

A NOTE ON PRODUCTS IN STABLE HOMOTOPY GROUPS OF SPHERES VIA THE CLASSICAL ADAMS SPECTRAL SEQUENCE

RYO KATO AND KATSUMI SHIMOMURA

ABSTRACT. In recent years, Liu and his collaborators found many non-trivial products of generators in the homotopy groups of the sphere spectrum. In this paper, we show a result which not only implies most of their results, but also extends a result of theirs.

1. INTRODUCTION

The homotopy groups $\pi_*(S^0)$ of the sphere spectrum S^0 form an algebra with multiplication given by composition. The determination of the structure of $\pi_*(S^0)$ is one of the most important problems in stable homotopy theory. We study the problem by considering the p -component ${}_p\pi_*(S^0)$ of the groups at a prime number p . The classical Adams spectral sequence (ASS) and the Adams-Novikov spectral sequence (ANSS) are typical and effective tools for calculating ${}_p\pi_*(S^0)$. We usually use the ANSS to study ${}_p\pi_*(S^0)$ at an odd prime p , and the ASS at the prime two. In recent years, Liu and his collaborators advocated that the ASS is sufficiently effective at $p > 2$ as well as at $p = 2$. Indeed, they derived out many results on the non-triviality of products of generators in ${}_p\pi_*(S^0)$ from the ASS at $p > 2$ by use of the May spectral sequence (MSS). Their method is simple as follows: for a product $\xi \in {}_p\pi_{t-s}(S^0)$ of generators, let $\bar{\xi}$ be an element of the E_2 -term ${}^A E_2^{s,t}$ of the ASS, which detects ξ . We also consider an element x in the E_1 -term ${}^M E_1^{s,t,*}$ of the MSS, which converges to $\bar{\xi}$. Then, they proceed their argument in the following steps:

- 1) The element x is not a coboundary of the first May differential $d_1^M : {}^M E_1^{s-1,t,*} \rightarrow {}^M E_1^{s,t,*}$.
- 2) For any $r \geq 2$, the domain of the May differential $d_r^M : {}^M E_r^{s-1,t,*} \rightarrow {}^M E_r^{s,t,*}$ is zero, and
- 3) For any $r \geq 2$, the domain of the Adams differential $d_r^A : {}^A E_r^{s-r,t-r+1} \rightarrow {}^A E_r^{s,t}$ is zero by use of the MSS.

The main theorem of this paper Theorem 1.1 is shown in a similar procedure (Proposition 4.1 and Corollary 4.2 for 1) and 2), and the proof of Theorem

Mathematics Subject Classification. Primary 55Q45; Secondary 55T15.

Key words and phrases. Stable homotopy of spheres, Adams spectral sequence, May spectral sequence .

1.1 for 3)) for the homotopy groups $\pi_*(V(2))$ of the second Smith-Toda spectrum $V(2)$ (cf. (1.1)). The result is new one, and implies most of results shown by Liu and his collaborators as a corollary.

From here on, we assume that the prime number p is greater than five. Let $H_*(X)$ denote the mod p reduced homology groups of a spectrum X represented by the mod p Eilenberg-MacLane spectrum H . The E_2 -term ${}^A E_2^{*,*}(X)$ of the ASS converging to the homotopy groups ${}_p \pi_*(X)$ of a spectrum X is the Ext group $\text{Ext}_{\mathcal{A}_*}^{*,*}(\mathbb{Z}/p, H_*(X))$ of the category of \mathcal{A}_* -comodules. Here $\mathcal{A}_* = H_*(H)$ denotes the dual of the Steenrod algebra, which is isomorphic as an algebra to the free algebra $P(\xi_i : i \geq 1) \otimes E(\tau_i : i \geq 0)$ over generators ξ_i 's and τ_i 's. Let $V(k)$ for $k \geq -1$ denotes the k -th Smith-Toda spectrum defined by $H_*(V(k)) = E(\tau_i : 0 \leq i \leq k)$. Then, for $k \leq 3$, $V(k)$ is known to exist if and only if $p \geq 2k + 1$ (Smith [32], Toda [33], Ravenel [31]). In particular, if $p \geq 7$, then $V(k)$ for $k \leq 3$ are given by the cofiber sequences

$$(1.1) \quad \begin{array}{c} S^0 \xrightarrow{p} S^0 \xrightarrow{i} V(0) \xrightarrow{j} \Sigma S^0, \\ \Sigma^q V(0) \xrightarrow{\alpha} V(0) \xrightarrow{i_1} V(1) \xrightarrow{j_1} \Sigma^{q+1} V(0), \\ \Sigma^{(p+1)q} V(1) \xrightarrow{\beta} V(1) \xrightarrow{i_2} V(2) \xrightarrow{j_2} \Sigma^{(p+1)q+1} V(1) \quad \text{and} \\ \Sigma^{(p^2+p+1)q} V(2) \xrightarrow{\gamma} V(2) \xrightarrow{i_3} V(3) \xrightarrow{j_3} \Sigma^{(p^2+p+1)q+1} V(2), \end{array}$$

in which α is the Adams v_1 -periodic map, and β and γ are the v_2 - and the v_3 -periodic maps given by Smith and Toda, respectively. Hereafter, q denotes the integer $2p - 2$, and $\pi_*(S^0)$ denotes ${}_p \pi_*(S^0)$. In this paper, we consider the Greek letter elements of $\pi_*(S^0)$ and $\pi_*(V(0))$ defined by

$$(1.2) \quad \begin{array}{l} \alpha_s = j\alpha^s i, \quad \beta_s = jj_1\beta^s i_1 i \quad \text{and} \quad \gamma_s = jj_1 j_2 \gamma^s i_2 i_1 i \in \pi_*(S^0); \quad \text{and} \\ \beta'_1 = j_1 \beta i_1 i \in \pi_*(V(0)). \end{array}$$

