

Dissection Method for a Regular Finite Element Mesh

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Synopsis

From the theoretical approach to the fill-in minimization problem we present one of the optimal vertex elimination process for a regular finite element mesh M ($n \times n$), and through a number of numerical experiments it is verified that the new process model can always lead to better numerical results comparing to other methods presently in use. Since the process here presented can't give the actual dissections of M but can clarify how the optimal elimination is, the process includes George's Nested Dissection Method and the method by Duff, Erisman and Reid. By this investigation we can conclude that 1) the concept of "Dissection" is necessary for minimizing the number of fill-ins, 2) the location of the dissection lines can be systematically decided even if n of M is odd or even number, and though the interior area of M is dissected as George's Method, the surrounding area of M is rather irregularly dissected, and 3) the model of the vertex elimination process given in this paper is applied to other kind of regular finite element mesh or finite difference mesh, too.

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1. Introduction

It is well known that if we avoid the operations of zeros included in a large sparse set of linear equations, the rearrangement of rows and columns of the matrix or the reordering of the unknowns of the system can affect the amount of computation and the storage required for the direct solution. As the actual reordering method we can apply, for example, Minimum Degree Algorithm and Minimum Defficiency Algorithm [1], and furthermore, there are some important studies on so-called Nested Dissection Method by George [2, 5], Birkhoff and George [3], Duff, Erisman and Reid [4], and Hoffman, Martin and Rose [6]. Minimum Degree and Deficiency Algorithms aim only to minimize the appearance of fill-ins at every elimination step, but Nested Dissection Method aims to decrease the total fill-ins by avoiding the appearance of bigger size of cliques. In other words, the latter decides the elimination ordering by taking the whole system into the considerations. On the other hand, Taniguchi [7] independently shows from the graph-theoretic approach that the elimination ordering for minimizing fill-ins necessarily produces the dissection of the graph, and Taniguchi and Numata [8] give an idea of the explanation of the optimal vertex elimination process.

According to [8] we clarify the vertex elimination process model, at first, and through a number of numerical experiments the process model is investigated. Furthermore, we give some considerations on George's Nested Dissection Method by comparing it with our results.

2. Dissection Method for Regular Mesh

2.1 Finite Element Mesh Structure

We consider the solution of the matrix equation

$$Ax = b$$

which arises from a set of finite element equations and involves a sparse matrix A which is symmetric and positive definite. We assume for the brevity that the coefficient matrix A has the nonzero pattern which results from the finite element discretization of regular $n \times n$ meshes of a square area. Therefore, the finite elements are also square, the corners are nodes, and there are 1 - 1 correspondence between nodes in the grid and the variables in the system of equations.

Thus, the matrix A has dimension $(n + 1)^2$. Furthermore, we assume that the finite element meshes are obtained by use of the nine point finite element discretization, that is every row (or column) of A has less than or equal to 8 off-diagonal nonzeros.

Let's denote this finite element mesh by M and the graph directly obtained from nonzero pattern of A by G_0 , respectively. Furthermore, we denote the set of vertices in G_0 by V and also the vertices locating on the boundary of M by ∂V , respectively. If the number of lines connected to a vertex, v , in G is denoted by $\deg.v$, then $\deg.v=8$ for $v \in V-\partial V$ and $\deg.v=3$ or 5 for $v \in \partial V$.

2.2 Dissection of a Graph

By an elimination of a vertex $v \in G_0$, all vertices adjacent to the vertex, denoted by $\text{adj}.v$, construct a clique in which all vertices are adjacent each other. For the construction of a clique a number of edges must be added for the subgraph $\{\text{adj}.v\}$, and these additional edges are called "Fill-in". A clique produced at each step of vertex eliminations is denoted by FVG (abbreviation of Frontal Vertex Group) in ref. [7, 8]. From these studies it is shown that any optimal vertex elimination process is generally equal to the repetitions of a). appearing, b). growing, c). stopping and d). joining of FVG's. It is evident that the elimination process for a subset of vertices surrounded by a stopped FVG gives no influence to the residual of G . Thus, if we find out the location of the stopped FVG's on G , then the fill-in minimization ordering for G may be replaced to the fill-in minimization ordering for a subgraph enclosed by a stopped FVG.

