A Multi-Sector Goodwin Model with n Kinds of Capital Goods

Ken-ichi ISHIYAMA

1. Introduction

In a work concerning stability analysis, Burmeister, Dobell, and Kuga(1968) showed the properties of Solow type growth model with many capital goods. They suggested that the one-sector Solow model is stable not because it has only one capital good, but because it has a particularly simple saving function. In the analysis, they proved the global stability of the multi-sector Solow model. The model is, however, unrealistic due to their neoclassical framework in which full employment is permanently kept. In this note, we allow for the existence of unemployment in their model. We substitute linear functions for Cobb-Douglas production functions. Also we introduce some elements of Goodwin's model (Goodwin (1965)), thus generalizing Sato's model in Sato (1985). The purpose of this paper is to analyze of the stability of a thus extended Goodwin model. We consider the relationship between the number of capital goods and the stability of the system.

This paper is organized as follows. In section 2, we propose an extended Goodwin model and show the existence of a positive equilibrium. In section 3 concerning a neighbourhood of the equilibrium the property of three dimensional case is compared with that of two dimensional case. Section 4 is devoted to a couple of analyses by computer simulations. Some comments are given in the final section.

2. The model

The notation and assumptions are as follows:

- Y_j denotes the output flow of the j-th commodity, with Y_0 designating the consumption good; $Y_1,\cdots,Y_n$ the capital goods;
- L_j denotes the labour input into sector j, $j=0,1,\cdots,n$;
- K_{ij} denotes the input of service of capital good i into sector j, $i=1,2,\cdots,n$; $j=0,1,\cdots,n$;
- L denotes the aggregated labour demand;
- N denotes the labour supply available;
- R denotes the real wage rate measured in the consumption good;

v denotes the employment ratio L/N.

(A1) In our economy, there are one consumption good and n capital goods. With no joint production we assume a linear production function for the j-th industry as follows.

$$Y_j = a_{0j} L_j = a_{1j} K_{1j} = \cdots = a_{nj} K_{nj} \ (j=0,1,\cdots,n), \tag{1}$$

where it is assumed that the parameters a_{ij} are positive constants and smaller than unity.

(A2) As in Sato(1985), we assume the real wage rate at period t, $R(t)$, is determined by the following difference equation:

$$R(t+1) = \{1 + c_1(v(t) - c_2)\} R(t), \tag{2}$$

where parameters c_1 and c_2 are positive constants, and the latter is smaller than unity. This equation reminds us of the Phillips Curve.[1]

(A3) Labour supply is greater than or equal to its demand[2].

(A4) Labour supply available is assumed to grow exogenously at a constant rate c_3.

(A5) We assume the classical saving function, that is, all profits are saved and automatically invested[3], while all wages are consumed.

(A6) We permit the capital mobility between industries. All capital goods are allocated so that they are utilized normally. It means that the economy is at an optimal position.

(A7) We ignore capital depreciation in order to simplify the discussion.

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[1] See Phillips(1958).

[2] Later we deal with the dynamics when the excess demand for labour may become positive.

[3] All capital goods produced in the current period are invested at the beginning of the next period.

From (A4) The dynamics of labour supply is determined as follows:

$$N(t+1)=(1+c_3)N(t). \tag{3}$$

From (A1) the aggregate labour demand and aggregate capital input are

$$L=\sum_{j=0}^{n} L_j, \text{ and} \tag{4}$$

$$K_i=\sum_{j=0}^{n} K_{ij} \quad (i=1,2,\cdots,n). \tag{5}$$

The condition that the demand for the consumption good is equal to its supply is represented by the equation

$$RL=Y_0. \tag{6}$$

From eq. (5) the equilibrium conditions for the capital good industries are

$$\sum_{j=0}^{n} K_{ij}=K_i \quad (i=1,2,\cdots,n).$$

Now, for reduction of dimension we rewrite the above conditions as follows:

$$\sum_{j=0}^{n} \frac{a_{1j}}{a_{ij}} K_{1j}=K_i\,. \tag{7}$$

Similarly, we rewrite eq. (6) as

$$R\sum_{j=0}^{n} \frac{a_{1j}}{a_{0j}} K_{1j}=a_{10}K_{10} \tag{6'}$$

Now we can show how the allocation of capital goods is determined in an arbitrary period when the real wage rate R and capital stocks K_i s are given:

