

A Consideration on Convergence Condition of Explicit Finite Element Analysis for Heat Transfer Equation

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SYNOPSIS

This paper treats the convergence condition of the explicit finite element method (i.e. the time and spatial axes are discretized by using the explicit finite difference method and the weighted residual method, respectively) which is applied for analyzing the heat problem in region with complex boundary configuration and also with several material properties. The main role of this study is the application of the Brauer's theorem. As the results we obtain that the usage of the Brauer's theorem is valid and that the application method of the theorem is presented in this paper.

1. INTRODUCTION

At the application of the explicit finite element method for any engineering problem the user has to decide the time-step, but at present the user has no tool for this determination. On the other hand, at the application of the explicit finite difference method the user has no trouble on this aspect, because its critical time-step is theoretically decided. Thus, we often apply the time-step obtained from FDM as that for the explicit FE analysis.

The characteristics of the finite element method comparing to FDM are that 1) its mesh pattern may be generally complex, 2) the

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introduction of the spatial irregularity is very easy comparing to FDM, and 3) the configuration of the region treated may be complex. These physical and geometrical irregularities actually affect on the critical value of the time-step, and, therefore, the usage of the value by the FDM to the explicit FEM may allow uneconomical computation.

The critical time-step is decided by the maximum eigenvalue of the governing equations. Though there exists no effective method to get the strict value, the Brauer's theorem can restrict the region where the all eigenvalues locate. The study by Taniguchi et al^(1,2) succeeded to show the effectivity of the theorem as a tool for the determination of the critical time-step for rather simple cases.

In this study the authors try to apply the Brauer's theorem to heat equation which governs the heat transfer problem in a region with several physical properties and complex geometrical configuration, and they present an evaluation method of the critical time-step for above cases.

2. EXPLICIT FINITE ELEMENT FORMULATION OF HEAT PROBLEM

Let

$$\kappa \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) = \frac{\partial \phi}{\partial t} \quad (1)$$

be the heat equation in which κ is the thermal conductivity and we assume that this coefficient is a function of the location. We assume that this equation subjects following boundary conditions;

$$\left. \begin{aligned} \phi &= \bar{\phi} && \text{on } S_1 \\ \kappa \frac{\partial \phi}{\partial n} &= -h(\phi - v) && \text{on } S_2 \end{aligned} \right\} \quad (2)$$

where h is the heat transfer coefficient and v is the value of ϕ out of the domain as shown in Fig. 1.

The time and spatial axes in eq's 1 and 2 are discretized by using the explicit FDM and WRM, respectively, and at the stage of the application of the latter the time axis is fixed as following;

$$\frac{\partial \phi}{\partial t} = T \quad (3)$$

We apply the Galerkin method as WRM, and its introduction to eq.1 yields to

$$\iint \kappa \left\{ \frac{\partial \phi}{\partial x} \frac{\partial \delta \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial \delta \phi}{\partial y} \right\} dx dy + \iint T \delta \phi dx dy = - \int_{S_2} h(\phi - v) \delta \phi ds \quad (4)$$

In order to express above equation by the matrix form we introduce

$$\phi = [N]^t \{\phi\} \quad (5)$$

in which $[N]$ is the shape function matrix and $\{\phi\}$ is the node vector of an element. From the property of the finite element method we may assume that κ is constant in the element. For this assumption the discretization of the domain

must be done by taking consideration of the differences of the physical property. Thus, eq.4 is newly expressed as following;

$$\begin{aligned} & \kappa \iint \left\{ \frac{\partial [N]}{\partial x} \frac{\partial [N]^t}{\partial x} + \frac{\partial [N]}{\partial y} \frac{\partial [N]^t}{\partial y} \right\} dx dy \{\phi\} + \iint [N] dx dy T \\ & = -h \int_{S_2} [N] [N]^t ds \{\phi\} + h \int_{S_2} [N] ds \cdot v \end{aligned} \quad (6)$$

Now, we discretize the time axis in eq.6, and for this purpose following expression is introduced;

$$T = \frac{\partial \phi}{\partial t} = \frac{\{\phi_{(i)}\} - \{\phi_{(i-1)}\}}{\Delta t} \quad (7)$$

where Δt is the time-step, and $\{\phi_{(i)}\}$ and $\{\phi_{(i-1)}\}$ are vectors at the (i)th and (i-1)st stage of the computation, respectively. Then eq.6 yields to

