{x}'$ intersects K. Then $b\Phi = b'$ and $b'\Phi = b$. If we replace the points a and a' by b and b' respectively, the same arguments as in the above are applicable to this case, i. e., the proposition also holds in this case. Thus we end the proof.

Under the above preparation Theorem (2. 1) is proved as follows :

$K \cap P$ has p as its affine center. Hence on P $K \cap P$ has the parallel $(n-2)$-flats tangent at the end points x and x' of a chord of $K \cap P$ through p. It follows from Proposition (2. 7) that the supporting planes at x and x' are parallel to each other. It is easy to show that there exists a motion of Γ such that the points a and a' are carried into x and x' respectively and which is an affine transformation. Hence $K \cap P$ is an $(n-2)$-dimensienal ellipsoid. We see from this that the intersection of a hyperplane through p with the surface K is an $(n-2)$-dimensional ellipsoid. The theorem follows from this.

Following H. Busemann [1] and W. C. Wang [7], we say that an n-dimensional H-space R^n admits a pairwise transitive group of motions, if the space has the following properties :

Let a, b, a' and b' be points such that $\rho(a, b) = \rho(a', b')$. Then there exists a motion under which the points a and b are carried into a' and b' respectively.

The following theorem is clear from the above proof.

Theorem (2. 8). *If an n-dimensional* H-*space* R^n *admits a pairwise transitive group of motions, the space is hyperbolic.*

It follows from Proposition (2. 7) that the relation between perpendicularity and transversality is symmetric at the point p. Under the assumption of the theorem this relation holds at every point. The theorem is also clear from the result of P. J. Kelly and L. J. Paige [6].

§ **3.** As we mentioned in the preceding paragraph, if the surface K is not strictly convex but convex, a geodesic arc from p to q is not necessarily unique but there exists a system of geodesics which covers simply the space except the point p. An L-space arises from such a metric space, i. e., we define an L-space R' as follows:

A. R is a metric space with distance $\rho(x, y)$ $(=\rho(y, x))$.

B. R is finitely compact, i. e., a bounded infinite set has an accumulation point in R.

C. R is of convex metric, i. e., for any two distinct points a and b there exists a point c different from a and b such that $\rho(a, c)+\rho(c, b)=\rho(a, b)$.

Under the above axioms A, B and C for any two points p and q there exists a shortest connection from p to q (or from q to p). Such an arc is said a geodesic arc.

D. Every point p has a neighborhood $S(p, \rho_p)$ $(\rho_p>0)$ such that there exists a system of geodesic arcs which covers simply $S(p, \rho_p)-p$.

If the space R satisfies the above axioms A, B, C and D, then the space is said an L-space.

Theorem (3. 1). *Let R be an* L-*space. If R satisfies "l'axiom du libre mobilité", the spase is locally elliptic, hyperbolic, or Euclidean. The universal covering space of R is spherical, hyperbolic or Euclidean.*

To prove the theorem it is sufficient to prove the first part, since the latter part is a direct result of the first part. To do this we define a positive number τ_p as follows:

$$\tau_p = \inf_{x\in S(p,\rho_p/2)} \rho_x/2.$$

Then the number τ_p is positive. We show that for a point q of $S(p, \tau_p/2)$ there exists a unique geodesic arc from p to q and the prolongation of such a geodesic arc is locally possible and unique. Then the space R is a G-space [1] and hence the theorem follows.

We put $\rho(p, q)=\varepsilon$. Let γ_1 and γ_2 be positive numbers such that $\gamma_1+\gamma_2>\varepsilon$ and $0<\gamma_1, \gamma_2\leqq\varepsilon$. We further put $K(x, \gamma)=\{y\,|\,\rho(x, y)=\gamma\}$ $(\gamma>0)$. Then $K(p, \gamma_1)\cap K(q, \gamma_2)\neq\phi$. To simply the notations we put $K(p, \gamma_1)=K_1$ and $K(q, \gamma_2)=K_2$. Then $S(p, \gamma_1)\cap K_2$ is an open set on K_2 with $K_1\cap K_2$ as its boundary. Let z be a point of K_2 which does not belong to $\overline{S(p, \gamma_1)}$ and x a point of $S(p, \gamma_1)\cap K_2$. Then an arc on K_2 from z to x has a common point with $K_1\cap K_2$. Let $\mathfrak{F}_p$ be a system of geodesic arcs which covers simply $S(p, \rho_p)-p$. Then the system of geodesic arcs of $\mathfrak{F}_p$ which connect p to the points of $K_1\cap K_2$ divides the interior into domains more than two. Let y be a point of the intersection of K_2 with a geodesic arc from p to q. Then y is an interior point of $K_1\cap K_2$ on K_2. We prove the theorem by proving some propositions.