We moreover consider some other generators:

$\zeta_n \in \pi_{(p^n+1)q-3}(S^0)$, $j\xi_n \in \pi_{(p^n+p)q-3}(S^0)$ and $\varpi_n \in \pi_{(p^n+2p+1)q-3}(S^0)$ given by Cohen [1], Lin [4] and Liu [19]. Lin and Zheng [7] and Liu [15] constructed generators $\lambda_{n,s} \in \pi_{(p^n+sp^2+sp+s)q-7}(S^0)$ for $n \geq 2$ and $3 \leq s < p - 2$. We now state our main theorem, which extends the results [20, Theorems 1.2 and 1.3] of Liu's. In this paper, n denotes a fixed integer > 4 .

Theorem 1.1. *Let n be an integer greater than four. The following products of elements of $\pi_*(S^0)$ and $\pi_*(V(0))$ are all non-trivial:*

$$\begin{array}{ll} \alpha_1 \varpi_n \gamma_s \beta_1, \quad j\xi_n \alpha_1 \beta_2 \gamma_s \in \pi_{(p^n+sp^2+(s+2)p+s)q-9}(S^0) & \text{for } 3 \leq s < p, \\ \zeta_n \beta_1 \beta_2 \gamma_s \in \pi_{(p^n+sp^2+(s+2)p+s)q-10}(S^0) & \text{for } 3 \leq s < p - 2, \text{ and} \\ \beta'_1 \lambda_{n,s} \beta_1 \in \pi_{(p^n+sp^2+(s+2)p+s)q-10}(V(0)) & \text{for } 3 \leq s < p - 2. \end{array}$$

The proof is given at the end of the paper.

Corollary 1.2. *Every factor of the elements $\alpha_1 \varpi_n \gamma_s \beta_1$, $j \xi_n \alpha_1 \beta_2 \gamma_s$, $\zeta_n \beta_1 \beta_2 \gamma_s$ of ${}_p \pi_*(S^0)$ and $\beta'_1 \lambda_{n,s} \beta_1$ of $\pi_*(V(0))$ in the theorem is also non-trivial in the homotopy groups.*

We note that the corollary contains almost of all results of Liu and his collaborators on the non-triviality of products of elements of $\pi_*(S^0)$: [2], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [34], [35], [36] and [37].

ACKNOWLEDGEMENT

The authors would like to thank the referee for many useful comments.

2. THE ADAMS SPECTRAL SEQUENCE FOR $\pi_*(V(2))$

Hereafter, $P(x_i)$ and $E(x_i)$ denote a polynomial and an exterior algebras on generators x_i over \mathbb{Z}/p , respectively. Let \mathcal{A}_* denote the dual of the Steenrod algebra isomorphic to $P(\xi_1, \xi_2, \dots) \otimes E(\tau_0, \tau_1, \dots)$ as a graded algebra, where $\deg \xi_m = 2(p^m - 1)$ and $\deg \tau_m = 2p^m - 1$. It is also a Hopf algebra with the coproduct $\Delta: \mathcal{A}_* \rightarrow \mathcal{A}_* \otimes \mathcal{A}_*$ given by

$$\Delta \xi_m = \sum_{i=0}^m \xi_{m-i}^{p^i} \otimes \xi_i \quad \text{and} \quad \Delta \tau_m = \tau_m \otimes 1 + \sum_{i=0}^m \xi_{m-i}^{p^i} \otimes \tau_i$$

($\xi_0 = 1$). Consider the Adams spectral sequence

$${}^A E_2^{s,t}(V(2)) = \text{Ext}_{\mathcal{A}_*}^{s,t}(\mathbb{Z}/p, H_*(V(2))) \Rightarrow \pi_{t-s}(V(2)).$$