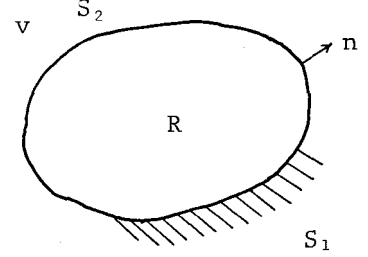


Fig.1 Domain and Boundary Conditions

$$\begin{aligned}
& [k^j]\{\phi_{(i-1)}\} + [m^j] \frac{\{\phi_{(i)}\} - \{\phi_{(i-1)}\}}{\Delta t} \\
& = -[b^j]\{\phi_{(i-1)}\} + \{p^j\}
\end{aligned} \tag{8}$$

for the (j)th finite element, in which

$$\begin{aligned}
[k^j] &= \kappa_j \iint \left\{ \frac{\partial [N]}{\partial x} \frac{\partial [N]^t}{\partial x} + \frac{\partial [N]}{\partial y} \frac{\partial [N]^t}{\partial y} \right\} dx dy \\
[m^j] &= \iint [N] dx dy \\
[b^j] &= h \int_{s_2} [N] [N]^t ds \\
[p^j] &= h.v. \int_{s_2} [N] ds
\end{aligned} \tag{9}$$

The assemblage of all finite elements gives following matrix expression;

$$\{\phi_{(i)}\} = \{I - [M]^{-1} [K^*] \Delta t\} \{\phi_{(i-1)}\} + \Delta t [M]^{-1} \{P\} \tag{10}$$

in which I is a unit matrix and

$$[K^*] = [K] + [B]$$

Note that the superscript in eq.9 indicates the number of the finite element.

3. CONVERGENCE CONDITION

Eq. 10 shows the explicit form and by using this equation the solution vector at each time-step is obtained by using only the solution of the previous time-step. But, in order to ensure these solutions numerical stability must be kept through the computations. The necessary and sufficient condition not to diverge the solution vector of eq.10 is expressed as following;

$$|\lambda_{\max}| \leq 1 \quad (11)$$

in which $\lambda_{\max}$ is the maximum eigenvalue of the coefficient matrix of the right hand of eq.10. This relation indicates that the discretization of the spatial and time axes decides the convergence condition. Since the discretization of the spatial axis is determined by the physical properties of the problem, the convergence condition is governed by the determination of the time-step, i.e. Δt .

Let's express the eigenvalue of $[M]^{-1}[K^*]$ by $\lambda_{(k/m)}$. Then, from eq.10 we obtain

$$\lambda_{\max} = 1 - \lambda_{(k/m)} \cdot \Delta t \quad (12)$$

From eq's 11 and 12 the time-step must satisfy following equation;

$$\Delta t \leq \frac{2}{\lambda_{(k/m)}} \quad (13)$$

Eq.13 indicates that the critical time-step, Δt_{cr} , is obtained as

$$\Delta t_{cr} = \frac{2}{\max. \text{ of } \lambda_{(k/m)}} \quad (14)$$

This equation suggests that the critical time-step is wholly governed only by the maximum eigenvalue of the coefficient matrix, i.e. $[M]^{-1}[K^*]$.

In general it is well known that it is very troublesome to find the maximum eigenvalue for a large set of linear equations. But, we can easily estimate the region where all eigenvalues locate by using the Brauer's theorem given as following;

[Brauer's Theorem]

Let A and $a(i,j)$ be a square matrix and its (i,j) entry. Then, all eigenvalues of A locate on or in the circle $|\lambda - a(s,s)| = P(s)$, where $a(ss)$ and $P(s)$ are the diagonal and $\sum |a(i,j)|$ for $j=1$ to $s-1$ and $s+1$ to n , respectively.

Direct application of this theorem to this investigation leads to the following relation;

$$|\lambda_{(k/m)} - a(s,s)| \leq P(s) \quad (15)$$

The introduction of eq.15 to eq.14 gives following relation;

$$\frac{2}{a(i,i)+P(i)} \leq \frac{2}{\max. \text{ of } \lambda_{(k/m)}} = \Delta t_{cr} \quad (16)$$

Eq. 16 indicates that the actual critical time-step is larger than the value which is decided by the row "s" whose entries gives the largest value of $\{a(i,i)+P(i)\}$. It is easy to select a row which minimizes the left hand of eq.16, but it may often give too small value for Δt_{cr} . This shows that the selection of "s" is the most important fact for the determination of Δt_{cr} .

