

Asymptotic convergence analysis of the proximal point algorithm for metrically regular mappings

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Abstract—This paper studies convergence properties of the proximal point algorithm when applied to a certain class of nonmonotone set-valued mappings. We consider an algorithm for solving an inclusion $0 \in T(x)$, where T is a metrically regular set-valued mapping acting from $\mathbb{R}^n$ into $\mathbb{R}^m$. The algorithm is given by the following iteration: $x_0 \in \mathbb{R}^n$ and

$$x_{k+1} = \alpha_k x_k + (1 - \alpha_k) y_k, \quad \text{for } k = 0, 1, 2, \dots,$$

where $\{\alpha_k\}$ is a sequence in $[0, 1]$ such that $\alpha_k \leq \bar{\alpha} < 1$, g_k is a Lipschitz mapping from $\mathbb{R}^n$ into $\mathbb{R}^m$ and y_k satisfies the following inclusion

$$0 \in g_k(y_k) - g_k(x_k) + T(y_k).$$

We prove that if the modulus of regularity of T is sufficiently small then the sequence generated by our algorithm converges to a solution to $0 \in T(x)$.

I. INTRODUCTION

We deal in this paper with methods for finding zeroes of set-valued mappings in Euclidean spaces, i.e., given Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ and a set-valued mapping $T : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^m}$, we study the convergence of iterative method for solving the inclusion

$$0 \in T(x). \quad (I.1)$$

Our study is devoted to metrically regular mappings, and we present an algorithm to solve (I.1), which is constructed on the basis of the classical proximal point algorithm [15]. The proximal point was first proposed by Martinet [12] and attained its current formulation in the works of Rockafellar [15], where its connection with the augmented Lagrangian method for constrained nonlinear optimization. In particular, Rockafellar studied the proximal point algorithm for the case when H is a Hilbert space and T is a monotone set-valued mapping from H into itself and showed that when x_{k+1} is an approximate solution of the following proximal point iteration, i.e.,

$$0 \in \mu_k(x_{k+1} - x_k) + T(x_{k+1}) \quad \text{for } k = 0, 1, 2, \dots, \quad (I.2)$$

and T is maximal monotone, then for a sequence of positive scalars μ_k the iteration (I.2) produces a sequence x_k that is

convergent to a solution to $0 \in T(x)$ for any starting point $x_0 \in H$. When T is monotone, i.e.

$$\langle x - y, u - v \rangle \geq 0,$$

for all $x, y \in H$, all $u \in T(x)$ and all $v \in T(y)$, and furthermore maximal monotone, i.e. $T = T'$ whenever $T' : H \rightarrow 2^H$ is monotone and $T(x) \subset T'(x)$ for all $x \in H$, it follows from Minty's theorem (see [13]) that $(I + \gamma T)$ is onto and $(I + \gamma T)^{-1}$ is single valued for all positive $\gamma \in \mathbb{R}$, so that the sequence defined by (I.2) is well defined.

In the past three decades, a number of authors have considered generalizations and modifications of the proximal point algorithm and have also found applications of this method to specific variational problems (see, for examples, [3], [14], [9], [16], [8], [10], [11], [1], [2]). In particular, the convergence to a zero point of a maximal monotone set-valued mapping T of the sequence

$$x_{k+1} = \alpha_k x_k + (1 - \alpha_k)(I + \gamma_k T)^{-1} x_k \quad \text{for } k = 0, 1, 2, \dots, \quad (I.3)$$

was observed by Eckstein and Bertsekas [3] (see also [9]), who showed that the sequence $\{x_k\}$ generated by (I.3) converges weakly to a solution $0 \in T(x)$ in the case that $\inf_k \alpha_k > -1$, $\sup_k \alpha_k < 1$ and $\inf_k \gamma_k > 0$.