The second Smith-Toda spectrum $V(2)$ satisfies $H_*(V(2)) = E(\tau_0, \tau_1, \tau_2) = \mathcal{A}_* \square_{\overline{\mathcal{A}}_*} \mathbb{Z}/p$ for the quotient Hopf algebra $\overline{\mathcal{A}}_* = P(\xi_1, \xi_2, \dots) \otimes E(\tau_3, \tau_4, \dots)$, and we have the isomorphisms

$$\begin{aligned} {}^A E_2^{s,t}(V(2)) &= \text{Ext}_{\mathcal{A}_*}^{s,t}(\mathbb{Z}/p, H^*(V(2))) \\ &= \text{Ext}_{\mathcal{A}_*}^{s,t}(\mathbb{Z}/p, \mathcal{A}_* \square_{\overline{\mathcal{A}}_*} \mathbb{Z}/p) = \text{Ext}_{\overline{\mathcal{A}}_*}^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p) \end{aligned}$$

by the change of rings theorem (*cf.* [31, A1.3.13]). The Ext group is determined as the cohomology of the cobar complex $C_{\overline{\mathcal{A}}_*}^*$ defined by $C_{\overline{\mathcal{A}}_*}^s = \overline{\mathcal{A}}_* \otimes \dots \otimes \overline{\mathcal{A}}_*$ (the s -fold tensor product of $\overline{\mathcal{A}}_*$) with coboundary $d_s: C_{\overline{\mathcal{A}}_*}^s \rightarrow C_{\overline{\mathcal{A}}_*}^{s+1}$ given by $d_s(x) = 1 \otimes x + \sum_{i=1}^s (-1)^i \Delta_i(x) + (-1)^{s+1} x \otimes 1$ for $\Delta_i(x_1 \otimes \dots \otimes x_s) = x_1 \otimes \dots \otimes \Delta(x_i) \otimes \dots \otimes x_s$. We consider the following generators:

$$(2.1) \quad \begin{aligned} h_i &= [\xi_1^{p^i}] \in {}^A E_2^{1,p^i q}(V(2)) \text{ and} \\ b_i &= \left[\sum_{k=1}^{p-1} \frac{1}{p} \binom{p}{k} \xi_1^{kp^i} \otimes \xi_1^{(p-k)p^i} \right] \in {}^A E_2^{2,p^{i+1}q}(V(2)) \end{aligned}$$

for $i \geq 0$, where $[x]$ denotes the cohomology class of a cocycle x of the cobar complex $C_{\mathcal{A}^*}^*$. We also have generators

$$(2.2) \quad \begin{aligned} g_0 &= \langle h_0, h_0, h_1 \rangle \in {}^A E_2^{2, (p+2)q}(V(2)) \text{ and} \\ k_0 &= \langle h_0, h_1, h_1 \rangle \in {}^A E_2^{2, (2p+1)q}(V(2)) \end{aligned}$$

given by the Massey products. By the juggling theorem of the Massey products, we have a well known relation:

$$(2.3) \quad g_0 h_1 = h_0 k_0 \in {}^A E_2^{3, 2(p+1)q}(V(2)).$$

3. THE MAY SPECTRAL SEQUENCE

Hereafter, we abbreviate ${}^A E_2^{*,*}(V(2))$ to ${}^A E_2^{*,*}$. In this section, we study the Adams E_2 -term by the May spectral sequence ${}^M E_1^{s,t,u} \Rightarrow {}^A E_2^{s,t}$ with

$${}^M E_1^{*,*,*} = A \otimes H_0 \otimes H \otimes B$$

and differential $d_r^M : {}^M E_r^{s,t,u} \rightarrow {}^M E_r^{s+1,t,u-r}$. Here,

$$(3.1) \quad \begin{aligned} A &= P(a_i : i \geq 3), & H_0 &= E(h_{i,0} : i > 0), \\ H &= E(h_{i,j} : i > 0, j > 0) & \text{and} & B = P(b_{i,j} : i > 0, j \geq 0) \end{aligned}$$

on the generators

$$\begin{aligned} a_i &\in {}^M E_1^{1, 2p^i-1, 2i+1}, \\ h_{i,j} &\in {}^M E_1^{1, 2(p^i-1)p^j, 2i-1} \quad \text{and} \quad b_{i,j} \in {}^M E_1^{2, 2(p^i-1)p^{j+1}, p(2i-1)}. \end{aligned}$$

We notice that the May E_1 -term is a graded commutative algebra and the May differentials are derivations. For each element $x \in {}^M E_1^{s,t,u}$, we denote by $\dim x$ and $\deg x$ the superscripts s and t , respectively. The first May differential d_1^M is given by

$$(3.2) \quad \begin{aligned} d_1^M(a_i) &= \sum_{3 \leq k < i} h_{i-k,k} a_k, \\ d_1^M(h_{i,j}) &= \sum_{0 < k < i} h_{i-k,k+j} h_{k,j} \quad \text{and} \quad d_1^M(b_{i,j}) = 0. \end{aligned}$$

By definition of the May E_1 -term, the generators $h_{1,i}$, $b_{1,i}$, $\widehat{g}_0 = h_{2,0}h_{1,0}$ and $\widehat{k}_0 = h_{2,0}h_{1,1}$ are obtained by the elements in (2.1) and (2.2). We also have a generator $\widehat{\gamma}_s$, see [8, Th. 1.1].