It is easily proved that any FVG stops its growth in general. Anyway, an appropriately selected vertex in G is eliminated for the first time, and it constructs a FVG. If the FVG does not stop its growth, then all of successive vertex eliminations are always to be b) type among four. But, this kind of the elimination process can give the minimum fill-ins only for some kinds of graphs like simple trees and complete graphs. Therefore, FVG stops its growth in general.

Above considerations on FVG indicate that the concept of "Dissection" of G is necessary for optimal vertex elimination ordering aiming the minimum fill-ins.

2.3 Dissection Pattern of $\{v \mid v \in V - \partial V\}$ of G

As mentioned in Section 2.1, any vertex in the subgraph $G_s = \{v \mid v \in$

$V - \partial V$ has the degree of 8. Assume the number n of M be sufficiently large. Then, the result in Section 2.2 indicates that a lot of vertices in G_s are eliminated independently from the surrounding subgraph, $G - G_s$. In this section we consider the optimal vertex elimination ordering for this subgraph G_s .

Arbitrarily selected vertex v_1 in G_s is, at first, eliminated. Now, find out the optimal vertex elimination ordering for a small subgraph including v_1 . If we take a subgraph including only 9 vertices (see Fig.1), then the optimal elimination ordering is as indicated in the same figure. That is, first four vertex eliminations produce four cliques each of which includes 8 vertices. And, the combination of these cliques yields to one bigger clique with 16 vertices. If the same elimination process is applied for bigger area, then the state after above process is as shown in Fig.2. That is, the graph consists of only clique with 16 vertices. Then, the repetition of above process is applied to this new graph and it yields to a new graph with clique with 32 vertices as shown in Fig.2. If n is big enough, this process is repeated and the modified graph at each elimination stage consists of only cliques with 4, 8, 16, 32, 64, ... vertices, respectively. The relation of these cliques is presented in Fig.3.

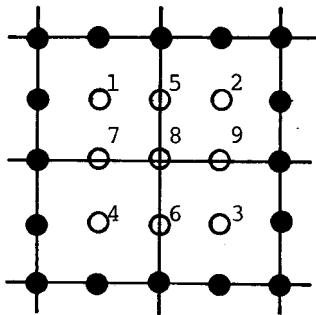


Fig. 1 Optimal Ordering
for G_s with 9
Vertices

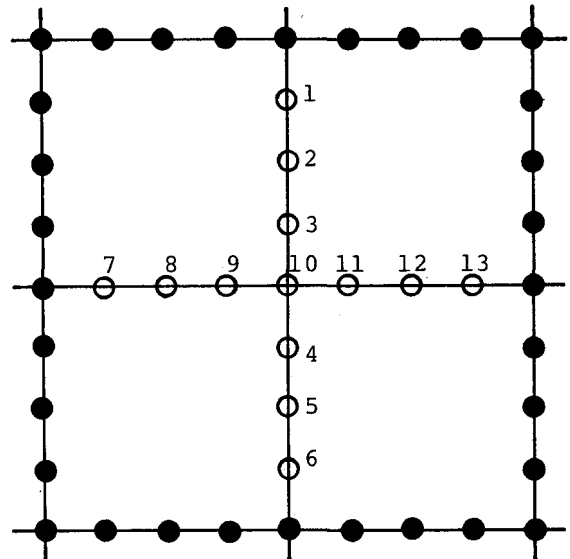


Fig. 2 Optimal Ordering for G_s
with 49 Vertices

If we denote the series of graphs appearing at above elimination process by $G_0, G_1, G_2, \dots, G_i$, then the vertex elimination for a subgraph of G_0 surrounded by a clique in G_i , for example, is independent from the other region of G_0 . Thus, the clique is equal to the dissection of G_0 . Since this elimination process is one of the optimal vertex elimination ordering for a subgraph G_s of G , we apply this process as fundamental pattern of eliminations for G with ∂G in next section.

Note that the set of above dissection lines is equivalent to the pattern of George's Nested Dissection.

2.4 General Dissection for Regular Mesh

By using the result in Section 2.3 we propose here new vertex elimination process for minimizing fill-ins locally, at least.