On the other hand, the situation becomes considerably more complicated when T fails to be monotone. A new approach to the subject was taken in [14], which deals with a class of nonmonotone mappings that, when restricted to a neighborhood of the solution set, are not far from being monotone. More recently, Aragón, Donchev and Geoffroy [1] considered the following proximal point algorithm for a certain class of a nonmonotone set-valued mappings.

$$0 \in g_k(x_{k+1} - x_k) + T(x_k), \quad \text{for } k = 0, 1, 2, \dots, \quad (I.4)$$

where g_k is a sequence of functions. They proved that if $\bar{x}$ is a solution of (I.1) and the mapping T is metrically regular at $\bar{x}$ for 0 with locally closed graph near $(\bar{x}, 0)$, then there exists a neighborhood O of $\bar{x}$ such that for each initial point $x_0 \in O$ one can find a sequence x_k satisfying (I.4) that is convergent to $\bar{x}$.

In this paper, motivated by (I.3) and (I.4), we will consider the following algorithm for finding zeroes of a metrically regular set-valued mapping. Given $x_0 \in \mathbb{R}^n$, find x_k such that

$$x_{k+1} = \alpha_k x_k + (1 - \alpha_k) y_k, \quad \text{for } k = 0, 1, 2, \dots, \quad (\text{I.5})$$

where $\{\alpha_k\}$ is a sequence in $[0, 1]$ such that $\alpha_k \leq \bar{\alpha} < 1$, g_k is a sequence of Lipschitz mappings and y_k satisfies the following inclusion

$$0 \in g_k(y_k) - g_k(x_k) + T(y_k). \quad (\text{I.6})$$

We show that if $\bar{x}$ is a solution of (I.1) and the mapping T is metrically regular at $\bar{x}$ for 0 with locally closed graph near $(\bar{x}, 0)$, then there exists a neighborhood O of $\bar{x}$ such that for each initial point $x_0 \in O$ one can find a sequence x_k satisfying (I.5) that is convergent to $\bar{x}$.

II. PRELIMINARIES

Let $\mathbb{R}^n$ be a Euclidean space, let S be a set-valued mapping from $\mathbb{R}^n$ into the subsets of $\mathbb{R}^m$, denoted $S : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^m}$. Let $(\bar{x}, \bar{y}) \in G(S)$. Here, $G(S) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in S(x)\}$ is the graph of S . Let $A, B \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$. The distance from a point x to a set A is defined by

$$d(x, A) = \inf_{y \in A} \rho(x, y)$$

and the Hausdorff semidistance from B to A is defined by

$$e(B, A) = \sup_{x \in B} d(x, A).$$

We denote by $B_r(a)$ the closed ball of radius r centered at a , and S^{-1} is the inverse of S defined as $x \in S^{-1}(y) \Leftrightarrow y \in S(x)$. We say that a set A is locally closed at $z \in A$ if there exists $\gamma > 0$ such that the set $A \cap B_\gamma(z)$ is closed.

Let $L > 0$. A mapping $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be Lipschitz continuous if

$$\|g(x) - g(y)\| \leq L\|x - y\| \quad \text{for all } x, y \in \mathbb{R}^n.$$

In this case, L is called the Lipschitz constant of g . The mapping S is said to be *metrically regular* at $\bar{x}$ for $\bar{y}$ if there exists a constant $\kappa > 0$ such that

$$d(x, S^{-1}(y)) \leq \kappa d(y, S(x)), \quad \text{for all } (x, y) \text{ close to } (\bar{x}, \bar{y}). \quad (\text{II.1})$$

The infimum of κ for which (II.1) holds is the *regularity modulus* denoted $\text{reg}S(\bar{x}|\bar{y})$; the case when S is not metrically regular at $\bar{x}$ for $\bar{y}$ corresponds to $\text{reg}S(\bar{x}|\bar{y}) = \infty$. The inequality (II.1) has direct use in providing an estimate for how far a point x is from being a solution to the variational inclusion $y \in S(x)$; the expression $d(y, S(x))$ measures the residual when $y \notin S(x)$. Smaller values of κ correspond to more favorable behavior. For recent advances on metric regularity and applications to variational problems, see [7], [5] and [6].