Lemma 3.1. *In the May E_1 -term, we have permanent cycles*

$$h_{1,i}, \quad b_{1,i}, \quad \widehat{g}_0, \quad \widehat{k}_0 \quad \text{and} \quad \widehat{\gamma}_s = a_3^{s-3} h_{3,0} h_{2,1} h_{1,2}$$

for $i \geq 0$ and $3 \leq s < p$, which detect h_i , b_i , g_0 , k_0 in (2.1) and (2.2), and $\overline{\gamma}_s \in {}^A E_2^{*,*}$, respectively. Here, $\overline{\gamma}_s$ is an element converging to $i_2 i_1 i \gamma_s \in \pi_{(sp^2+(s-1)p+s-2)q-3}(V(2))$ for the element γ_s in (1.2)

Throughout this paper, the word ‘monomial’ means a (nonzero) product of algebraic generators of the May E_1 -term up to sign, that is, a monomial xy is identified as yx (without sign) for generators x and y . A monomial $x \in {}^M E_1^{*,*,*}$ is expressed as

$$(3.3) \quad x = \prod_{x_i \in G} x_i \text{ for a subset } G \subset \{a_{k'}, h_{l,k}, b_{l,k} \mid k' \geq 3, k \geq 0, l \geq 1\}.$$

In particular, if $G = \emptyset$, then $x = 1$. A monomial x of ${}^M E_1^{*,*,*}$ has a factorization

$$(3.4) \quad x = a(x)h_0(x)f(x) \text{ for } a(x) \in A, h_0(x) \in H_0, f(x) \in H \otimes B.$$

Let M denote the set of all monomials of ${}^M E_1^{*,*,*}$. We define mappings $c, c', c_k: M \rightarrow \mathbb{Z}$ for $k \geq 0$ so that

$$\begin{aligned} c'(a_i) &= 1, & c'(h_{i,j}) &= 0, & c'(b_{i,j}) &= 0, \\ c_k(a_i) &= \begin{cases} 1 & 0 \leq k < i \\ 0 & \text{otherwise} \end{cases}, & c_k(h_{i,j}) &= \begin{cases} 1 & j \leq k < i + j \\ 0 & \text{otherwise} \end{cases}, \\ c_k(b_{i,j}) &= \begin{cases} 1 & j < k \leq i + j \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

for the generators of ${}^M E_1^{*,*,*}$, and for a monomial $x = \prod_i x_i$,

$$c'(x) = \sum_i c'(x_i), \quad c_k(x) = \sum_i c_k(x_i)$$

and

$$(3.5) \quad c(x) = \left(\sum_{k \geq 0} c_k(x)p^k \right) q + c'(x).$$

Under the notation, we see that

$$(3.6) \quad \deg x = c(x).$$

We note that the part $\sum_{k \geq 0} c_k(x)p^k$ of (3.5) is not always the p -adic expansion of c in $\deg x = cq + c'(x)$. We notice that

$$(3.7) \quad \begin{aligned} c'(x) = c_0(a(x)) = c_1(a(x)) = c_2(a(x)) &= \dim a(x), \\ c_0(h_0(x)) &= \dim h_0(x) \end{aligned}$$

and

$$(3.8) \quad c_0(x) = c_0(a(x)h_0(x)) = c'(x) + \dim h_0(x) = \dim a(x)h_0(x).$$

Furthermore, we have the following relations on $c_k(x)$:

Lemma 3.2. *Let $x \in {}^M E_1^{*,*,*}$ be a monomial. Then,*

- 1) For integers s, t and u with $s > t > u$, we have $c_s(x) + c_u(x) - c_t(x) \leq \dim x$.
- 2) For $r \geq 0$, $\dim h_0(x) - r \leq c_r(x)$.

Proof. 1) For a monomial $x = \prod_{x_i \in G} x_i$ in (3.3), we put $C_s(x) = \{x_i \in G \mid c_s(x_i) = 1\}$. We notice that $c_s(x) = \#C_s(x)$ and $C_s(x) \cap C_u(x) \subset C_t(x)$. It follows that $c_s(x) + c_u(x) - c_t(x) \leq c_s(x) + c_u(x) - \#(C_s(x) \cap C_u(x)) = \#(C_s(x) \cup C_u(x)) \leq \dim x$.

2) We note that $\dim h_{i,0} = 1$ and $c_r(h_{i,0}) = 1$ if $i > r$. For a monomial $x = \prod_{x_i \in G} x_i$, we have

$$\dim h_0(x) = \dim \prod_{h_{i,0} \in G, i \leq r} h_{i,0} + \dim \prod_{h_{i,0} \in G, i > r} h_{i,0} \leq r + c_r(x).$$

□

We introduce a notation:

$$(3.9) \quad \mathbf{c}_i(x) = (c_{i-1}(x), c_{i-2}(x), \dots, c_0(x))$$

for $i \geq 1$ and a monomial x .