(3. 2). *The set $K_1\cap K_2$ is a connected closed set on K_2.*

Proof. Let O be an open set on K_2 enclosed by the set $K_1\cap K_2$ and L

its boundary. Obviously $L \subset K_1 \cap K_2$ and we can suppose that the set O does not contain a point of $K_1 \cap K_2$. We further assume that O is the maximal connected open set. Then the set O is clearly contained in $S(p, \gamma_1)$. The boundary L of O is connected. For if L has components L_1 and L_2, the open set enclosed by one of these contains the other. Suppose that L_1 is such a component, i. e., the open set O' enclosed by L_1 contains L_2. Let x and y be points of L_1 and L_2 respectively. The $\rho(p, x) = \rho(p, y) = \gamma_1$ and $\rho(q, x) = \rho(q, y) = \gamma_2$. Hence there exists a motion Φ under which the points p, x and q are carried into p, y and q respectively. The sets K_1, K_2 and $K_1 \cap K_2$ are carried onto themselves under Φ. But L_1 cannot be carried onto L_2 nnder Φ which is a contradiction.

We prove that $K_1 \cap K_2$ consists of only one component. Suppose this is not true. Let M_1 and M_2 be two different components of $K_1 \cap K_2$. Then M_1 and M_2 enclose open sets O_1 and O_2 respectively. These open sets are both simply connected. Let γ'_1 be a positive number such that $\gamma_1 < \gamma'_1 < \gamma_2 + \varepsilon$. Then there exist on K_2 open sets O'_1 and O'_2 whose boundaries M'_1 and M'_2 are contained in $K(p, \gamma'_1) \cap K_2$ and such that $O_1 \subset O'_1$ and $O_2 \subset O'_2$. If the number γ'_1 is sufficiently near γ_1, then $M'_1 \cap M'_2 = \phi$. Hence there exists a positive number γ'_1 such that $O'_1 \cap O'_2 = \phi$ but $M'_1 \cap M'_2 \neq \phi$. Let x' and y' be two distinct points of M'_1. Then $\rho(p, x') = \rho(p, y') = \gamma'_1$ aud $\rho(q, x') = \rho(q, y') = \gamma_2$. Hence there exists a motion Ψ under which the points p, x' and q are carried into p, y' and q respectively. From the fact mentioned above there exists on K_2 an open set O''_1 such that $O''_1 \subset S(p, \gamma'_1)$ and $O''_1 \supset O'_1$ and which does not contain a point of $K(p, \gamma'_1) \cap K_2$. But this is impossible. Thus the proposition is proved.

From the above proposition we see that $K_1 \cap K_2$ encloses on K_2 an open set which is contained in $S(p, \gamma_1)$.

(3. 3). *The set $K_1 \cap K_2$ coincides with the intersection of K_1 with a sphere S whose center lies on K_2.*

Proof. Let $S(x, \gamma)$ be a sphere whose center x is on K_2 and which contains $K_1 \cap K_2$. We denote by δ the lower bound of such number's γ. The number δ is clearly positive. Since the sphere K_2 is compact, there exists a point x_0 such that $S(x_0, \delta)$ contains the interior O of $K_1 \cap K_2$. We show that $K(x_0, \delta) \cap K_2$ coincides with $K_1 \cap K_2$.

Suppose that this is not the case. There exists a sphere $S(y_0, \delta)$ distinct from $S(x_0, \delta)$ whose center y_0 lies on K_2 and such that $S(y_0, \delta) \cap K_2$ contains the open set O. In this case $K(x_0, \delta) \cap K(y_0, \delta) \cap K_2$ does not contain a point of $K_1 \cap K_2$. For if so, $K_1 \cap K_2$ is contained in the intersection of two such spheres whose centers lie on K_2 and with radii δ. It is easy to show that there exist a positive number δ' smaller than δ and a point x of K_2

such that $S(x, \delta') \cap K_2 \supset O$. But this contradicts the definition of thə number δ. On the other hand, if $K(x_0, \delta) \cap K(y_0, \delta) \cap K_2$ does not contain a point of $K_1 \cap K_2$, there exist a positive number δ' and a point x such as in the above, which is also a contradiction. The proposition follows from this.

We see from the above proposition that, if γ_1 is a positive number such that $\gamma_1 + \gamma_2 = \varepsilon$, then $K_1 \cap K_2$ coincides with a point. Thus it follows hat there exists a unique geodesic arc from p to q.

(3. 4). *The prolongation of a geodesic arc issuing from the point p is locally possible and unique.*

The proposition is obvious. Thus we see that the space is a G-space from which the theorem follows.

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