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THE d-SMITH SETS OF DIRECT PRODUCTS OF DIHEDRAL GROUPS

Kohei Seita

Abstract. Let G be a finite group and let V and W be real G-modules. We call V and W *dim-equivalent* if for each subgroup H of G, the H-fixed point sets of V and W have the same dimension. We call V and W are *Smith equivalent* if there is a smooth G-action on a homotopy sphere Σ with exactly two G-fixed points, say a and b, such that the tangential G-representations at a and b of Σ are respectively isomorphic to V and W. Moreover, We call V and W are *d-Smith equivalent* if they are dim-equivalent and Smith equivalent. The differences of d-Smith equivalent real G-modules make up a subset, called the *d-Smith set*, of the real representation ring $\mathrm{RO}(G)$. We call V and W $\mathcal{P}(G)$*-matched* if they are isomorphic whenever the actions are restricted to subgroups with prime power order of G. Let N be a normal subgroup. For a subset $\mathcal{F}$ of G, we say that a real G-module is $\mathcal{F}$*-free* if the H-fixed point set of the G-module is trivial for all elements H of $\mathcal{F}$. We study the d-Smith set by means of the submodule of $\mathrm{RO}(G)$ consisting of the differences of dim-equivalent, $\mathcal{P}(G)$-matched, $\{N\}$-free real G-modules. In particular, we give a rank formula for the submodule in order to see how the d-Smith set is large.

1. Introduction

Throughout this paper, let G be a finite group and N a normal subgroup of G. Let $\mathcal{S}(G)$, $\mathrm{R}_{\mathbb{Q}}(G)$, $\mathrm{RO}(G)$ and $\mathrm{R}(G)$ denote the set of all subgroups, the rational representation ring, the real representation ring, and the complex representation ring, respectively, of G. We mean by a *real G-module* a real G-representation space of finite dimension. By canonical homomorphisms, we regard

$$\mathrm{R}_{\mathbb{Q}}(G) \subset \mathrm{RO}(G) \subset \mathrm{R}(G).$$

Real G-modules V and W are called *dim-equivalent* if $\dim V^H = \dim W^H$ holds for any subgroup H of G. Real G-modules V and W are called *Smith equivalent* and written $V \sim_{\mathfrak{S}} W$ if there exists a homotopy sphere Σ with a smooth G-action such that $\Sigma^G = \{a, b\}$ $(a \neq b)$, $T_a(\Sigma) \cong V$ and $T_b(\Sigma) \cong W$ (as real G-modules). Moreover, real G-modules V and W are called *d-Smith*

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equivalent and written $V \sim_{\mathfrak{d}\mathfrak{S}} W$ if V and W are Smith equivalent and dim-equivalent. Define the Smith set $\mathfrak{S}(G)$ and the d-Smith set $\mathfrak{d}\mathfrak{S}(G)$ by

$$\mathfrak{S}(G) = \{[V] - [W] \in \mathrm{RO}(G) \mid V \sim_{\mathfrak{S}} W\},$$
$$\mathfrak{d}\mathfrak{S}(G) = \{[V] - [W] \in \mathrm{RO}(G) \mid V \sim_{\mathfrak{d}\mathfrak{S}} W\}.$$

In 1960, P. A. Smith [14] asked the next question. If there exists a smooth G-action on a sphere S such that $S^G = \{a, b\}$, then are the tangent spaces $T_a(S)$ and $T_b(S)$ isomorphic? It is an interesting research subject whether $\mathfrak{S}(G)$ is 0 or not. Since this problem was proposed, it has been studied by various researchers. Let C_n, A_n, and S_n denote a cyclic group of order n, the alternating group of degree n, and the symmetric group of degree n, respectively. The following affirmative results are known. M. F. Atiyah–R. Bott [1] proved $\mathfrak{S}(C_p) = 0$ for any prime p. C. U. Sanchez [13] proved $\mathfrak{S}(C_{p^k}) = 0$ for any odd prime p and any integer $k \geq 1$. It is known that $\mathfrak{S}(G) = 0$ for each $G = A_n$, S_n with $n \leq 5$, (cf. [5], [9]). On the other hand, the following negative results are known. T. Petrie [10, 11, 12] proved $\mathfrak{S}(G) \neq 0$ for abelian groups G having at least 4 noncyclic Sylow subgroups. S. E. Cappel–J. L. Shaneson [2] proved $\mathfrak{S}(C_{4k}) \neq 0$ for any integer $k \geq 2$. X. -M. Ju [4] proved that neither $\mathfrak{S}(A_5 \times C_2^n)$ nor $\mathfrak{S}(S_5 \times C_2^n)$ is 0 for any integer $n \geq 1$, where $C_2^n = C_2 \times \cdots \times C_2$ (n-fold). For $A \subset \mathrm{RO}(G)$ and $\mathcal{F}$, $\mathcal{G} \subset \mathcal{S}(G)$, we set

$$A^{\mathcal{F}} = \{[V] - [W] \in A \mid V^H = W^H = 0 \text{ for all } H \in \mathcal{F}\},$$
$$A_{\mathcal{G}} = \{[V] - [W] \in A \mid \mathrm{res}^G_K V \cong \mathrm{res}^G_K W \text{ for all } K \in \mathcal{G}\},$$
$$A^{\mathcal{F}}_{\mathcal{G}} = (A^{\mathcal{F}})_{\mathcal{G}}.$$

A real G-module V is called *$\mathcal{F}$-free* if $V^H = 0$ for all $H \in \mathcal{F}$. Real G-modules V and W are called *$\mathcal{G}$-matched* if $\mathrm{res}^G_K V \cong \mathrm{res}^G_K W$ for all $K \in \mathcal{G}$. We use the following notation.

E : the trivial group.

$\mathcal{C}(G)$: the set of all cyclic subgroups of G.

$\mathcal{P}(G)$: the set of all subgroups of G of prime power order.