As mentioned in Section 2.1 a finite element mesh M consists of "Interior Area M_i " where the degree of any vertex is equal to 8 and "Surrounding Area M_s ". It is obvious from Fig.3 that as the elimination process is continued, the clique in G_i becomes bigger and bigger. Suppose that n is sufficiently large. Then, the elimination ordering of the subgraph of M_i , denoted by G_i^0 , may be determined without taking the influence of M_s into consideration. That is, G_i^0 consisting of cliques with 4 vertices is replaced to G_i^1 which consists of only cliques with 8 vertices. For the surrounding area M_s appropriate vertex eliminations are also done not to produce bigger cliques, because the number of fill-ins depends on the square of the number of vertices in a clique, and a new graph G_s^1 appears at the area. Then we obtain new graph $G^1 (= G_i^1 \cup G_s^1)$.

For this new graph G^1 above mentioned process is repeated. But, notice that the interior area of G^1 consists of cliques with 8 vertices. Fig.3 indicates

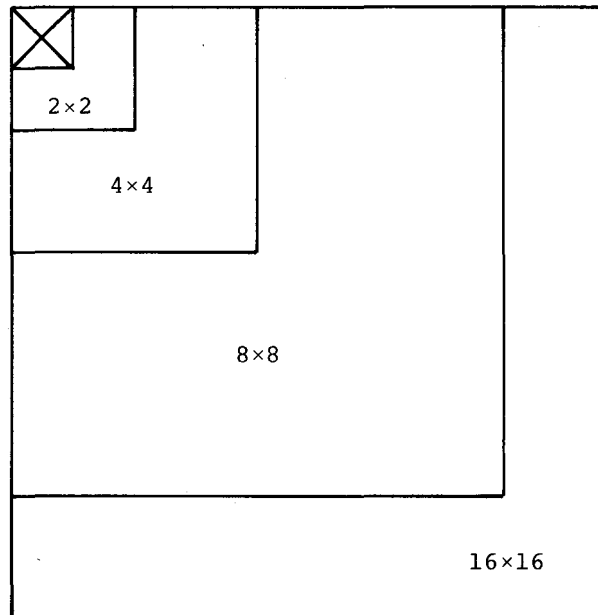


Fig.3 Fundamental Elimination Pattern for the Interior Area of M

that G_i^2 which is obtained from G^1 consists of more bigger cliques than that in G_i^1 . Therefore, the area of G_i^2 becomes smaller than the area of G_i^1 . As the elimination is continued, the area of G_i^k is smaller than that of G_i^{k-1} , and at the end of the elimination process the interior area disappears. That is, the whole graph must be treated as the surrounding area. Fig. 4 illustrates the process of above elimination steps. All the crossing lines show the dissection lines, and the hatched area presents the surrounding area.

From above consideration on the vertex elimination ordering it is evident that the dissection line of G^k gives influence to the dissection of G^{k-1} , G^{k-2} , Thus, the setting of the dissection line for the last stage of graph G^k is most important. On the setting of this dissection line in G^k the approach by use of numerical experiments is taken, and some theoretical considerations are given in successive section.

3. Numerical Experiments

In this section we survey the actual location of the dissection lines appearing in finite element mesh M according to the vertex elimination process model in the former section, and we compare the results with other methods. For this comparison we use George's Nested Dissection Method, Dissection Method by Duff, Erisman and Reid and also Minimum Degree Algorithm.

As obvious from the contents in Section 2.4, since the elimination process model can suggest only the guide line for setting the dissection lines, the numerical experiments are carefully done by hands as 1) to satisfy the results in Section 2.4, 2) to give the minimum fill-ins, and also 3) to get some informations which are important for the proposal of the actual dissection method.