In the Adams spectral sequence, we write

$$\xi = (y)^\sim$$

if a permanent cycle y of the E_2 -term detects a homotopy element ξ . This is well defined up to higher filtration of the ASS. The Greek letter elements we consider here are

$$(3.10) \quad \begin{aligned} \alpha_1 &= (h_0)^\sim \in \pi_{q-1}(S^0), & \beta_1 &= (b_0)^\sim \in \pi_{pq-2}(S^0), \\ \beta_2 &= (k_0)^\sim \in \pi_{(2p+1)q-2}(S^0); & \text{and } \beta'_1 &= (h_1)^\sim \in \pi_{pq-1}(V(0)), \end{aligned}$$

and Cohen's [1], Lin's [4] and Liu's elements [19] :

$$(3.11) \quad \begin{aligned} \zeta_n &= (h_0 b_{n-1})^\sim \in \pi_{(p^n+1)q-3}(S^0) \text{ for } n \geq 1, \\ j\zeta_n &= (b_0 h_n + h_1 b_{n-1})^\sim \in \pi_{(p^n+p)q-3}(S^0) \text{ for } n \geq 3, \quad \text{and} \\ \varpi_n &= (k_0 h_n)^\sim \in \pi_{(p^n+2p+1)q-3}(S^0) \text{ for } n \geq 3. \end{aligned}$$

Lin and Zheng [7] constructed a generator

$$\lambda_n = \langle \zeta_{n-1}'' i_1, \alpha, \beta'_1 \rangle = (b_{n-1} g_0)^\sim \in \pi_{(p^n+p+2)q-4}(V(1))$$

(Toda bracket), where $\zeta_{n-1}'' \in [V(1), V(1)]_{(p^n+1)q-4}$ satisfies $j_1 \zeta_{n-1}'' = i j j_1 (\zeta_{n-1} \wedge V(1))$. Lin and Zheng [7] and Liu [15] showed that the composite $\lambda_{n,s} = j j_1 j_2 \gamma^s i_2 \lambda_n$ satisfying

$$(3.12) \quad \lambda_{n,s} = (b_{n-1} g_0 \bar{\gamma}_s)^\sim \in \pi_{(p^n+s(p^2+p+1))q-4-s}(S^0)$$

is essential for $n \geq 4$ and $3 \leq s < p - 2$.

For a monomial $x \in {}^M E_1^{*,*,*}$, we denote by \tilde{x} the set of monomials, each of these has degree $\deg x$. Hereafter, we consider a monomial

$$l_{i,j} \in \{h_{i,j}, b_{i,j-1}\}.$$

We see that $\tilde{l}_{i,j} = \tilde{h}_{i,j} = \tilde{b}_{i,j-1}$. For example,

$$\tilde{l}_{2,1} = \{h_{2,1}, b_{2,0}, h_{1,2}h_{1,1}, h_{1,1}b_{1,1}, h_{1,2}b_{1,0}, b_{1,1}b_{1,0}, h_{1,1}b_{1,0}^p, b_{1,0}^{p+1}\}$$

and

$$\tilde{a}_4 = \{a_4, a_3h_{1,3}, a_3b_{1,2}, a_3h_{1,2}b_{1,1}^{p-1}, a_3b_{1,1}^p\}.$$

Lemma 3.3. *For $u > 0$ and $k \geq 0$, we consider a monomial x of ${}^M E_1^{s,c(x),*}$ such that*

$$(3.13) \quad c_i(x) = \begin{cases} u & k \leq i < n \\ 0 & i \geq n \end{cases}.$$

If $l_{a,b}$ with $k < a + b < n$ (resp. a_b with $k < b < n$) is a factor of x , then x has a factor in $\tilde{l}_{n-b,b}$ (resp. \tilde{a}_n).

Proof. Consider an element $l_{a,b}$ with $k < a + b < n$ such that $x = x_0 l_{a,b}$ for a monomial x_0 . Then, $c_{a+b-\varepsilon}(x_0) = c_{a+b-\varepsilon}(x) - \varepsilon = u - \varepsilon$ for $\varepsilon = 0, 1$, which shows that x_0 has a factor $l_{\iota_1, a+b}$ for an integer $\iota_1 > 0$. Therefore, x has a factor $l_{\iota_1, a+b} l_{a,b} \in \tilde{l}_{a+\iota_1, b}$. Inductively, we see that x has a factorization

$$l_{\iota_\ell, s_\ell} l_{\iota_{\ell-1}, s_{\ell-1}} \cdots l_{\iota_1, s_1} l_{a,b} \quad \text{for some } \ell > 0 \text{ and } s_j = a + b + \sum_{i=1}^{j-1} \iota_i,$$

which is in $\tilde{l}_{n-b,b}$ if $\iota_\ell + s_\ell = n$.

The statement for \tilde{a}_n is verified similarly. □

For sets S_k for $1 \leq k \leq \ell$ of monomials in the May E_1 -terms, we consider a set

$$S_1 S_2 \cdots S_\ell = \{x_1 x_2 \cdots x_\ell \mid x_k \in S_k\}$$

of monomials. In particular, we write $S^e = S \cdots S$ (e factors) if $e > 0$, and $S^0 = \emptyset$ for a set S . We also define

$$S^{(d)} = \{x \in S \mid \dim x = d\}$$

and

$$\underline{\dim} S = \begin{cases} 0 & S = \emptyset, \\ \min\{\dim x \mid x \in S\} & \text{otherwise.} \end{cases}$$

In particular, we have

$$(3.14) \quad \underline{\dim} \tilde{l}_{n-\iota, \iota}^e = \begin{cases} 0 & \iota = 0 \text{ and } e > n, \text{ or } e = 0 \\ 2e - 1 & \text{otherwise.} \end{cases}$$

Indeed, if $e \geq 1$ and $\tilde{l}_{n-i,i}^e \neq \emptyset$, then the dimension of a monomial of the subset

$$(3.15) \quad h_{n-i,i}(\tilde{l}_{n-i,i}^{(2)})^{e-1} \subset \tilde{l}_{n-i,i}$$

is $2e - 1$ and implies $\underline{\dim} \tilde{l}_{n-i,i}^e = 2e - 1$ since $h_{i,j}^2 = 0$.