$$\begin{bmatrix} R\frac{a_{10}}{a_{00}}-a_{10} & R\frac{a_{11}}{a_{01}} & \cdots & R\frac{a_{1n}}{a_{0n}} \\ \frac{a_{10}}{a_{10}} & \frac{a_{11}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{1n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{10}}{a_{n0}} & \frac{a_{11}}{a_{n1}} & \cdots & \frac{a_{1n}}{a_{nn}} \end{bmatrix} \begin{bmatrix} K_{10} \\ K_{11} \\ \vdots \\ K_{1n} \end{bmatrix} = \begin{bmatrix} 0 \\ K_1 \\ \vdots \\ K_n \end{bmatrix}. \tag{8}$$

On the other hand, from (A5) and (A7), the growth of capital stock K_i is represented as

$$K_i(t+1)=K_i(t)+a_{1i}K_{1i}, \tag{9}$$

where the set $(K_{10}, K_{11}, \cdots K_{1n})$ is the solution of eqs. (8) in which $R(t)$ and $K_i(t)$s are given.

From eqs. (2), (3), (5) and (9), the dynamics of our model is summarized as follows:

$$R(t+1)=R(t)+\alpha\left\{-c_1c_2R(t)+a_{10}c_1\left(k_1(t)-\sum_{i=1}^{n}k_{1i}(t)\right)\right\}, \tag{10}$$

$$k_i(t+1)=k_i(t)+\alpha\left(\frac{a_{1i}k_{1i}(t)-c_3k_i(t)}{1+c_3}\right), \tag{11}$$

where parameter α is a speed of adjustment and by definitions

$$k_i\equiv\frac{K_i}{N},\quad k_{1j}\equiv\frac{K_{1j}}{N}\qquad (i=1,2,\cdots,n,\ j=0,1,2,\cdots,n). \tag{12}$$

We reduced the dimension of the system to $n+1$.

The existence of a balanced growth path

Here we consider the equilibrium point of the system represented by eqs. (10) and (11). The system may have a positive equilibrium[1] at which the economy keeps on a balanced growth trajectory; that is, each variable grows with the same constant ratio.

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[1] Although the origin is also the equilibrium point, we ignore it because of the lack of its economic significance.

Now, we define

$$x_{1j} \equiv \frac{K_{1j}}{L}, \text{ and } x_i \equiv \frac{K_i}{L} \qquad (i=1,2,\cdots,n,\ j=0,1,\cdots,n)\ . \tag{13}$$

From eqs. (8) and (13), we obtain

$$\begin{bmatrix} \frac{a_{10}}{a_{00}} & \frac{a_{11}}{a_{01}} & \cdots & \frac{a_{1n}}{a_{0n}} \\ \frac{a_{10}}{a_{10}} & \frac{a_{11}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{1n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{10}}{a_{n0}} & \frac{a_{11}}{a_{n1}} & \cdots & \frac{a_{1n}}{a_{nn}} \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{11} \\ \vdots \\ x_{1n} \end{bmatrix} = \begin{bmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{bmatrix} \tag{14}$$

On a balanced growth trajectory, both the growth rate of employed labour and the rate of change in the capital stock are equivalent to the population growth rate c_3.

The following equations hold at the equilibrium point $E(R^*, x_1^*,\cdots,x_n^*)$.

$$a_{10}x_{10}^* = R^*, \text{ and} \tag{15}$$

$$a_{1i}x_{1i}^* = c_3 x_i^*\ . \tag{16}$$

Then R^* is determined by the n-dimensional vector $x^* \equiv (x_1^*, x_2^*,\cdots,x_n^*)$ as follows

$$R^* = a_{00} - c_3 a_{00} \sum_{i=1}^{n} \frac{1}{a_{0i}} x_i^*. \tag{17}$$

It is clear that if and only if the rate of population growth c_3 is sufficiently small (which, in the case of Sato's two-sector model, is smaller than the productivity of the capital of capital good sector.), the equilibrium wage rate R^* is positive.

From these equations, we can obtain the following system of simultaneous equations which determine the equilibrium vector x^*:

$$x_i^* = \frac{a_{00}}{a_{i0}} + \sum_{j=1}^{n} b_{ij} x_j^*, \tag{18}$$

where

$$b_{ij} \equiv c_3\left(\frac{1}{a_{ij}} - \frac{a_{00}}{a_{i0}\,a_{0j}}\right). \tag{19}$$

It should be noted that the parameter b_{ij} has the same sign as the formula

$$\frac{K_{ij}}{L_j} - \frac{K_{i0}}{L_0}.$$

That is, the parameter b_{ij} is positive if and only if concerning i-th capital the capital labour ratio of the j-th capital good industry is higher than that of consumption good industry.