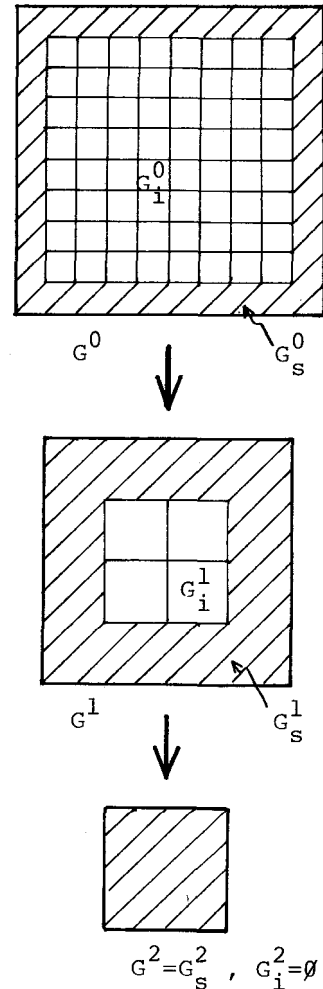


Fig.4 Vertex Elimination Process Model

The numerical experiments are done for 31 cases of finite element mesh $M(n \times n)$. The comparison of the number of fill-in with other three methods is summarized in Table 1. And Fig.A shows the location of the actual dissection lines for each M for n equal from 7 to 32. Table 1 clarifies that we can find out better elimination orderings for all test examples.

From these numerical experiments we can summarize the properties of the dissection lines as follows:

- 1). Fundamental dissection lines of M locate so as to dissect M into four subareas with the same number of vertices, even if n is odd or even number.
- 2). At the stage when n is equal to 16 and 32, new dissection lines according to bigger cliques appear for the first time as the process model suggests. But, instead of $n=8$, $M(9 \times 9)$ produces new dissection lines.
- 3). Though the dissection method in the interior area is just the same as George's Nested Dissection, the dissection of the surrounding area is different. But, we may say that the latter dissection is also sufficiently systematic.

Let's give some considerations on above three items.

For 1). The location of the fundamental dissection lines seems to be determined by the surrounding configuration of G_0 . The reason are as follows; a). all of the test examples have the square surrounding configuration, and b). since the number of fill-ins almost depends on the square of the number of vertices of clique, the maximum clique appearing through $G_0, G_1, G_2, \dots$ is minimized when M is dissected into four subareas with the same size.

For 2). Except for the case of $n=9$, the theoretical results in previous section is recognized through the numerical experiments, too. For this exceptional case we can guess that a). n is still small so as to be affected by the elimination of ∂M and b). the size of clique appearing in the area of M_s is not determined uniquely, for example of $n=9$ and 10 we find cliques with 7, 8 and 9 vertices.

For 3). George's Dissection Method is equivalent to the discussion in Section 2.3, and it pays no attention to the difference of degrees between vertices on the boundary and the vertices in the interior area. Thus, his method produces too many cliques in the surrounding area.

The difference of the number of fill-ins between New Dissections and George's one depends wholly on the difference of the dissection lines in the surrounding area.

2	2	1	1	1	
3	3	11		/	
4	4	28	28	28	
5	5	64		/	
6	6	113	113	/	
7	7	191		/	218
8	8	288	300	300	349
9	9	416		/	473
10	10	573		/	677
11	11	755	777	/	898
12	12	976		/	1280
13	13	1236		/	1367
14	14	1529	1649	/	1861
15	15	1871		/	2316
16	16	2256	2280	2280	2898
17	17	2636		/	3080
18	18	3093		/	3857
19	19	3571		/	4667
20	20	4084		/	5468
21	21	4680	4848	/	5575
22	22	5309		/	6550
23	23	5967		/	7214
24	24	6740		/	8644
25	25	7480		/	9015
26	26	8297	8421	/	10038
27	27	9155		/	11960
28	28	10048		/	13841
29	29	11068		/	13724
30	30	12121	12793	/	14811
31	31	13487		/	16671
32	32	14512	14588	14588	18982
65	65		83744	83733	
n × n		N.O.	G.D.	N.D.	MIN. DEG.

Table 1. Results of Numerical Experiments

Note ; N.O., G.D., N.D. and MIN.DEG. indicate New Ordering by the authors, General Dissection by Duff et al, Nested Dissection by George, and Min. Deg. Algorithm, respectively. Numerals are the number of fill-ins.