$\mathcal{P}_{\mathrm{odd}}(G)$: the set of all $P \in \mathcal{S}(G)$ of odd prime power order.

$G^{\{p\}}$: the smallest normal subgroup $H \leq G$ such that $|G/H|$ is a power of p (p a prime).

$\mathcal{L}(G)$: the set of all $H \in \mathcal{S}(G)$ such that $H \supset G^{\{p\}}$ for some prime p.

G^{nil} : the smallest normal subgroup $H \leq G$ such that G/H is nilpotent.

$G^{\cap 2}$: the intersection of all normal subgroups H of G such that $|G/H| \leq 2$.

It is known that $G^{\rm nil} = \bigcap_p G^{\{p\}}$ where p runs over the set of all primes dividing $|G|$. Let $\mathrm{RO}_0(G)$ denote the set of all $[V]-[W] \in \mathrm{RO}(G)$ such that V and W are dim-equivalent. $\mathrm{RO}_0(G)$ is a $\mathbb{Z}$-submodule of $\mathrm{RO}(G)$. We note that if $G^{\rm nil} = G^{\{p\}}$ for some prime p, then

$$\mathrm{R}_{\mathbb{Q}}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} = \mathrm{R}_{\mathbb{Q}}(G)^{\{G^{\{p\}}\}}_{\mathcal{P}(G)} \quad \text{and} \quad \mathrm{RO}_0(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} = \mathrm{RO}_0(G)^{\{G^{\{p\}}\}}_{\mathcal{P}(G)}.$$

A finite group G is called an *Oliver group* if there never exists a normal series $P \trianglelefteq H \trianglelefteq G$ such that $P \in \mathcal{P}(G)$, H/P is cyclic, and G/H is of prime power order. For $g \in G$, the real conjugacy class $(g)^{\pm}$ is defined to be the set $(g)\cup(g^{-1})$, where $(g) = \{xgx^{-1} | x \in G\}$. For $H \in \mathcal{S}(G)$, let $(H)_G$ denote the G-conjugacy class of H. Let $\lambda(G, N)$ denote the number of all real conjugacy classes $(gN)^{\pm}$ such that g is an element of G not of prime power order, and let $\nu(G, N)$ denote the number of all G/N-conjugacy classes $(HN/N)_{G/N}$ for all cyclic subgroups H of G not of prime power order.

Theorem 1.1. *Let G be a finite group containing an element not of prime power order. Then, the $\mathbb{Z}$-rank of $\mathrm{R}_{\mathbb{Q}}(G)^{\{N\}}_{\mathcal{P}(G)}$ is equal to $\nu(G, E) - \nu(G, N)$.*

Corollary 1.2. *Let G be a finite group containing an element not of prime power order. Then the inequalities*

$$\nu(G, E) - \nu(G, G^{\rm nil}) \le \mathrm{rank}_{\mathbb{Z}} \mathrm{R}_{\mathbb{Q}}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \le \nu(G, E) - \max_{p:prime} \{\nu(G, G^{\{p\}})\}$$

hold.

Let $\overline{\mathrm{RO}}_{\mathbb{Q}}(G)$ (resp. $\overline{\mathrm{R}}_{\mathbb{Q}}(G)$) denote the submodule of $\mathrm{RO}(G)$ (resp. $\mathrm{R}(G)$) consisting of $x \in \mathrm{RO}(G)$ (resp. $x \in \mathrm{R}(G)$) such that $nx \in \mathrm{R}_{\mathbb{Q}}(G)$ for some $n \in \mathbb{N}$. Let $\mu(G, N)$ denote the $\mathbb{Z}$-rank of $\mathrm{RO}_0(G)^{\{N\}}_{\mathcal{P}(G)}$.

Theorem 1.3. *Let G be a finite group containing an element not of prime power order. Then, $\mu(G, N)$ is equal to $(\lambda(G, E) - \lambda(G, N)) - (\nu(G, E) - \nu(G, N))$.*

We remark that for an arbitrary Oliver group G, the inequality

$$\lambda(G, E) - \lambda(G, G^{\rm nil}) > \nu(G, E) - \nu(G, G^{\rm nil})$$

holds if and only if $\mathfrak{d}\mathfrak{S}(G)^{\{G^{\rm nil}\}}_{\mathcal{P}(G)}$ is an infinite set.

Corollary 1.4. *Let G be a finite group containing an element not of prime power order. Then the inequalities*

$$\mu(G, G^{\rm nil}) \le \mathrm{rank}_{\mathbb{Z}} \mathrm{RO}_0(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \le \min_{p:prime} \{\mu(G, G^{\{p\}})\}$$

hold.

For a natural number u, let D_{2u} denote the dihedral group of order $2u$, i.e.

$$D_{2u} = \langle x,\, y \mid x^u,\, y^2,\, yxyx \rangle.$$

Throughout this paper, let m be a natural number with $m \geq 2$, and let $p_1, p_2, \ldots, p_m$ be distinct odd primes.