From eqs. (17) and (18) it is clear that the equilibrium E is in the positive orthant if the parameters b_{ij} satisfy the following conditions:

$$\sum_{j=1}^{n}\left|b_{ij}\right| < 1. \tag{20}$$

In fact it is a simple application of the Brouwer fixed point theorem to prove the existence of an equilibrium for eq. (18), and the condition (20) represents an upper limit of the row sum norm shown by Solow(1952). In this paper we consider the economy in which (20) is satisfied.

3. Stability near the equilibrium

To analyze the effect of generalization of the model on the stability near the equilibrium without difficulty, we concentrate on the case in which $n=2$, $a_{11}=a_{12}=a_{21}\geq a_{22}$, $a_{01}=a_{02}$, $a_{10}=a_{20}$, and $\frac{a_{10}}{a_{00}}<\frac{a_{11}}{a_{01}}$. The last inequality means that if parameter a_{22} is also equal to a_{11}, a_{12}, and a_{21}, all the capital good sectors are less capital-intensive than the consumption good sector. When $a_{11}=a_{12}=a_{21}=a_{22}$, the system is locally stable. (See Sato (1985).)

Theorem:

If in one capital good industry the productivity of its capital is sufficiently smaller than any other capital productivity in both capital good sector, which are assumed to be identical, then the system is locally unstable.

Proof:

First we obtain a Jacobian matrix J from linear approximation.

$$J \equiv \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}. \quad (21)$$

It is clear that J_{13} and J_{31} are equal to zero from the conditions concerning parameters. The characteristic equation is as follows:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \quad (22)$$

where

$$a_1 \equiv -(J_{11} + J_{22} + J_{33}), \quad (23)$$

$$a_2 \equiv J_{11}J_{22} + J_{22}J_{33} + J_{33}J_{11} - J_{12}J_{21} - J_{23}J_{32}, \text{ and} \quad (24)$$

$$a_3 \equiv -J_{11}J_{22}J_{33} + J_{12}J_{21}J_{33} - J_{23}J_{32}J_{11}. \quad (25)$$

If the speed of adjustment is sufficiently small, the left hand side of eq. (22) in the case $\lambda = 1$ has the same sign as the following term:

$$c_2\{a_{10} + (d_2 - d_1)R^*\}^2 - a_{10}^2 d_2 k_1^*, \quad (26)$$

where

$$R^* = \frac{a_{10}c_2\left\{\left(\frac{a_{11}}{c_3} - 1\right)\left(\frac{a_{11}}{c_3} - d_3\right) - 1\right\}}{\left\{d_2 - d_1 + \frac{a_{11}d_1}{c_3}\right\}\left(1 + \frac{a_{11}}{c_3} - d_3\right) + \frac{a_{11}}{c_3}(d_2 - d_1)}, \quad (27)$$

$$K_1^* = \frac{a_{11}c_2\left(1 + \frac{a_{11}}{c_3} - d_3\right)}{c_3\left[\left\{d_2 - d_1 + \frac{a_{11}d_1}{c_3}\right\}\left(1 + \frac{a_{11}}{c_3} - d_3\right) + \frac{a_{11}}{c_3}(d_2 - d_1)\right]}, \quad (28)$$

$$d_1 \equiv \frac{a_{10}}{a_{00}}, \ d_2 \equiv \frac{a_{11}}{a_{01}} = \frac{a_{12}}{a_{02}}, \ d_3 \equiv \frac{a_{12}}{a_{22}}, \text{ and } d_2 > d_1.$$

From eqs. (27) and (28),

$$\frac{\partial k_1^*}{\partial d_3} = -\frac{a_{11}^2 c_2 (d_2 - d_1)}{c_3^2 \xi} < 0, \text{ and} \tag{29}$$

$$\frac{\partial R^*}{\partial d_3} = -\frac{a_{10}\left(d_2 - d_1 + \frac{a_{11} d_1}{c_3}\right)}{\xi} < 0, \tag{30}$$

where

$$\xi \equiv \left[\left\{ d_2 - d_1 + \frac{a_{11} d_1}{c_3} \right\} \left(1 + \frac{a_{11}}{c_3} - d_3 \right) + \frac{a_{11}}{c_3} (d_2 - d_1) \right]^2. \tag{31}$$

Thus when a_{22} is sufficiently small, or d_3 is sufficiently large, the following condition holds:

$$c_2 \{a_{10} + (d_2 - d_1) R^*\}^2 - a_{10}^2 d_2 k_1^* > 0. \tag{32}$$

This means that the characteristic equation (22) has a real root which is smaller than unity. Therefore the system is locally unstable.