Assume that a row "s" is appropriately selected so as to be less or equal to Δt_{cr} . Then, we can easily estimate Δt_{cr} as following;

$$\Delta t_{cr} \geq \delta t_{cr} = \frac{2}{a(s,s)+P(s)} \quad (17)$$

Another expression of above equation is given as following;

$$\delta t_{cr} \geq \frac{2m(s,s)}{k(s,s)+P(k)} \quad (18)$$

where $k(s,s)$ is the diagonal entry of $[K^*]$, $P(k)$ is $\{\sum |k(s,i)| - k(s,s)\}$ for $i=1$ to n , and $m(s,s)$ is the diagonal entry of $[M]$.

From above consideration it is classified that how to select a row used for the estimation of δt_{cr} is most important.

Assume that the region in Fig.1 consists of several subareas with physical materials, namely $\kappa_1, \kappa_2, \dots, \kappa_m$, and that each area is divided into finite elements. Furthermore, we assume that, at the subdivision of the region into finite element mesh system the change of the properties on S_2 boundary is also considered. Then, the critical time-step, δt_{cr} , can be estimated by using a row of the coefficient matrix of eq.10 which represents the property of a node located on S_1 , S_2 or in R . Eq.17 indicates that a row "s" maximizing the value of $\{a(s,s)+P(s)\}$ gives the lower band of δt_{cr} .

Before treating the general case of δt_{cr} we try to treat rather simple case as shown in Fig.2. In this example R is a rectangle region which is subdivided into regular finite elements. If we assume

being adjacent to S_1 boundary does not concern with the critical its value of time-step, because $a(s,s)+P(s)$ is less than the value of any node of type 4 in Fig.2. Thus, eq.20 is newly written as following;

$$\delta t_{cr} = \min\{\delta t_{cr}^{S_2}, \delta t_{cr}^R\} \quad (21)$$

in which $\delta t_{cr}^{S_2}$ and δt_{cr}^R indicates the estimate δ values for nodes on S_2 and in R , respectively.

This expression of eq.21 is directly applied to the estimation of the critical time-step for general cases. Since, in general finite element problem, the configuration of R is complex and R consists of several physical properties, the region of R is subdivided into irregular meshes. Since all of these affect on the determination of values of entries in the coefficient matrix, both of $\delta t_{cr}^{S_2}$ and δt_{cr}^R in eq.21 can't be expressed explicitly as shown in eq.19 for regular finite element system. But, eq.19 suggests following important items for the determination of δt_{cr} ;

- (1) Since $\delta t_{cr}^{(i)}$ for $i=1$ to 4 depends on the area of the finite element mesh, any node surrounded by the smallest finite elements may almost decide the critical time-step.
- (2) If the region of R consists of only one physical property, then $\delta t_{cr}^{(R)}$ is decided by a node which is surrounded by the smallest finite elements. Even if it consists of several properties, $\delta t_{cr}^{(R)}$ is estimated by selecting a node surrounded by smallest elements in each subregion in which κ is constant and comparing them.
- (3) $\delta t_{cr}^{S_2}$ is governed not only by the area of elements but also by the physical properties, i.e. κ and h .

Thus, by selecting a node surrounded by the smallest meshes for each subregion with constant κ , and comparing them we can easily estimate $\delta t_{cr}^{S_2}$.

Generally speaking it is preferable from the computational viewpoint that the region is subdivided by using only acute-edged triangular meshes. If we can assume this kind of mesh system, and constant κ for all region, a node of type 1 determines δt_{cr} .

Summarization of above considerations leads to following procedure for the theoretical estimation of the critical time-step by using the Brauer's theorem. Select a node surrounded by the smallest finite element meshes in each subregion with constant κ and find the smallest value of δt among them.

4. NUMERICAL EXPERIMENTS

In preceding section the theoretical estimation method of the critical time-step is presented by the introduction of the Brauer's theorem. By using the result we can obtain the time-step for the computation. But we have to notice that the Brauer's theorem can indicate only the region where all eigenvalues exist. This suggests that the method presented in preceding section may often give too short time-step. In this section we examine the accuracy of the method through a number of numerical experiments.