Proposition 3.4. *Suppose that a monomial $x \in {}^M E_1^{s,c(x),*}$ satisfies (3.13) for integers $u > 0$ and $k \geq 0$. Then,*

$$x = lz \quad \text{for } l \in \tilde{a}_n^{e_0} \tilde{l}_{n-\iota_1, \iota_1}^{e_1} \cdots \tilde{l}_{n-\iota_m, \iota_m}^{e_m},$$

in which $k \geq \iota_1 > \iota_2 > \cdots > \iota_m \geq 0$ for $m \geq 0$, $e_0 \geq 0$, $e_i > 0$ for each $i \geq 1$, $\sum_{i=0}^m e_i = u = c_{n-1}(x)$, and z is a monomial which has no factor of the form $l_{\iota_i-\ell, \ell}$ nor a_{ι_i} . Furthermore, $c_i(z) = 0$ for $i \geq k$ and $c_{\iota_i-1}(z) \leq c_{\iota_i}(z)$.

Note that we do not claim the uniqueness of the factorization of the proposition.

Proof. By Lemma 3.3, we have an integer $\iota_0 \leq k$ and an element $y_0 \in \tilde{l}_{n-\iota_0, \iota_0} \cup \tilde{a}_n$ such that $x = x_0 y_0$. The factor x_0 also satisfies (3.13) for $k \geq 0$ and $u - 1$ unless $u = 1$. Inductively, we obtain a factorization

$$x = z y_{u-1} y_{u-2} \cdots y_0,$$

for $y_i \in \tilde{l}_{n-\iota_i, \iota_i} \cup \tilde{a}_n$ with $\iota_i \leq k$, and z has no factor of the form $l_{\iota_i-\ell, \ell}$ nor a_{ι_i} . Put $l = y_{u-1} \cdots y_0$, and we may consider $l \in \tilde{a}_n^{e_0} \tilde{l}_{n-\iota_1, \iota_1}^{e_1} \cdots \tilde{l}_{n-\iota_m, \iota_m}^{e_m}$ and $\iota_1 > \iota_2 > \cdots > \iota_m \geq 0$. We also obtain the equality $\sum_{j=0}^m e_j = u$. The element z satisfies $c_i(z) = 0$ for $i \geq k$, since $c_i(z) = c_i(x) - c_i(y_{u-1} y_{u-2} \cdots y_0) = u - u = 0$.

We also have $c_{\iota_i-1}(z) \leq c_{\iota_i}(z)$. Indeed, if $c_{\iota_i-1}(z) > c_{\iota_i}(z)$, then z should have a factor $z' \in \tilde{l}_{\iota_i-\ell, \ell} \cup \tilde{a}_{\iota_i}$, which implies $y_i z' \in \tilde{l}_{\iota_i-\ell, \ell} \cup \tilde{a}_n$. Hence we may replace y_i with $y_i z'$ as a factor of l . \square

Now consider the internal degree

$$(3.16) \quad t_0 = (p^n + p^3 + 2p - 1)q + p - 4.$$

We put

$$(3.17) \quad u_s = \deg a_3^s = (sp^2 + sp + s)q + s \quad \text{for } s \geq 0.$$

Lemma 3.5. *Consider a monomial x of the May E_1 -term ${}^M E_1^{p+5+\varepsilon-s-r, t_0-u_s-r+1, *}$ with $\varepsilon \in \{0, 1\}$, $0 \leq s \leq p - 4$, and $r \geq 1$. Then $\mathbf{c}_{n+1}(x)$ in (3.9) is*

$$(3.18) \quad \begin{aligned} \mathbf{c}_{n+1}^0(s) &= (1, 0, \dots, 0, p-1-s, p+1-s, p-1-s) \text{ or} \\ \mathbf{c}_{n+1}^1(s) &= (0, p-1, \dots, p-1, p, p-1-s, p+1-s, p-1-s). \end{aligned}$$

Proof. We first note that

$$(3.19) \quad \dim x \leq p + 5 - s < 2p - 1 - s$$

by $p \geq 7$. We also note that

$$(3.20) \quad \begin{aligned} \deg x &= t_0 - u_s - r + 1 \\ &= (p^n + p^3 - sp^2 + (2 - s)p - 1 - s)q + p - 3 - s - r \\ &= (\sum_{k \geq 0} c_k(x)p^k)q + c'(x) \end{aligned}$$

by (3.5) and (3.6). Consider the factorization (3.4). By (3.7), we obtain $\dim a(x) = c'(x) \equiv p - 3 - s - r \pmod q$. The inequality

$$q + p - 3 - s - r > p + 5 + \varepsilon - s - r = \dim x$$

implies

$$(3.21) \quad \dim a(x) = c'(x) = p - 3 - s - r.$$

Notice that $c_0(x) \equiv -1 - s \pmod p$ by (3.20), $0 \leq c_0(x) \leq \dim x$ and $c_0(x) = \dim a(x) + \dim h_0(x)$ by (3.8), and we obtain