We state the following set-valued generalization of the Banach fixed point theorem proved by Donchev and Hager

[4] in a complete metric space that we employ to prove our main result (Theorem 3.2).

Lemma 2.1: (Donchev and Hager [4]) *Let (X, ρ) be a complete metric space and $\Phi : X \rightarrow 2^X$ be a set-valued mapping. Let $\bar{x} \in X$, $\alpha > 0$ and $0 \leq \theta < 1$ such that $\Phi(x) \cap B_\alpha(\bar{x})$ is closed for all $x \in B_\alpha(\bar{x})$ and the following conditions hold:*

- (i) $d(\bar{x}, \Phi(\bar{x})) \leq \alpha(1 - \theta)$;
- (ii) $e(\Phi(u) \cap B_\alpha(\bar{x}), \Phi(v)) \leq \theta\rho(u, v)$ for all $u, v \in B_\alpha(\bar{x})$.

Then there exists $x_0 \in B_\alpha(\bar{x})$ such that $x_0 \in \Phi(x_0)$.

III. CONVERGENCE THEOREM

First, we recall the algorithm we consider to solve (I.1). Given a starting point x_0 , find a sequence x_k by applying the iteration

$$x_{k+1} = \alpha_k x_k + (1 - \alpha_k) y_k, \quad \text{for } k = 0, 1, 2, \dots,$$

where $\{\alpha_k\}$ is a sequence in $[0, 1]$ such that $\alpha_k \leq \bar{\alpha} < 1$, g_k is a sequence of Lipschitz mappings and y_k satisfies the following inclusion

$$0 \in g_k(y_k) - g_k(x_k) + T(y_k).$$

The main result of this section reads as follows:

Theorem 3.1: *Let $T : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^m}$ be a set-valued mapping and $\bar{x} \in T^{-1}(0)$. Assume that $G(T)$ is locally closed at $(\bar{x}, 0)$ and T is metrically regular at $\bar{x}$ for 0. Choose a sequence of functions $g_k : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $g_k(0) = 0$ which are Lipschitz continuous in a neighborhood U of 0 and Lipschitz constants λ_k satisfying*

$$\sup_k \lambda_k < \frac{1}{2\text{reg}T(\bar{x}|0)}. \quad (\text{III.1})$$

Then there exists a neighborhood O of $\bar{x}$ such that for any $x_0 \in O$ there exists a sequence $\{x_k\}$ generated by (I.5) and (I.6) is well-defined and $\{x_k\}$ converges to $\bar{x}$.

Proof: We first show that well-definedness of the sequence generated by our algorithm.

Let $\lambda = \sup_k \lambda_k$, then from (III.1) there exists $\kappa > \text{reg}T(\bar{x}|0)$ such that $\kappa\lambda < \frac{1}{2}$ and

$$d(x, T^{-1}(y)) \leq \kappa d(y, T(x)) \quad (\text{III.2})$$

for all (x, y) close to $(\bar{x}, 0)$. Let $\gamma > 0$ be such that $((\kappa\lambda)^{-1} - 1)^{-1} < \gamma < 1$. From (III.2), there exists $a > 0$ such that the mapping T is metrically regular on $B_a(\bar{x}) \times B_{2\lambda a}(0)$ with constant κ and $B_{2a}(0) \subset U$.

Let $x_0 \in B_a(\bar{x})$. For any $x \in B_a(\bar{x})$, we have

$$\begin{aligned} \|(g_0(x) - g_0(x_0))\| &= \|g_0(x_0) - g_0(x)\| \\ &\leq \lambda_0 \|x_0 - x\| \\ &\leq 2\lambda_0 a \\ &\leq 2\lambda a. \end{aligned}$$