The area treated for numerical experiments is a unit square region, and each edge subjects to one of the boundary conditions of eq.2. The constant strain triangular finite element is used for the subdivision of the area. In the case where there exist more than two physical properties in the region, the area is firstly subdivided into subregions where only one physical property exists, and each subregion is divided into finite element meshes.

4-1. REGULAR MESH SYSTEM

It is well known that there appears the discretization error at the application of the finite element meshed. In order to decrease the effect of this kind of error to the results of our numerical experiments we examine the size of a finite element by Numerical Test presented as following;

In order to examine only the effect of the size of finite element to the results a unit square region is subdivided into N and M elements along x and y axes, respectively. All edges subject to S_1 boundary condition, and the physical property, κ , is constant for whole area of the problem region. The results are summarized in Table 2, and it indicates that the maximum size of an element should be, at least, less than $1/10$ of the width of problem region. Henceforce, for all finite element meshes used in this paper we set $N = M = 20$, by considering above results.

4-2. REGULAR MESH SYSTEM WITH VARIOUS BOUNDARY CONDITIONS

In this section, 6 cases in Table 1 (Type 1 to Type 6) are treated in order to survey the influence of the boundary conditions to the critical time-step. The mesh system treated here is the one

$\Delta y \backslash \Delta x$		0.025	0.050	0.100	0.167
0.025	E	1.561	-	-	-
	T	1.561			
0.050	E	2.500	6.250	-	-
	T	2.500	6.250		
0.100	E	2.940	10.27	25.00	-
	T	2.940	10.00	25.00	
0.167	E	3.093	11.77	37.90	77.96
	T	3.093	11.46	36.80	69.44

($\times 10^{-4}$)

E ; Observed Value T ; Theoretical Value

Table 2

presented in Section 4-1.

The results are summarized in Fig.3. As indicated in Table 1 there are, at most, four nodes which are used for the estimation of the critical time-step by use of the Brauer's Theorem, and all of them are presented in the figure. The lateral axis indicates the value of the heat transfer coefficient, h , on the boundary, and the vertical axis indicates the ratio of the estimated value by the Brauer's Theorem (the observed value of the critical time-step). Therefore, for the ratio ≤ 1 we can obtain the convergent solution, and for the ratio > 1 the solution is divergent.

From Fig.3 we can remark that node 3 can give the best estimated value of the time-step even though it may lead to divergent calculation for Type 1 with $h \geq 100$. Moreover, node 3 always exists in all 6 types of boundary conditions presented in Fig.3. For Type 7 in Table 1 there exists only node 4. Therefore, for further numerical experiments only these two nodes are used for estimating the theoretical value of the critical time-step.

4-3. REGULAR MESH SYSTEM WITH TWO PHYSICAL PROPERTIES

In this section we survey the influence of the physical properties to the critical time-step. For this purpose we assume that the unit area consists of two properties, K_1 and K_2 , and that one half of the area has one property. The finite element model treated in this