Theorem 1.5. *Let G be the group $D_{2u} \times D_{2u}$ with $u = p_1 p_2 \cdots p_m$, where $m \geq 2$. Then, $\mathfrak{dS}(G)$ coincides with $\mathrm{RO}_0(G)^{\{G^{\mathrm{nil}}\}}_{\mathcal{P}(G)}$ and the $\mathbb{Z}$-rank of $\mathrm{RO}_0(G)^{\{G^{\mathrm{nil}}\}}_{\mathcal{P}(G)}$ is equal to*

$$\left(\frac{p_1 p_2 \cdots p_m + 3}{2}\right)^2 - \sum_{i=1}^{m} \frac{p_i^2 - 9}{4} - \sum_{k=1}^{m} \frac{3^{m-k}}{2} \sum_{1 \leq t_1 < \cdots < t_k \leq m} \prod_{i=1}^{k} (p_{t_i} - 1) - 3^m - 2^{m+1} - 1.$$

Theorem 1.6. *Let G be the group $D^n_{2p_1p_2}$ for distinct odd primes p_1, p_2 and a natural number n with $n \geq 2$. Then, the following holds.*

(1) *$\mathfrak{dS}(G)$ coincides with $\mathrm{RO}_0(G)^{\{G^{\mathrm{nil}}\}}_{\mathcal{P}(G)}$, and the $\mathbb{Z}$-rank of $\mathrm{RO}_0(G)^{\{G^{\mathrm{nil}}\}}_{\mathcal{P}(G)}$ is equal to $\lambda(G, E) - \nu(G, E)$.*

(2) $\lambda(G, E) = \left(\frac{p_1 p_2 + 3}{2}\right)^n - \left(\frac{p_1 + 1}{2}\right)^n - \left(\frac{p_2 + 1}{2}\right)^n - 2^n + 2.$

(3)

$$\begin{aligned}\nu(G, E) = &\sum_{i=1}^{2} \frac{2}{p_i - 1} \left(\left(\frac{p_i + 3}{2}\right)^n - \left(\frac{p_i + 1}{2}\right)^n - 2^n + 1 \right) \\ &+ \frac{4}{(p_1 - 1)(p_2 - 1)} \left(2 \left(\frac{p_1 p_2 + 3}{2}\right)^n - \left(\frac{p_1 + p_2 + 2}{2}\right)^n \right. \\ &\left. - \left(\frac{p_1 + 3}{2}\right)^n - \left(\frac{p_2 + 3}{2}\right)^n + 2^n \right)\end{aligned}$$

2. Proof of Theorem 1.1

For $g \in G$, let $\langle g \rangle$ denote the cyclic subgroup of G generated by g. For a G-conjugation invariant subset A of G, let $\mathfrak{M}(G, A)$ denote the set of all G-conjugation invariant functions $f : A \to \mathbb{Q}$ such that $f(a) = f(b)$ for elements a and b of A satisfying $\langle a \rangle = \langle b \rangle$. Let $\mathfrak{M}(G, A)_{\mathcal{P}(G)}$ denote the kernel of $\mathrm{res}^G_{\mathcal{P}(G)} : \mathfrak{M}(G, A) \to \prod_{P \in \mathcal{P}(G)} \mathfrak{M}(P, A)$. The homomorphism $\mathrm{fix}^G_{G/N} : \mathfrak{M}(G, A) \to \mathfrak{M}(G/N, AN/N)$ is defined by

$$\left(\mathrm{fix}^G_{G/N}\right) f(aN) = \frac{1}{|N|} \sum_{x \in N} f(ax)$$

for $f \in \mathfrak{M}(G, A)$ and $a \in A$. Let $\mathfrak{M}(G, A)^{\{N\}}$ denote the kernel of $\mathrm{fix}^G_{G/N} : \mathfrak{M}(G, A) \to \mathfrak{M}(G/N, AN/N)$. For $C \in \mathcal{C}(G)$, we have the associated map $f_C : G \to \mathbb{Q}$ by

$$f_C(g) = \begin{cases} 1 & (\langle g \rangle \in (C)_G) \\ 0 & (\langle g \rangle \notin (C)_G) \end{cases}$$

for $g \in G$.

Proposition 2.1. *For $a \in G$ and $C \in \mathcal{C}(G)$, the value $\mathrm{fix}^G_{G/N} f_C(aN)$ is positive if and only if the cyclic subgroup $\langle aN \rangle$ of G/N is G/N-conjugate to the cyclic group CN/N.*

Proof. We have

$$\begin{aligned} |N| \mathrm{fix}^G_{G/N} f(aN) &= \sum_{x \in N} f_C(ax) \\ &= |\{x \in N \mid \langle ax \rangle \in (C)_G\}| \\ &= \left| \left(\bigcup_{g \in G} gCg^{-1} \right) \cap aN \right|. \end{aligned}$$

The set $\left(\bigcup_{g \in G} gCg^{-1}\right) \cap aN$ is not empty if and only if $(C)_G \cap aN$ is not empty. $(C)_G \cap aN$ is not empty if and only if $C \cap (aN)_G$ is not empty. The set $C \cap (aN)_G$ is not empty if and only if C is a cyclic group with $gabg^{-1}$ as a generator for some $b \in N$ and $g \in G$. □

For a G-representation space V, let $\rho_V : G \to \mathrm{Aut}(V)$ be the homomorphism associated with V, and let χ_V denote the character of ρ_V. For any G-representation space V, define the homomorphism $\rho_{V^N} : G/N \to \mathrm{Aut}(V^N)$ by $\rho_{V^N}(aN) = \rho_V(a)|_{V^N}$ for $a \in G$. Then, the following fact is obtained from [9, p. 857].

Lemma 2.2. *For $g \in G$, $\chi_{V^N}(gN)$ is equal to*

$$\frac{1}{|N|} \sum_{x \in N} \chi_V(gx).$$

Let $Q(G)$ denote the set of all elements of G of prime power order. By Lemma 2.2, the diagram

$$\begin{array}{ccc} \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{R}_{\mathbb{Q}}(G)_{\mathcal{P}(G)} & \xrightarrow{\mathrm{fix}^G_{G/N}} & \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{R}_{\mathbb{Q}}(G/N) \\ \downarrow \tau_G & & \downarrow \tau_{G/N} \\ \mathfrak{M}(G, G \setminus Q(G)) & \xrightarrow{\mathrm{fix}^G_{G/N}} & \mathfrak{M}(G/N, (G \setminus Q(G))N/N) \end{array}$$

commutes, where the homomorphisms τ_G and $\mathrm{fix}^G_{G/N} : \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{R}_{\mathbb{Q}}(G)_{\mathcal{P}(G)} \to \mathbb{Q}\otimes_{\mathbb{Z}}\mathrm{R}_{\mathbb{Q}}(G/N)$ are defined by $\tau_G\left(\sum_i (r_i \otimes [V_i])\right) = \sum_i r_i\chi_{V_i}$ and $\mathrm{fix}^G_{G/N}\left(\sum_i (r_i \otimes [V_i])\right) = \sum_i (r_i \otimes [V_i^N])$ for all non-isomorphic irreducible G-representation spaces V_i and $r_i \in \mathbb{Q}$, respectively.