Q.E.D.

4. Simulations

Now we make a comparison between the stability of Sato's model and that of our multi-sector economy. (We define the stability as the property that the economy approaches or reaches the equilibrium in the long run.) In this section we use computer simulations. It can exhibit not only the local stability but also the behaviour in the large.

Here we verify the result of the last section and examine in detail. Simulations are executed repeatedly changing the values of parameters, where parameters a_{11}, a_{12}, a_{21}, and a_{22} are variable and others are fixed. The augmentations of these four parameters make the corresponding capital good sector less capita-intensive. Formulas $a_{0j}/a_{ij} = K_{ij}/L_j$ $(i=1,2; j=0,1,2)$ stand for these intensities corresponding to sector j. Again we assume that fixed parameters a_{10} and a_{20} are equal; a_{01} and a_{02} are also identical. From these assumptions, if four variable parameters are identical, two capital goods are homogeneous.

In addition we restrict the values of parameters considered with certain ranges, because extremely small or large values may be economically meaningless. Hence the set of the possible combinations of the parameters can be represented as a hyper-cube in R^{4}_{++}. It should be noted that if a combination of parameters is on the diagonal of the hyper-cube which links the lowest and highest point, then the economy can be reduced to the two-sector economy since the heterogeneity of capital goods has disappeared.

Positive excess demand in a capital goods market may occur in the case of the heterogeneity of capital good. In other words, the solution of eq. (8) is not always non-negative. (It may occur when the economy is far away from the equilibrium point E.) In this case the allocation of capital goods is determined so that the goods which is short is preferentially produced. The allocations corresponding to possible three cases are illustrated by Fig. 1, Fig. 2, and Fig. 3 respectively.