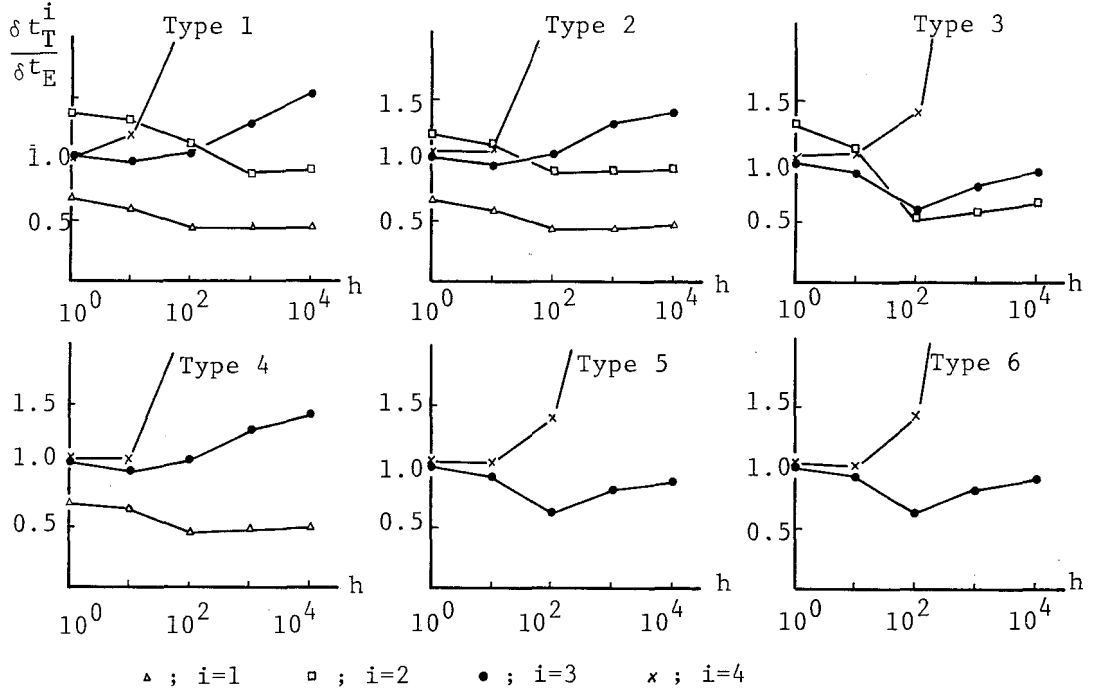


Fig. 3 Estimation of Critical Time-Step by Proposed Method

investigation is presented in Table 3.

The results are summarized in Table 3. The values given in the table indicate the ratio of the estimated value (the observed value). According to the table node 3 can give good estimation for $h=1 \sim 100$ of Type 1 to 6 but it overestimates the critical value for $h > 1000$ of Type 1 to 3. Node 4 can give good estimated value for Type 7.

From this experiment we can remark that Node 3 and 4 are useful to estimate the critical time-step but we should newly introduce the safety factor, s , in order to decide the critical time-step as following;

$$\delta t_{cr} = \text{the estimated value} / s$$

where $1 \leq s \leq 1.5$.

4-4. IRREGULAR MESH SYSTEM WITH ONE PHYSICAL PROPERTY

In this experiment we survey the influence of the irregularity of the mesh size to the critical value. For this experiment it is assumed that whole area consists of only one physical property. The mesh system treated in this experiment is presented in Table 4, and the results are summarized in the same table.

According to Table 4, node 3 and 4 can give good estimation value of the critical time-step. Thus, we can conclude that these two nodes are valid for estimating the theoretical time-step.

4-5. IRREGULAR MESH SYSTEM WITH TWO PHYSICAL PROPERTIES

In this section we survey the influence of both properties, the irregularity of mesh size and the physical properties, to the critical time-step. The models are presented in Table 5, and the results are also summarized in the table.

According to Table 5 we can conclude that node 3 and 4 are useful for the evaluation of the critical time-step, and the estimated value can always lead to convergent solution.

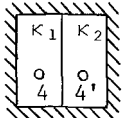
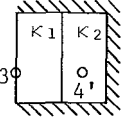
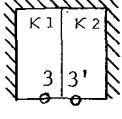
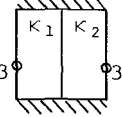
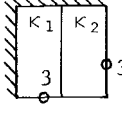
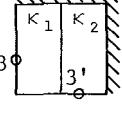
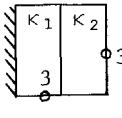
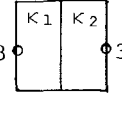
5. CONCLUDING REMARKS

In this investigation how to determine the critical time-step was newly proposed and its validity was surveyed through a number of numerical experiments. According to one proposed method how to select a node in the finite element mesh system decides its validity, and the experiments given in this paper clarified that a node on S_2 boundary and surrounded by the smallest elements should be selected as for the evaluation of the critical time-step. If there exists no S_2 boundary in the problem area, we should select a node in the area which is surrounded by the smallest finite elements.