Proposition 2.3. *The $\mathbb{Q}$-vector space $\mathfrak{M}(G,G)_{\mathcal{P}(G)}$ is canonically identified with $\mathfrak{M}(G, G \setminus Q(G))$, and the homomorphisms τ_G and $\tau_{G/N}$ are isomorphisms.*

Proof. The map $\mathfrak{M}(G,G)_{\mathcal{P}(G)} \to \mathfrak{M}(G, G \setminus Q(G))$ which is defined by $f \mapsto f|_{G\setminus Q(G)}$ is injective. Additionally, The map $\mathfrak{M}(G, G\setminus Q(G)) \to \mathfrak{M}(G,G)_{\mathcal{P}(G)}$ which is defined by

$$h \longmapsto \bar{h}\ ;\ \bar{h}(x) = \begin{cases} h(x) & (x \in G \setminus Q(G)) \\ 0 & (x \in Q(G)) \end{cases}$$

is injective. Hence $\mathfrak{M}(G,G)_{\mathcal{P}(G)} = \mathfrak{M}(G, G \setminus Q(G))$. For real G-modules V, W, $[V] = [W]$ if and only if $\chi_V = \chi_W$. Therefore, the homomorphisms τ_G and $\tau_{G/N}$ are isomorphisms. □

Let $\mathrm{Conj}(G,\mathcal{C})$ denote the set of all G-conjugacy classes of cyclic subgroups of G, and let $\mathrm{Conj}(G,\mathcal{C}_{\mathcal{P}})$ denote the set of all $(C)_G \in \mathrm{Conj}(G,\mathcal{C})$ such that C is a cyclic subgroup of prime power order.

Proposition 2.4. *Let G be a finite group containing an element not of prime power order. Then, the $\mathbb{Z}$-rank of $\mathrm{R}_{\mathbb{Q}}(G)_{\mathcal{P}(G)}$ is equal to $\nu(G,E)$.*

Proof. We have the exact sequence

$$0 \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{R}_{\mathbb{Q}}(G)_{\mathcal{P}(G)} \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{R}_{\mathbb{Q}}(G) \xrightarrow{\mathrm{res}^G_{\mathcal{P}(G)}} \prod_{P\in\mathcal{P}(G)} \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{R}_{\mathbb{Q}}(P).$$

Set $\mathrm{Conj}(G,\mathcal{C}) = \{(H_1)_G, (H_2)_G, \ldots, (H_t)_G\}$. For $i = 1, 2, \ldots, t$, define the map $\varphi_i : \mathrm{Conj}(G,\mathcal{C}) \to \mathbb{Q}$ by $\varphi_i((H_j)_G) = \delta_{ij}$ where δ_{ij} is the Kronecker delta. Since $\mathrm{Map}(\mathrm{Conj}(G,\mathcal{C}),\mathbb{Q})$ and $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{R}_{\mathbb{Q}}(G)$ are isomorphic and $\{\varphi_i \mid (H_i)_G \in \mathrm{Conj}(G,\mathcal{C})\}$ is a basis of $\mathrm{Map}(\mathrm{Conj}(G,\mathcal{C}),\mathbb{Q})$, we have $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{R}_{\mathbb{Q}}(G)) = |\mathrm{Conj}(G,\mathcal{C})|$. Since $\{\mathrm{res}^G_{\mathcal{P}(G)}\varphi_i | (H_i)_G \in \mathrm{Conj}(G,\mathcal{C}_{\mathcal{P}})\}$ is linearly independent, we have $\dim_{\mathbb{Q}} \mathrm{Im}(\mathrm{res}^G_{\mathcal{P}(G)}) = |\mathrm{Conj}(G,\mathcal{C}_{\mathcal{P}})|$. Therefore,

$$\begin{aligned}\mathrm{rank}_{\mathbb{Z}}\mathrm{R}_{\mathbb{Q}}(G)_{\mathcal{P}(G)} &= \dim_{\mathbb{Q}}\left(\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{R}_{\mathbb{Q}}(G)_{\mathcal{P}(G)}\right) \\ &= |\mathrm{Conj}(G,\mathcal{C})| - |\mathrm{Conj}(G,\mathcal{C}_{\mathcal{P}})| \\ &= \nu(G,E).\end{aligned}$$

□

Proposition 2.5. *The set* $\{f_C \mid (C)_G \in \mathrm{Conj}(G,\mathcal{C}) \setminus \mathrm{Conj}(G,\mathcal{C}_{\mathcal{P}})\}$ *(resp.* $\{f_D \mid (D)_{G/N} \in \mathrm{Conj}(G/N,\mathcal{C})\}$*) is a basis of the* $\mathbb{Q}$*-vector space* $\mathfrak{M}(G, G \setminus Q(G))$ *(resp.* $\mathfrak{M}(G/N, G/N)$*).*

Proof. For each $(C)_G \in \mathrm{Conj}(G,\mathcal{C})\backslash\mathrm{Conj}(G,\mathcal{C}_{\mathcal{P}})$ (resp. $(D)_{G/N} \in \mathrm{Conj}(G/N,\mathcal{C})$), f_C (resp. f_D) belongs to $\mathfrak{M}(G, G \setminus Q(G))$ (resp. $\mathfrak{M}(G/N, G/N)$). Since the set $\{f_C|(C)_G \in \mathrm{Conj}(G,\mathcal{C})\backslash\mathrm{Conj}(G,\mathcal{C}_{\mathcal{P}})\}$ (resp $\{f_D|(D)_{G/N} \in \mathrm{Conj}(G/N,\mathcal{C})\}$) is linear independent and $\dim_{\mathbb{Q}} \mathfrak{M}(G, G\backslash Q(G)) = |\mathrm{Conj}(G,\mathcal{C})| - |\mathrm{Conj}(G,\mathcal{C}_{\mathcal{P}})|$ (resp. $\dim_{\mathbb{Q}} \mathfrak{M}(G, G/N) = |\mathrm{Conj}(G/N,\mathcal{C})|$), we obtain the proposition. □

The next proposition immediately follows from Proposition 2.1.