$$(3.22) \quad c_0(x) = p - 1 - s \quad \text{and} \quad \dim h_0(x) = 2 + r.$$

It follows that

$$(3.23) \quad \dim f(x) = 6 + \varepsilon - r.$$

Since $c_1(x) \equiv 1 - s \pmod p$ by (3.20), and $2 \leq r + 1 = \dim h_0(x) - 1 \leq c_1(x)$ by (3.22) and Lemma 3.2 2), we deduce

$$c_1(x) = p + 1 - s$$

under the condition (3.19), and so

$$c_2(x) = p - 1 - s \quad \text{and} \quad c_3(x) \equiv 0 \pmod p.$$

We also see that $c_n(x) = 1$ or $= 0$. If $c_n(x) = 1$, then $c_i(x) = 0$ for $3 \leq i < n$ by degree reason. Therefore, we have $\mathbf{c}_{n+1}(x) = \mathbf{c}_{n+1}^0(s)$ in this case.

Suppose that $c_n(x) = 0$. Then, we have an integer j with $3 \leq j < n$ such that

$$c_i(x) = \begin{cases} 0 & 3 \leq i < j \\ p & i = j \\ p - 1 & j < i < n \end{cases}.$$

If $j \neq 3$, then Lemma 3.2 1) shows that $p + 5 + \varepsilon - s - r \geq c_j(x) + c_1(x) - c_3(x) = 2p + 1 - s$, which contradicts to (3.19). Thus, $j = 3$ and we have $\mathbf{c}_{n+1}(x) = \mathbf{c}_{n+1}^1(s)$. \square

Lemma 3.6. *Let x be a monomial such that $\mathbf{c}_{n+1}(x) = \mathbf{c}_{n+1}^1(s)$ in (3.18). Then,*

$$x = lz \text{ for } l \in \tilde{a}_n^e \tilde{l}_{n-3,3}^{e_3} \tilde{l}_{n-1,1}^{e_1} \tilde{l}_{n,0}^{e_0},$$

where e, e_3, e_1 and e_0 are non-negative integers such that

$$(3.24) \quad e + e_3 + e_1 + e_0 = p - 1,$$

$e_0 \leq n, e_3 \in \{s, s+1\}$ and $e_1 \in \{0, 1, 2\}$. The factor z satisfies $c_i(z) = 0$ for $i > 3, c'(z) \leq 3$,

$$(3.25) \quad \mathbf{c}_4(z) = (1, e_3 - s, 2 + e_3 - s, e_3 + e_1 - s)$$

and $\dim z \geq 3$. Furthermore, $s + r \leq \frac{4 + w + \varepsilon - c'(z) - \dim z}{2} < 3$, where w denotes the number of i 's with $e_i \neq 0$.

Proof. Consider a factorization

$$x = lz$$

in Proposition 3.4. Since the integer k in Lemma 3.3 is four in our case,

$$l \in \tilde{a}_n^e \tilde{l}_{n-4,4}^{e_4} \tilde{l}_{n-3,3}^{e_3} \tilde{l}_{n-2,2}^{e_2} \tilde{l}_{n-1,1}^{e_1} \tilde{l}_{n,0}^{e_0} \quad \text{for } e \geq 0 \text{ and } e_i \geq 0 \ (0 \leq i \leq 4), \quad \text{and} \\ c_i(z) = 0 \quad \text{for } i \geq 4.$$

We may assume that $e_0 \leq n$. Indeed, if $e_0 > n$, then $\tilde{l}_{n,0}^{e_0} = \emptyset$. Furthermore, the fact $c_{n-1}(x) = p - 1$ implies $e + \sum_{i=0}^4 e_i = p - 1$, and so

$$\mathbf{c}_4(z) = \left(1 + e_4, e_4 + e_3 - s, 2 + \sum_{i=2}^4 e_i - s, \sum_{i=1}^4 e_i - s \right)$$

since $\mathbf{c}_n(l) = (p - 1, \dots, p - 1, \sum_{i=0}^4 e_i, \sum_{i=0}^3 e_i, \sum_{i=0}^2 e_i, e_1 + e_0, e_0)$. Notice that $c_3(z) > 0 = c_4(z)$ and $c_1(z) > c_2(z)$. Then, the last statement in Proposition 3.4 implies $e_4 = 0$ and $e_2 = 0$. Thus, we obtain (3.24) and (3.25). By (3.25), $c_1(z) = 2 + c_2(z) \geq 2$. If $c_1(z) \geq 3$, then $\dim z \geq 3$. If $c_1(z) = 2$, then $c_2(z) = 0$. Therefore, z has a factor $l_{1,3} \in \tilde{l}_{1,3}$ and two factors whose coefficient c_1 is one, and so $\dim z \geq 3$.