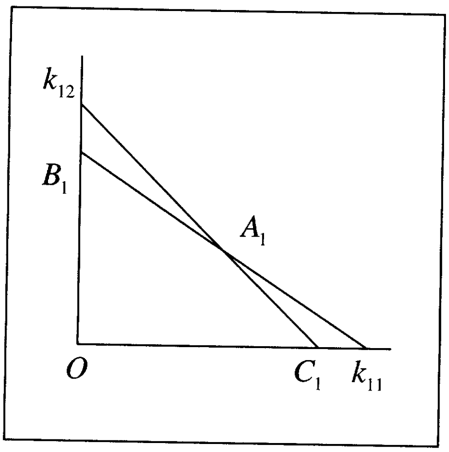

Fig. 1

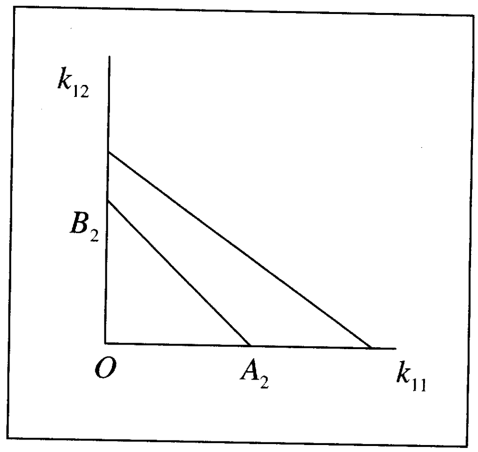

Fig. 2

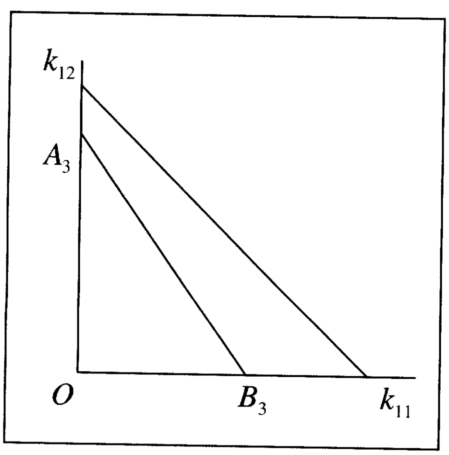

Fig. 3

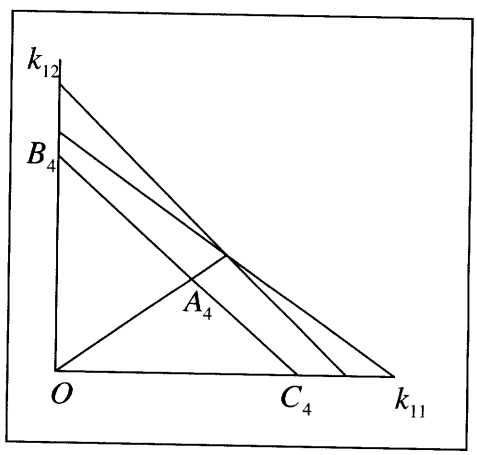

Fig. 4

Formulas of two lines in each figure are derived from eq. (8) as follows:

$$k_{12}=-k_{11}+\frac{(a_{10}-d_1R)k_1}{a_{10}+(d_2-d_1)R}, \text{ and} \quad (33)$$

$$k_{12}=-\frac{a_{10}+(d_2-d_1)R}{a_{10}d_3+(d_2-d_1d_3)R}+\frac{(a_{10}-d_1R)k_2}{a_{10}d_3+(d_2-d_1d_3)R}. \quad (34)$$

Eq. (33) is the condition of the normal utilization of K_1, and eq. (34) is that of K_2. And the production possibility boundaries in three cases are respectively represented as $OB_1A_1C_1$, OA_2B_2, and OA_3B_3. Paying attention to the point that the slope of (33) is always higher than that of (34) in modulus, we can understand that possible allocation is A_1, A_2, or A_3. (If the number of capital goods is greater than two, this problem becomes more difficult.)

On the other hand, the full employment ceiling, which is drawn as the line B_4C_4 in Fig. 4, may cut these boundaries. If excess demand for labour is positive, then there are idle capacities with each capital. In Fig. 4 the allocation of capital 1 is at the point A_4.

The results of simulations are shown in Table 1.

We consider two different hyper-cubes in the parameter space to make it clear how the stable area occupies the space. Table 1 displays proportions of the stable domain to the total in percentages, concerning both the whole cube and the diagonal. Table2 shows the coordinates and the sizes of those cubes. Other parameters are given as $c_1=0.2289$, $c_2=0.9462$, $c_3=0.04$, and the speed of adjustment $\alpha=0.1$.

In case 1, all the points on the diagonal satisfy the local stability condition. On the other hand, in case 2 we can see the stable area spreads from the lowest to the highest point. The most important point in these results is that if two kinds of capital goods are heterogeneous, the system is likely to be more unstable than the case in which capital goods are homogeneous.

	Stability		Fixed Parameters		
	to hyper-cube	to diagonal	a00	a01(=a02)	a10(=a20)
Case 1	65.20	100.00	1.35	2.13	0.29
Case 2	7.02	30.00	1.06	2.70	0.29

Table 1

	Lowest point	Highest point	Size
	a11(=a12=a21=a22)	a11(=a12=a21=a22)	
Case1	0.6200	0.8641	1000
Case2	0.6200	1.1083	8000

Table 2

5. A temporary summary

We have generalized Goodwin's model by introducing a plural number of capital goods as Burmeister Dobell, and Kuga(1968) extended Solow's model. (See Solow(1952).) And we examine the stability of the system by using computer simulations. Especially, our interest is in a stability condition which is found by Shinkai(1960) or Uzawa(1961) . That is, the economy is always stable if the consumption-goods sector is more capital-intensive than the investment-goods sector. Sato (1985) showed that it is also applicable to a two-sector Goodwin model. Our result is that even if not all the capital-goods sectors are less capital-intensive than the consumption-goods sector, the economy can be stable. However, in general the heterogeneity of capital affects the stability of the system. Thus we obtain a different result from those neoclassical models. In addition we showed that in our model the balanced growth path as an equilibrium can be economically meaningful if and only if the rate of population growth is sufficiently small.

Our next purpose is to analyze higher dimensional models, and to examine the relationships between the dimension and the stability in a more detailed way.

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