The results given in this paper show that the method may sometimes overestimate the critical time-step. In order to avoid the numerical divergency according to the introduction of the overestimated critical time-step we should apply the safety factor s , i.e. $1 \leq s \leq 1.5$. That is,

$$\text{the critical time-step} = \text{the estimated value}/s.$$

It should be noted that the estimated value by the proposed

	$\delta t_{T_3} / \delta t_E$		$\delta t_{T_4} / \delta t_E$	
	$h=1 \sim 100$	$h=1000$	$h=1 \sim 100$	$h=1000$
	-	-	0.9~1.0	0.9~1.0
	0.6~1.0	0.8~1.0	0.9~1.0	0.9~1.0
	0.6~1.0	0.6~1.0		
	0.5~1.0	0.6~1.0		
	0.6~1.0	0.6~1.0		
	0.8~1.1	0.9~1.4		
	0.8~1.1	0.9~1.4		
	0.9~1.1	0.9~1.4		

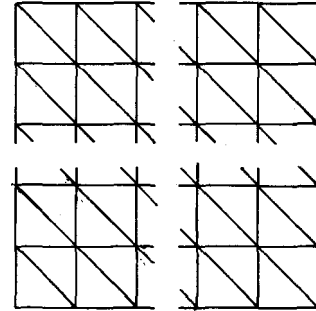
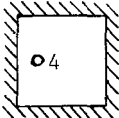
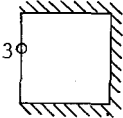
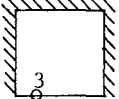
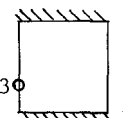
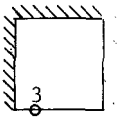
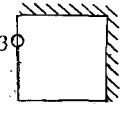
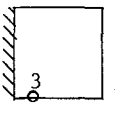
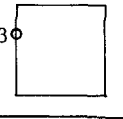


Table 3

	$\delta t_{T_3} / \delta t_E$	$\delta t_{T_4} / \delta t_E$
	$h=1\sim 1000$	$h=1\sim 1000$
	-	0.9~1.0
	0.8~1.0	
	0.8~1.0	
	0.8~1.0	
	0.6~1.0	
	0.8~1.0	
	0.6~1.0	
	0.8~1.0	

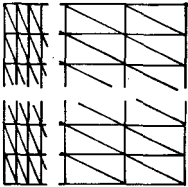
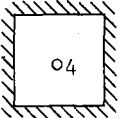
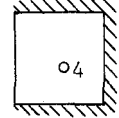
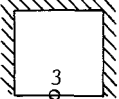
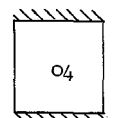
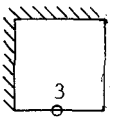
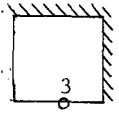
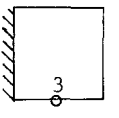
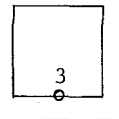


Table 4-1

	$\delta t_{T_3} / \delta t_E$	$\delta t_{T_4} / \delta t_E$
	$h=1\sim 1000$	$h=1\sim 1000$
	-	0.9~1.0
		0.9~1.0
	0.6~1.0	
		0.9~1.0
	0.6~1.0	
	0.6~1.0	
	0.6~1.0	
	0.6~1.0	

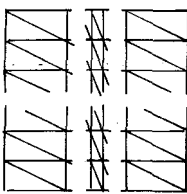


Table 4-2

	$\delta t_{T_3} / \delta t_E$	$\delta t_{T_4} / \delta t_E$
	$h=1 \sim 1000$	$h=1 \sim 1000$
	-	0.9~1.0
	0.7~1.0	0.9~1.0
	0.7~1.0	
	0.7~1.0	
	0.6~1.0	
	0.8~1.0	
	0.6~1.0	
	0.8~1.0	

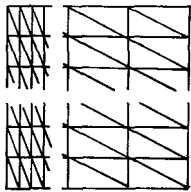


Table 5-1

	$\delta t_{T_3} / \delta t_E$	$\delta t_{T_4} / \delta t_E$
	$h=1 \sim 1000$	$h=1 \sim 1000$
	-	0.9~1.0
		0.9~1.0
	0.6~1.0	
		0.9~1.0
	0.6~1.0	
	0.6~1.0	
	0.6~1.0	
	0.6~1.0	

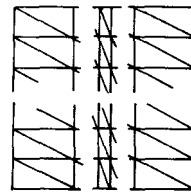


Table 5-2

method may become worse for coarse finite element mesh systems.

On the application of the proposed method to general engineering problem one example is presented in Ref.2. The problem treated in the reference has rather complex surrounding configuration comparing to the examples given in this paper, but the proposed method can give good critical value.

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References

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