Proposition 2.6. *The* $\mathbb{Q}$*-dimension of* $\mathrm{fix}^G_{G/N}(\mathfrak{M}(G, G \setminus Q(G)))$ *is equal to* $\nu(G,N)$*.*

Proof of Theorem 1.1. By Proposition 2.3, we have

$$\mathrm{rank}_{\mathbb{Z}}\mathrm{R}_{\mathbb{Q}}(G)^{\{N\}}_{\mathcal{P}(G)} = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{R}_{\mathbb{Q}}(G)^{\{N\}}_{\mathcal{P}(G)}) = \dim_{\mathbb{Q}} \mathfrak{M}(G, G \setminus Q(G))^{\{N\}}.$$

We note that $\nu(G,E) = |\mathrm{Conj}(G,\mathcal{C})| - |\mathrm{Conj}(G,\mathcal{C}_{\mathcal{P}})|$. By Propositions 2.5, 2.6, it holds that $\mathrm{rank}_{\mathbb{Z}}\mathrm{R}_{\mathbb{Q}}(G)^{\{N\}}_{\mathcal{P}(G)} = \nu(G,E) - \nu(G,N)$. □

3. Proof of Theorem 1.3

Let Γ denote the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$, where ζ is a primitive $|G|$-th root of 1. The group ring $\mathbb{Z}[\Gamma]$ has the exact sequence

$$0 \longrightarrow I_\Gamma \xrightarrow{\ i\ } \mathbb{Z}[\Gamma] \xrightarrow{\ \varepsilon\ } \mathbb{Z} \longrightarrow 0$$

where ε is the augmentation homomorphism, I_Γ is the kernel of ε and i is the inclusion map. We set $\Sigma_\Gamma = \sum_{\gamma\in\Gamma} \gamma$. We have $\mathbb{Z}[\Gamma]^\Gamma = \mathbb{Z} \cdot \Sigma_\Gamma$ and $\varepsilon(\Sigma_\Gamma) = |\Gamma|$. Thus

$$\mathbb{Q}[\Gamma] = (\mathbb{Q} \cdot I_\Gamma) \oplus (\mathbb{Q} \cdot \Sigma_\Gamma) = (\mathbb{Q} \cdot I_\Gamma) \oplus \mathbb{Q}[\Gamma]^\Gamma.$$

The next fact is well known.

Proposition 3.1 ([3, Proposition 9.2.6])**.** $\mathrm{RO}(G)$ *is the direct sum of* $\overline{\mathrm{RO}}_{\mathbb{Q}}(G)$ *and* $\mathrm{RO}_0(G)$*.*

Since $\overline{\mathrm{RO}}_{\mathbb{Q}}(G) = \mathrm{RO}(G)^\Gamma$ and $\overline{\mathrm{R}}_{\mathbb{Q}}(G) = \mathrm{R}(G)^\Gamma$, it holds that $|\mathrm{RO}(G)^\Gamma : \mathrm{R}_{\mathbb{Q}}(G)| < \infty$ and $|\mathrm{R}(G)^\Gamma : \mathrm{R}_{\mathbb{Q}}(G)| < \infty$.

Proposition 3.2. *Let* N *be a normal subgroup of* G*. Then,* $\mathbb{Q}\otimes_{\mathbb{Z}}\mathrm{RO}(G)^{\{N\}}_{\mathcal{P}(G)}$ *is canonically isomorphic to* $\left(\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{R}_{\mathbb{Q}}(G)^{\{N\}}_{\mathcal{P}(G)}\right) \oplus \left(\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{RO}_0(G)^{\{N\}}_{\mathcal{P}(G)}\right)$*.*

Proof. Let $x \in \mathrm{RO}(G)^{\{N\}}_{\mathcal{P}(G)}$, then

$$|\Gamma| x = \Sigma_\Gamma x + \sum_{\gamma \in \Gamma} (\mathrm{id} - \gamma) x \in \left(\mathrm{RO}(G)^{\{N\}}_{\mathcal{P}(G)}\right)^\Gamma + \mathrm{RO}_0(G)^{\{N\}}_{\mathcal{P}(G)}.$$

By Proposition 3.1, we have

$$\begin{aligned}\left(\mathrm{RO}(G)^{\{N\}}_{\mathcal{P}(G)}\right)^\Gamma + \mathrm{RO}_0(G)^{\{N\}}_{\mathcal{P}(G)} &= \left(\mathrm{RO}(G)^\Gamma\right)^{\{N\}}_{\mathcal{P}(G)} + \mathrm{RO}_0(G)^{\{N\}}_{\mathcal{P}(G)} \\ &= \overline{\mathrm{RO}}_{\mathbb{Q}}(G)^{\{N\}}_{\mathcal{P}(G)} + \mathrm{RO}_0(G)^{\{N\}}_{\mathcal{P}(G)} \\ &= \overline{\mathrm{RO}}_{\mathbb{Q}}(G)^{\{N\}}_{\mathcal{P}(G)} \oplus \mathrm{RO}_0(G)^{\{N\}}_{\mathcal{P}(G)}.\end{aligned}$$