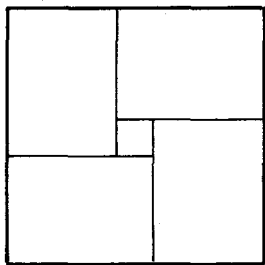
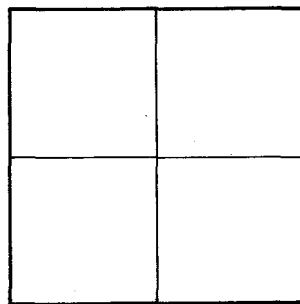
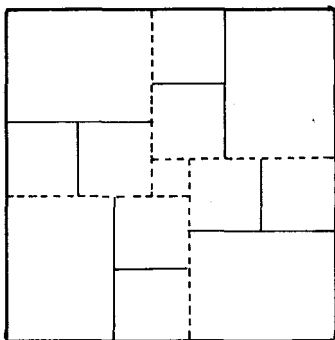
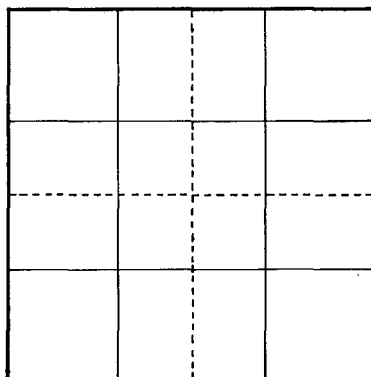
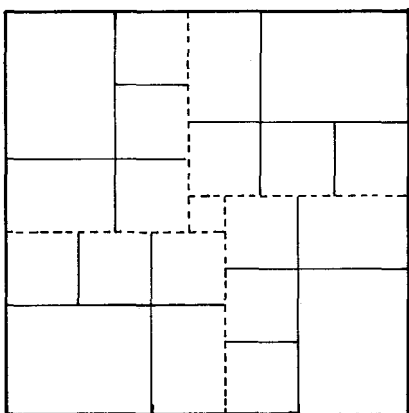
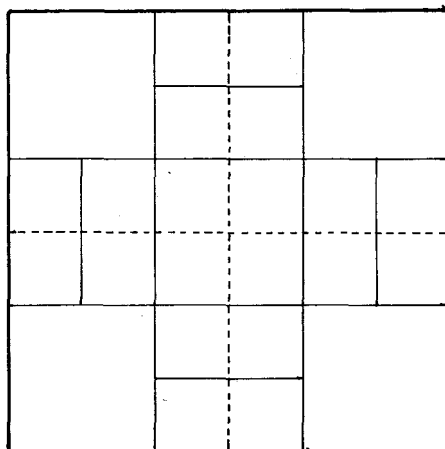
4. Concluding Remarks

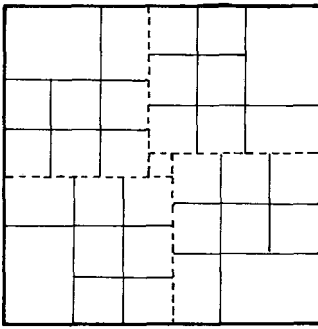
First of all we may conclude that the elimination process model and the numerical experiments clarify the proposal of a better re-ordering method being possible. Furthermore, since the model is derived not for a specific type of graphs, the concept is applicable not only for more general finite element mesh of $M(n \times m)$ for $n \neq m$ but also for M whose ∂M configuration is complicated.

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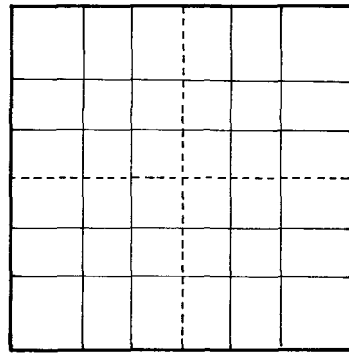
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Note for Fig.A : Dissection lines which are indicated as "thin lines", "thin dotted lines", "thick lines" and "thick dotted lines" appear in this ordering when the vertex eliminations succeeds for $M(n \times n)$.