We will show that the mapping $\phi_0(y) = T^{-1}(-(g_0(y) - g_0(x_0)))$ satisfies the assumptions of the fixed point result in Lemma 2.1. First, by using the assumptions that T is

metrically regular at $\bar{x}$ for 0, $0 \in T(\bar{x})$ and $g_k(0) = 0$, we have

$$\begin{aligned} d(\bar{x}, \phi_0(\bar{x})) &= d(\bar{x}, T^{-1}(-(g_0(\bar{x}) - g_0(x)))) \\ &\leq \kappa d(-(g_0(\bar{x}) - g_0(x_0)), T(\bar{x})) \\ &\leq \kappa \|g(x_0) - g(\bar{x})\| \\ &\leq \kappa \lambda_0 \|x_0 - \bar{x}\| \\ &\leq \kappa \lambda_0 a \\ &< a(1 - \kappa \lambda_0). \end{aligned}$$

Further, for any $u, v \in B_a(\bar{x})$, by the metric regularity of T ,

$$\begin{aligned} e(\phi_0(u) \cap B_a(\bar{x}), \phi_0(v)) &= \sup_{x \in T^{-1}(-(g_0(u) - g_0(x_0))) \cap B_a(\bar{x})} d(x, T^{-1}(-(g_0(u) - g_0(x_0)))) \\ &\leq \sup_{x \in T^{-1}(-(g_0(u) - g_0(x_0))) \cap B_a(\bar{x})} \kappa d(-(g_0(u) - g_0(x_0)), T(x)) \\ &\leq \kappa \|-(g_0(u) - g_0(x_0)) - (-(g_0(v) - g_0(x_0)))\| \\ &\leq \kappa \lambda_0 \|u - v\|. \end{aligned}$$

To apply Lemma 2.1, it remains to see that the sets $\phi_0(y) \cap B_a(\bar{x})$ are closed for all $y \in B_a(\bar{x})$. Keeping in mind that T is locally closed graph, adjusting a needed, this can be easily shown. Hence by Lemma 2.1, there exists $y_0 \in \phi_0(y_0) \cap B_a(\bar{x})$, that is

$$y_0 \in B_a(\bar{x}) \quad \text{and} \quad 0 \in g_0(y_0) - g_0(x_0) + T(y_0).$$

Let

$$x_1 = \alpha_0 x_0 + (1 - \alpha_0) y_0.$$

For any $x \in B_a(\bar{x})$, we have

$$\begin{aligned} &\|-(g_1(x) - g_1(x_1))\| \\ &\leq \lambda_1 \|x_1 - x\| \\ &= \lambda_1 \|\alpha_0 x_0 + (1 - \alpha_0) y_0 - x\| \\ &= \lambda_1 \|\alpha_0(x_0 - \bar{x}) + (1 - \alpha_0)(y_0 - \bar{x}) + \bar{x} - x\| \\ &\leq \lambda_1 \{\alpha_0 \|x_0 - \bar{x}\| + (1 - \alpha_0) \|y_0 - \bar{x}\| + \|\bar{x} - x\|\} \\ &\leq 2\lambda_1 a \\ &\leq 2\lambda a. \end{aligned}$$

Let

$$a_1 = \gamma \|x_1 - \bar{x}\|. \quad (\text{III.3})$$

Since $\gamma < 1$, we have $a_1 < a$. We consider the mapping $\phi_1(y) = T^{-1}(-(g_1(y) - g_1(x_1)))$. By (III.3), the metric regularity of T and the choice of γ

$$\begin{aligned} d(\bar{x}, \phi_1(\bar{x})) &\leq d(\bar{x}, T^{-1}(-(g_1(\bar{x}) - g_1(x_1)))) \\ &\leq \kappa d(-(g_1(\bar{x}) - g_1(x_1)), T(\bar{x})) \\ &\leq \kappa \|-(g_1(\bar{x}) - g_1(x_1))\| \\ &\leq \kappa \lambda_1 \|x_1 - \bar{x}\| \\ &\leq a_1(1 - \kappa \gamma). \end{aligned}$$