Since $\mathbb{Q} \otimes_{\mathbb{Z}} \overline{\mathrm{RO}}_{\mathbb{Q}}(G)^{\{N\}}_{\mathcal{P}(G)} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{R}_{\mathbb{Q}}(G)^{\{N\}}_{\mathcal{P}(G)}$, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{RO}(G)^{\{N\}}_{\mathcal{P}(G)}$ is contained in

$$\left(\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{R}_{\mathbb{Q}}(G)^{\{N\}}_{\mathcal{P}(G)}\right) \oplus \left(\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{RO}_0(G)^{\{N\}}_{\mathcal{P}(G)}\right).$$

On the other hand, it is clear that

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{RO}(G)^{\{N\}}_{\mathcal{P}(G)} \supset \left(\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{R}_{\mathbb{Q}}(G)^{\{N\}}_{\mathcal{P}(G)}\right) \oplus \left(\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{RO}_0(G)^{\{N\}}_{\mathcal{P}(G)}\right).$$

□

Lemma 3.3 ([9, Second Rank Lemma])**.** *The $\mathbb{Z}$-rank of $\mathrm{RO}(G)^{\{N\}}_{\mathcal{P}(G)}$ is equal to $\lambda(G, E) - \lambda(G, N)$.*

Theorem 1.3 immediately follows from Proposition 3.2, Lemma 3.3, and Theorem 1.1.

4. Proofs of Theorems 1.5 and 1.6

Let m and n are natural numbers. Let $p_1, p_2, \ldots, p_m$ be m distinct odd primes, and let $u_m = p_1 p_2 \ldots p_m$. We note that $D^n_{2u_m}$ is an Oliver group if $m \geq 2$ and $n \geq 2$. It is easy to see that

$$\begin{aligned}&(D^n_{2u_m})^{\{p_i\}} = D^n_{2u_m} \quad (i = 1,\ 2, \ldots,\ m), \\ &(D^n_{2u_m})^{\mathrm{nil}} = (D^n_{2u_m})^{\{2\}} \cong C^n_{u_m}, \\ &D^n_{2u_m}/(D^n_{2u_m})^{\mathrm{nil}} \cong C^n_2.\end{aligned} \tag{4.1}$$

For D_{2u_m}, the order of element is 1, 2 or $p_{t_1} p_{t_2} \cdots p_{t_k}$ for $1 \leq t_1 < t_2 < \cdots < t_k \leq m$. Moreover, the numbers of conjugacy classes of elements of order 2 and $p_{t_1} p_{t_2} \ldots p_{t_k}$ is 1 and $\left(\prod_{i=1}^k (p_{t_i} - 1)\right)/2$, respectively.

For a group element g, let $o(g)$ be the order of g. For $D^n_{2u_m}$, let Z be the set of cyclic subgroups H of $D^n_{2u_m}$ generated by $(g_1, g_2, \ldots, g_n)$ such that $o(g_1) = \cdots = o(g_n) = 2$ or $o(g_1) = \cdots = o(g_n) = p_{t_1} p_{t_2} \cdots p_{t_k}$ for $1 \leq t_1 < t_2 < \cdots < t_k \leq m$. Then, the number of $D^n_{2u_m}$-conjugacy classes of elements in Z is 1 in former case and $\left(\left(\prod_{i=1}^{k}(p_{t_i}-1)\right)/2\right)^{n-1}$ in the latter case.

For natural numbers a_1 and a_2, let $\gcd(a_1, a_2)$ denote the greatest common divisor of a_1 and a_2.

Fact 4.1. Let $G = D_{2u_m}^2$. For $j = 0, 1$ and $0 \le k \le m$, let Y_k^j be the subset of $\mathcal{C}(G)$ consisting of $H = \langle (g_1, g_2) \rangle$ such that $|H| \equiv j \mod 2$ and $\gcd(o(g_1), o(g_2))$ is the product of k primes. Then, $|H|$ is 1 or a prime if and only if $(o(g_1), o(g_2))$ is $(1,1)$, $(1, p_i)$, $(p_i, 1)$ or (p_i, p_i) for some i, or $(2,1)$, $(1,2)$ or $(2,2)$. Moreover, the number of G-conjugacy classes of elements H in Y_k^j such that $|H|$ is not prime power is as follows.

$$\begin{cases} 3^m - 2m - 1 & \text{if } j = 1 \text{ and } k = 0, \\ (3^{m-1} - 1)\sum_{i=1}^{m}(p_i - 1)/2 & \text{if } j = 1 \text{ and } k = 1, \\ 3^{m-k}\sum_{1 \le t_1 < \cdots < t_k \le m}\left(\prod_{i=1}^{k}(p_{t_i} - 1)\right)/2 & \text{if } j = 1 \text{ and } k > 1, \\ 2(2^m - 1) & \text{if } j = 0 \text{ and } k = 0, \\ 0 & \text{if } j = 0 \text{ and } k > 0. \end{cases}$$

Fact 4.2. Let a, b, c, d and e be non-negative integers such that $a + b + c + d + e = n$. For $G = D_{2u_2}^n$, let X be the set of cyclic subgroups H of G generated by $(g_1, g_2, \ldots, g_n)$ such that $o(g_1) = \cdots = o(g_a) = 1$, $o(g_{a+1}) = \cdots = o(g_{a+b}) = p_1$, $o(g_{a+b+1}) = \cdots = o(g_{a+b+c}) = p_2$, $o(g_{a+b+c+1}) = \cdots = o(g_{a+b+c+d}) = p_1p_2$ and $o(g_{a+b+c+d+1}) = \cdots = o(g_n) = 2$. Then, $|H|$ is 1 or a prime if and only if $c = d = e = 0$, $b = d = e = 0$ or $b = c = d = 0$. Moreover, the number of G-conjugacy classes of elements in X under certain conditions are as follows.