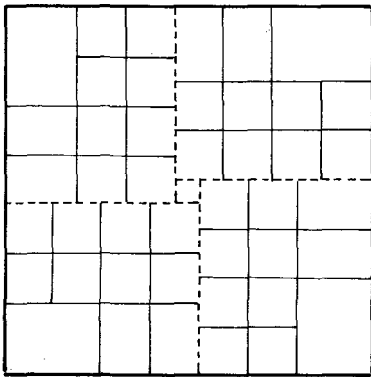
 $M(7 \times 7)$  $M(8 \times 8)$  $M(9 \times 9)$  $M(10 \times 10)$  $M(11 \times 11)$  $M(12 \times 12)$ Fig. A Dissection Lines for $M(n \times n)$



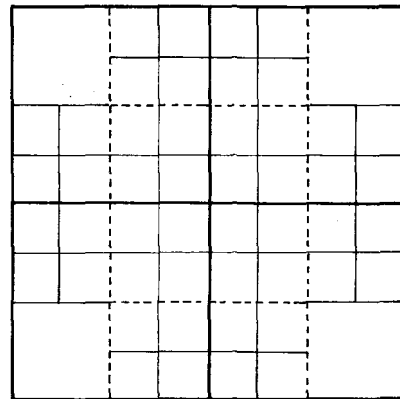
$M(13 \times 13)$



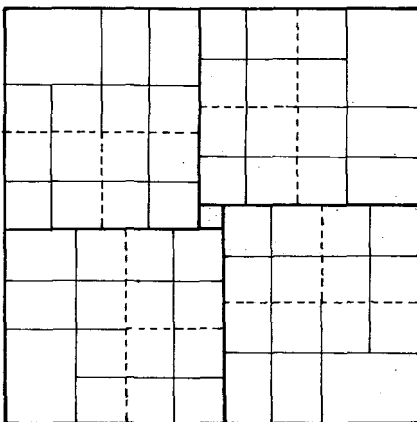
$M(14 \times 14)$



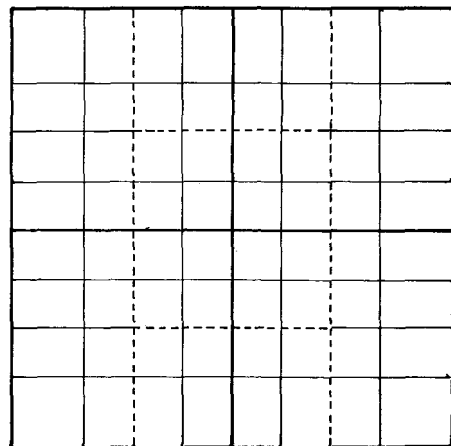
$M(15 \times 15)$



$M(16 \times 16)$

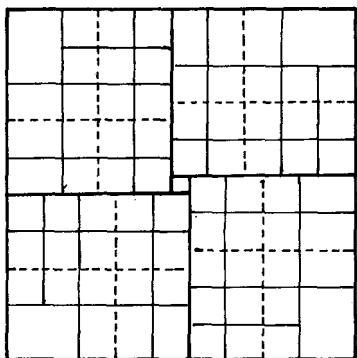
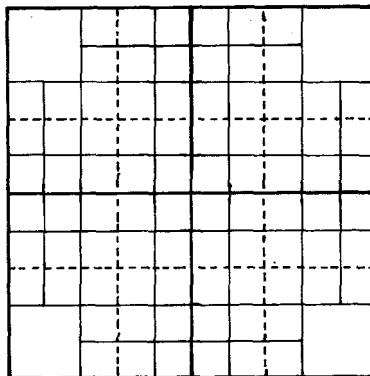
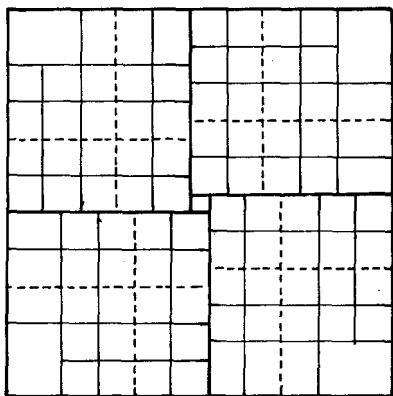
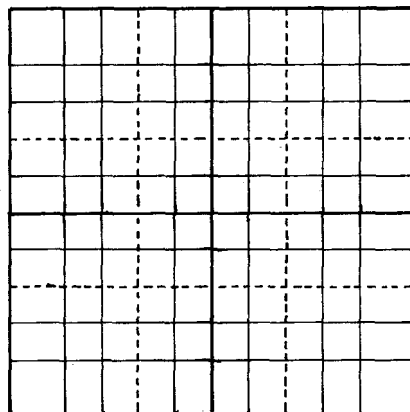
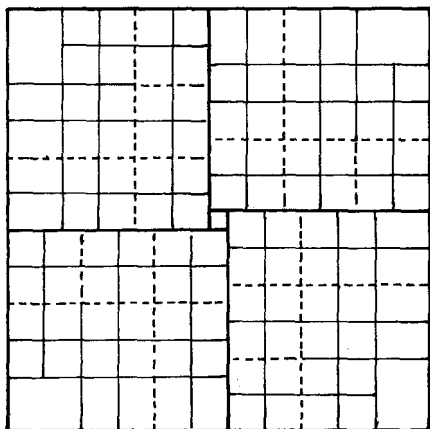
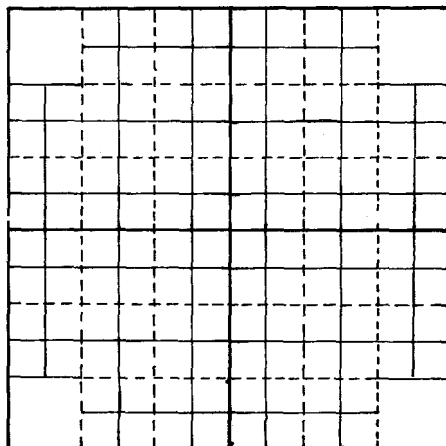