For $u, v \in B_{a_1}(\bar{x})$, again by the metric regularity of T , we obtain

$$\begin{aligned} e(\phi_1(u) \cap B_{a_1}(\bar{x}), \phi_1(v)) &= \sup_{x \in T^{-1}(-(g_1(u) - g_1(x_1))) \cap B_{a_1}(\bar{x})} d(x, T^{-1}(-(g_1(u) - g_1(x_1)))) \\ &\leq \sup_{x \in T^{-1}(-(g_1(u) - g_1(x_1))) \cap B_{a_1}(\bar{x})} \kappa d(-(g_1(u) - g_1(x_1)), T(x)) \\ &\leq \kappa \|-(g_1(u) - g_1(x_1)) - (-(g_1(v) - g_1(x_1)))\| \\ &\leq \kappa \lambda_1 \|u - v\|. \end{aligned}$$

Because $\phi_1(y) \cap B_{a_1}(\bar{x})$ is closed for any $y \in B_{a_1}(\bar{x})$, by Lemma 2.1, there exists $y_1 \in \phi_1(y_1) \cap B_{a_1}(\bar{x})$, which by (III.3), satisfies

$$\|y_1 - \bar{x}\| \leq \gamma \|x_1 - \bar{x}\|.$$

Let

$$x_2 = \alpha_1 x_1 + (1 - \alpha_1) y_1.$$

It follows that

$$\begin{aligned} \|x_2 - \bar{x}\| &= \|\alpha_1 x_1 + (1 - \alpha_1) y_1 - \bar{x}\| \\ &\leq \alpha_1 \|x_1 - \bar{x}\| + (1 - \alpha_1) \|y_1 - \bar{x}\| \\ &\leq \alpha_1 \|x_1 - \bar{x}\| + \gamma(1 - \alpha_1) \|x_1 - \bar{x}\| \\ &= (\alpha_1 + \gamma(1 - \alpha_1)) \|x_1 - \bar{x}\| \\ &\leq ((1 - \gamma)\bar{\alpha} + \gamma) \|x_1 - \bar{x}\| \end{aligned}$$

The induction step is now clear. Let $x_k \in B_a(\bar{x})$. Then for $\alpha_k = \gamma \|x_k - \bar{x}\|$, by applying Lemma 2.1 to $\phi_k : y \rightarrow T^{-1}(-(g_k(y) - g_k(x_k)))$ we obtain the existence of $y_k \in B_{a_k}(\bar{x})$ such that $0 \in g_k(y_k) - g_k(x_k) + T(y_k)$. And hence,

$$\|y_k - \bar{x}\| \leq \gamma \|x_k - \bar{x}\| \quad \text{for all } k = 1, 2, \dots$$

Let

$$x_{k+1} = \alpha_k x_k + (1 - \alpha_k) y_k.$$

Thus, we establish that

$$\|x_{k+1} - \bar{x}\| \leq ((1 - \gamma)\bar{\alpha} + \gamma) \|x_k - \bar{x}\|.$$

Since $(1 - \gamma)\bar{\alpha} + \gamma < 1 - \gamma + \gamma = 1$, the sequence x_k converges to $\bar{x}$. $\blacksquare$

Note that if $\alpha_k = 0$ for all $k = 0, 1, 2, \dots$, then we can consider the following particular case of (I.5) and (I.6).

$$0 \in g_k(x_{k+1}) - g_k(x_k) + T(x_{k+1}), \quad \text{for } k = 0, 1, 2, \dots \quad (\text{III.4})$$

Now, we are able to state the following result.

Theorem 3.2: Let $T : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^m}$ be a set-valued mapping and $\bar{x} \in T^{-1}(0)$. Assume that $G(T)$ is locally closed at $(\bar{x}, 0)$ and T is metrically regular at $\bar{x}$ for 0. Choose a sequence of functions $g_k : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $g_k(0) = 0$ which are Lipschitz continuous in a neighborhood U of 0 and Lipschitz constants λ_k satisfying

$$\sup_k \lambda_k < \frac{1}{2\text{reg}T(\bar{x}|0)}.$$

Then there exists a neighborhood O of $\bar{x}$ such that for any $x_0 \in O$ there exists a sequence $\{x_k\}$ generated by (III.4) is well-defined and $\{x_k\}$ converges to $\bar{x}$.

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