$$\begin{cases} ((p_1 - 1)/2)^{b-1} & H \text{ with } b > 0,\ c = d = 0,\ e > 0, \\ ((p_2 - 1)/2)^{c-1} & H \text{ with } c > 0,\ b = d = 0,\ e > 0, \\ ((p_1 - 1)/2)^{b-1}((p_2 - 1)/2)^{c-1} & H \text{ with } b > 0,\ c > 0,\ d = 0, \\ ((p_1 - 1)/2)^{b}((p_2 - 1)/2)^{c}((p_1 - 1)(p_2 - 1)/2)^{d-1} & H \text{ with } d > 0. \end{cases}$$

Proposition 4.3. *Let $G = D_{2u_m}^n$ for $m \ge 2$. Then $\lambda(G, E)$ is equal to*

$$\left(\frac{p_1p_2\cdots p_m + 3}{2}\right)^n - \sum_{i=1}^{m}\left(\frac{p_i + 1}{2}\right)^n + m - 2^n.$$

Proof. We note that $(g)^{\pm} = (g)$ holds for any element g of G. It suffices to calculate the number of conjugacy classes (g) of $g \in G$ which is not of prime power order. By the facts of the number of conjugacy classes (g) with $g \in D_{2u_m}$ of the beginning of this section, the number of conjugacy classes of elements of D_{2u_m} is

$$2+\sum_{1\leq t_1<\cdots<t_k\leq m}\frac{1}{2}\prod_{i=1}^{k}(p_{t_i}-1)=2+\frac{1}{2}\left(\prod_{i=1}^{m}((p_i-1)+1)-1\right)$$

which is equal to $(p_1p_2\cdots p_m+3)/2$, and hence G has $((p_1p_2\cdots p_m+3)/2)^n$ conjugacy classes. Moreover, since the numbers of conjugacy classes of elements of orders p_i and 2 in D_{2u_m} are $(p_i-1)/2$ and 1, respectively, those for G are

$$\sum_{k=1}^{m}{}_n\mathrm{C}_k\left(\frac{p_i-1}{2}\right)^k=\left(\frac{p_i-1}{2}+1\right)^n-1=\left(\frac{p_i+1}{2}\right)^n-1$$

and $\sum_{k=1}^{n}{}_n\mathrm{C}_k=2^n-1$, respectively, where ${}_n\mathrm{C}_k$ is the binomial coefficient. Therefore, we obtain

$$\begin{aligned}\lambda(G,E)&=\left(\frac{p_1p_2\cdots p_m+3}{2}\right)^n-\sum_{i=1}^{m}\left(\left(\frac{p_i+1}{2}\right)^n-1\right)-(2^n-1)-1\\&=\left(\frac{p_1p_2\cdots p_m+3}{2}\right)^n-\sum_{i=1}^{m}\left(\frac{p_i+1}{2}\right)^n+m-2^n.\end{aligned}$$

□

Theorem 1.6 (2) is obtained immediately from Proposition 4.3.

For a real G-module V, let $V^{\mathcal{L}(G)}$ denote the submodule $\sum_{L\in\mathcal{L}(G)}V^L$ and let $V_{\mathcal{L}(G)}$ denote the orthogonal complement of $V^{\mathcal{L}(G)}$ in V, with respect to a G-invariant inner-product on V.

The next lemma follows from [7, Theorem 6.7].

Lemma 4.4. *Let G be an Oliver group. If $x=[V]-[W]$ is an element of $\mathrm{RO}_0(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}$, then there exists an $\mathcal{L}(G)$-free real G-module U such that $V\oplus U\oplus\mathbb{R}[G]^{\oplus m}_{\mathcal{L}(G)}$ and $W\oplus U\oplus\mathbb{R}[G]^{\oplus m}_{\mathcal{L}(G)}$ are Smith equivalent for any $m\in\mathbb{N}$, and therefore x belongs to $\mathfrak{d}\mathfrak{S}(G)$.*

Since $\mathfrak{S}(G)\subset\mathrm{RO}(G)_{\mathcal{P}_{\mathrm{odd}}(G)}$ by C. U. Sanchez [13] and $\mathfrak{S}(G)\subset\mathrm{RO}(G)^{\{G^{\cap 2}\}}$ by M. Morimoto–Y. Qi [8], we have $\mathfrak{S}(G)\subset\mathrm{RO}(G)^{\{G^{\cap 2}\}}_{\mathcal{P}_{\mathrm{odd}}(G)}$. By [6, Section 1, p.3684], we get $\mathfrak{S}(G)\subset\mathrm{RO}(G)_{\mathcal{P}^*(G)}$ where $\mathcal{P}^*(G)$ is the subset of $\mathcal{P}(G)$ consisting of P such that $|P|$ is odd or $|P|\leq 4$ if 2 divides $|P|$. Therefore we have

$$\mathfrak{S}(G)\subset\mathrm{RO}(G)^{\{G^{\cap 2}\}}_{\mathcal{P}^*(G)}\quad\text{and}\quad\mathfrak{d}\mathfrak{S}(G)\subset\mathrm{RO}_0(G)^{\{G^{\cap 2}\}}_{\mathcal{P}^*(G)}.$$

The next fact follows from Lemma 4.4.

Proposition 4.5. *If G is an Oliver group, then*

$$\mathrm{RO}_0(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \subset \mathfrak{dS}(G) \subset \mathrm{RO}_0(G)^{\{G^{\cap 2}\}}_{\mathcal{P}^*(G)}.$$

Since $\mathfrak{dS}(G) \subset \mathrm{RO}_0(G)^{\{G^{\cap 2}\}}$, the following fact is obtained from Proposition 4.5.