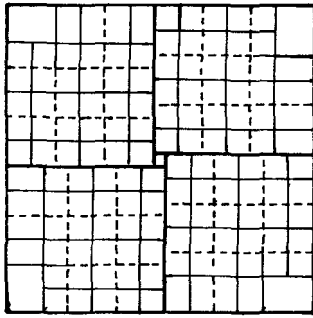
$M(17 \times 17)$



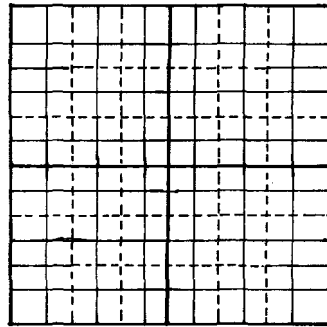
$M(18 \times 18)$

Fig. A Dissection Lines for $M(n \times n)$

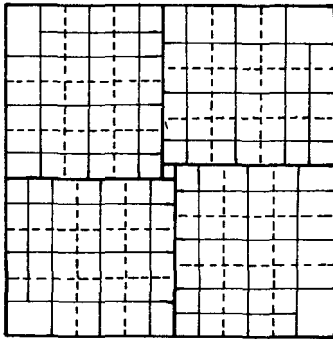
 $M(19 \times 19)$  $M(20 \times 20)$  $M(21 \times 21)$  $M(22 \times 22)$  $M(23 \times 23)$  $M(24 \times 24)$ Fig. A Dissection Lines for $M(n \times n)$