Proposition 4.6. *Let G be an Oliver group such that $G^{\cap 2} = G^{\mathrm{nil}}$. Then, $\mathfrak{dS}(G)_{\mathcal{P}(G)}$ coincides with $\mathrm{RO}_0(G)^{\{G^{\mathrm{nil}}\}}_{\mathcal{P}(G)}$.*

Proposition 4.7. *Let G be as in Proposition 4.6. If G^{nil} is of odd order, then $\mathfrak{dS}(G)$ coincides with $\mathrm{RO}_0(G)^{\{G^{\mathrm{nil}}\}}_{\mathcal{P}(G)}$.*

Proof. Since $\mathcal{P}(G) = \mathcal{P}^*(G)$, we get it immediately from Propositions 4.5, 4.6. □

It is easy to see the next fact.

Proposition 4.8. *Let G be a finite group and let N be a normal subgroup of G. If G/N is isomorphic to C_2^n for some natural number n, then $\lambda(G,N)$ is equal to $\nu(G,N)$.*

By Corollary 1.2, (4.1), and Propositions 4.7, 4.8, the next proposition immediately follows.

Proposition 4.9. *Let $G = D^n_{2u_m}$. If m, $n \geq 2$, then $\mathfrak{dS}(G)$ coincides with $\mathrm{RO}_0(G)^{\{G^{\mathrm{nil}}\}}_{\mathcal{P}(G)}$, and the $\mathbb{Z}$-rank of $\mathrm{RO}_0(G)^{\{G^{\mathrm{nil}}\}}_{\mathcal{P}(G)}$ is equal to $\lambda(G,E) - \nu(G,E)$.*

Theorem 1.6 (1) is obtained immediately from Proposition 4.9.

Proof of Theorem 1.6 (3). Let $G = D^n_{2u_2}$. In Sections 1 and 2, we defined $\mathrm{Conj}(G,\mathcal{C})$ and $\mathrm{Conj}(G,\mathcal{C}_{\mathcal{P}})$. For $i = 1, 2$, let X_i denote the set of all G-conjugacy classes $(H)_G$ of subgroups H of G with $H \cong C_{2p_i}$. Let X_3 (resp. X_4) denote the set of all G-conjugacy classes $(H)_G$ of cyclic subgroups $H = \langle (g_1, g_2, \ldots, g_n) \rangle$ of G such that $p_1p_2 \mid |H|$ and $o(g_i) \neq p_1p_2$ for all i (resp. $o(g_i) = p_1p_2$ for some i). Let B_1, B_2, B_3 and B_4 be the sets

$B_1 = \{(a,b,e) \mid a \in \mathbb{N} \cup \{0\},\ b,\ e \in \mathbb{N},\ a+b+e = n\}$,
$B_2 = \{(a,c,e) \mid a \in \mathbb{N} \cup \{0\},\ c,\ e \in \mathbb{N},\ a+c+e = n\}$,
$B_3 = \{(a,b,c,e) \mid a,e \in \mathbb{N} \cup \{0\},\ b,c \in \mathbb{N},\ a+b+c+e = n\}$, and
$B_4 = \{(a,b,c,d,e) \mid d \in \mathbb{N},\ a,\ b,\ c,\ e \in \mathbb{N} \cup \{0\},\ a+b+c+d+e = n\}$,

respectively. By Fact 4.2 and the multinomial theorem, we obtain that

$$\begin{aligned}
|X_1| &= \sum_{(a,b,e)\in B_1} \frac{n!}{a!b!e!}\left(\frac{p_1-1}{2}\right)^{b-1} \\
&= \frac{2}{p_1-1}\left(\left(\frac{p_1+3}{2}\right)^n - \left(\frac{p_1+1}{2}\right)^n - 2^n + 1\right), \\
|X_2| &= \sum_{(a,c,e)\in B_2} \frac{n!}{a!c!e!}\left(\frac{p_2-1}{2}\right)^{c-1} \\
&= \frac{2}{p_2-1}\left(\left(\frac{p_2+3}{2}\right)^n - \left(\frac{p_2+1}{2}\right)^n - 2^n + 1\right), \\
|X_3| &= \sum_{(a,b,c,e)\in B_3} \frac{n!}{a!b!c!e!}\left(\frac{p_1-1}{2}\right)^{b-1}\left(\frac{p_2-1}{2}\right)^{c-1} \\
&= \frac{4}{(p_1-1)(p_2-1)}\left(\left(\frac{p_1+p_2+2}{2}\right)^n - \left(\frac{p_1+3}{2}\right)^n - \left(\frac{p_2+3}{2}\right)^n + 2^n\right), \\
|X_4| &= \sum_{(a,b,c,d,e)\in B_4} \frac{n!}{a!b!c!d!e!}\left(\frac{p_1-1}{2}\right)^{b}\left(\frac{p_2-1}{2}\right)^{c}\left(\frac{(p_1-1)(p_2-1)}{2}\right)^{d-1} \\
&= \frac{2}{(p_1-1)(p_2-1)}\left(\left(\frac{p_1p_2+3}{2}\right)^n - \left(\frac{p_1+p_2+2}{2}\right)^n\right).
\end{aligned}$$

Since $\nu(G,E) = |X_1| + |X_2| + |X_3| + |X_4|$, Theorem 1.6 (3) is obtained. □

Proof of Theorem 1.5. Let $G = D^2_{2u_m}$. By Fact 4.1, we obtain that

$$\nu(G,E) = \sum_{k=1}^{m} \frac{3^{m-k}}{2} \sum_{1\le t_1<\cdots<t_k\le m} \prod_{i=1}^{k}(p_{t_i}-1) - \sum_{i=1}^{m} \frac{p_i+5}{2} + 3^m + 2^{m+1} - 3.$$

Therefore, Theorem 1.5 immediately follows from Propositions 4.3, 4.9. □

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Kohei Seita
Department of Mathematics, Graduate School of Natural Science and Technology, Okayama University
3-1-1 Tsushimanaka, Kitaku, Okayama, 700-8530 Japan
e-mail address: p77e5xzo@s.okayama-u.ac.jp

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