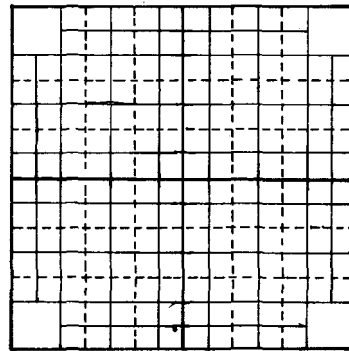
$M(25 \times 25)$



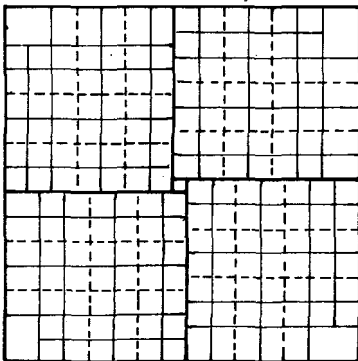
$M(26 \times 26)$



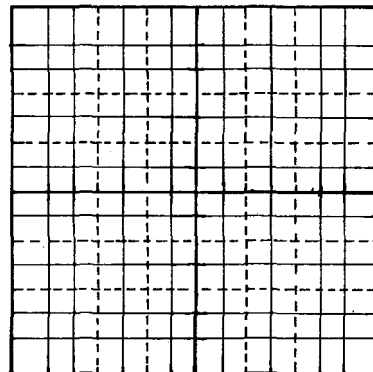
$M(27 \times 27)$



$M(28 \times 28)$

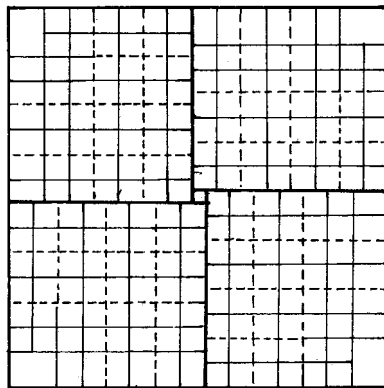
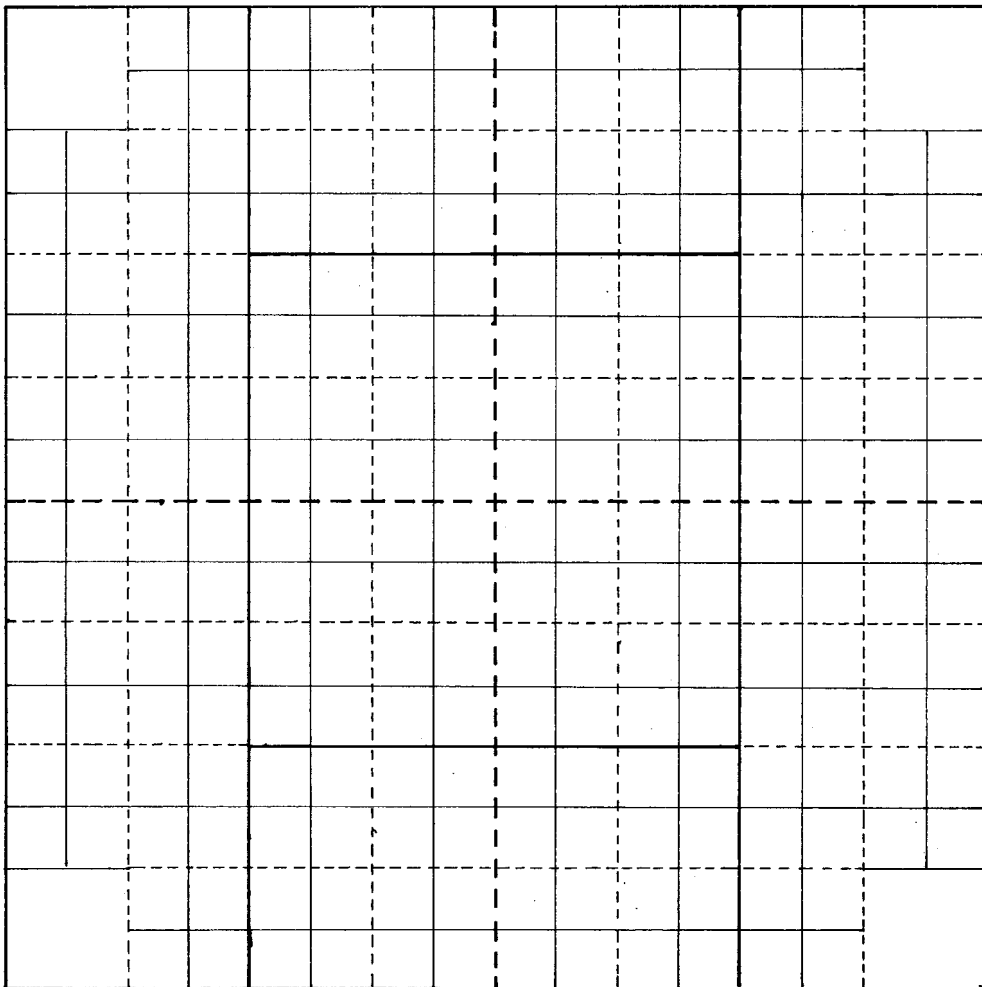


$M(29 \times 29)$



$M(30 \times 30)$

Fig. A Dissection Lines for $M(n \times n)$

 $M(31 \times 31)$  $M(32 \times 32)$ Fig. A Dissection Lines for $M(n \times n)$