Proposition 3.4 implies that $2 \geq e_1$ by (3.25) if $e_1 \neq 0$, and that $0 \leq c_2(z) = e_3 - s \leq c_3(z) = 1$ if $e_3 \neq 0$. We also see $c_2(z) = -s \geq 0$ if $e_3 = 0$. These show $e_1 \in \{0, 1, 2\}$, and $e_3 \in \{s, s+1\}$. Now, $c'(z) = c_1(a(z)) \leq c_1(z) \leq 3$ by (3.7) and (3.25).

Note that $e_0 \leq n$. By (3.14), we compute

$$\begin{aligned} \dim x &\geq e + 2(e_3 + e_1 + e_0) - w + \dim z \\ &= e + 2(p - 1 - e) - w + \dim z \quad (\text{by (3.24)}) \\ &= 2(p - 1) - (p - 3 - s - r - \dim a(z)) - w + \dim z \\ &\quad (\text{by } c'(x) = e + \dim a(z) \text{ and (3.21)}). \end{aligned}$$

Since $\dim x = p + 5 + \varepsilon - s - r$, $w \leq 3$ and $\dim z \geq 3$, we obtain the last inequality. \square

4. PROOF OF THE MAIN THEOREM

In this section, we also abbreviate $AE_2^{*,*}(V(2))$ to $AE_2^{*,*}$. Put $m_s(x) = x\bar{\gamma}_s g_0 h_1 b_0$ for $x \in AE_2^{*,*}$. Then $m_s(h_n) \in AE_2^{s+6, (p^n + sp^2 + (s+2)p + s)q + s}$ and $m_s(b_{n-1}) \in AE_2^{s+7, (p^n + sp^2 + (s+2)p + s)q + s}$. We notice that

$$(4.1) \quad \text{the elements } m_s(h_n) \text{ and } m_s(b_{n-1}) \text{ are permanent cycles,}$$

since

$$(4.2) \quad i_2 i_1 i(\alpha_1 \varpi_n \gamma_s \beta_1) = (m_s(h_n))^\sim \text{ and } i_2 i_1 i(\zeta_n \beta_1 \beta_2 \gamma_s) = (m_s(b_{n-1}))^\sim.$$

Indeed, we have

$$\begin{aligned} m_s(h_n) &= h_n \bar{\gamma}_s g_0 h_1 b_0 = b_0 k_0 h_n h_0 \bar{\gamma}_s = (b_0 h_n + h_1 b_{n-1}) k_0 h_0 \bar{\gamma}_s \text{ and} \\ m_s(b_{n-1}) &= b_{n-1} \bar{\gamma}_s g_0 h_1 b_0 = h_0 b_{n-1} b_0 k_0 \bar{\gamma}_s = h_1 b_{n-1} g_0 \bar{\gamma}_s b_0 \end{aligned}$$

by (2.3), and also (3.10), (3.11) and (3.12) imply

$$(4.3) \quad \begin{aligned} i_2 i_1 i(\alpha_1 \varpi_n \gamma_s \beta_1) &= (h_0 k_0 h_n \bar{\gamma}_s b_0)^\sim \\ &= -(b_0 h_n + h_1 b_{n-1}) h_0 k_0 \bar{\gamma}_s)^\sim \\ &= -i_2 i_1 i(j \xi_n \alpha_1 \beta_2 \gamma_s) \text{ and} \\ i_2 i_1 i(\zeta_n \beta_1 \beta_2 \gamma_s) &= (h_0 b_{n-1} b_0 k_0 \bar{\gamma}_s)^\sim \\ &= (h_1 b_{n-1} g_0 \bar{\gamma}_s b_0)^\sim \\ &= i_2 i_1 (\beta'_1 \lambda_{n,s} \beta_1) \end{aligned}$$

in $\pi_*(V(2))$. In particular,

$$i_2 i_1 i(\alpha_1 \varpi_n \gamma_s \beta_1) = -i_2 i_1 i(j \xi_n \alpha_1 \beta_2 \gamma_s)$$

and

$$i_2 i_1 i(\zeta_n \beta_1 \beta_2 \gamma_s) = i_2 i_1 (\beta'_1 \lambda_{n,s} \beta_1)$$

up to Adams filtration. In this section, we show that the elements in (4.2) are non-trivial.

Proposition 4.1. *The elements $m_{p-1}(h_n)$ and $m_{p-1}(b_{n-1})$ of the Adams E_2 -term are non-trivial.*

Proof. Let $y_\varepsilon \in AE_2^{p+5+\varepsilon, t_0}$ denote $m_{p-1}(h_n)$ if $\varepsilon = 0$, and $m_{p-1}(b_{n-1})$ if $\varepsilon = 1$. We also take an element \bar{y}_ε in $ME_1^{p+5+\varepsilon, t_0, *}$, which detects y_ε . If $y_\varepsilon = 0$, then there exists $\bar{x}_\varepsilon \in ME_r^{p+4+\varepsilon, t_0, *}$ such that $d_r^M(\bar{x}_\varepsilon) = \bar{y}_\varepsilon$ for some r . We denote by $x_\varepsilon \in ME_1^{p+4+\varepsilon, t_0, *}$ a monomial appearing in a term of a representative of \bar{x}_ε . By Lemma 3.5 at $(s, r) = (0, 1)$, the n -tuple $\mathbf{c}_{n+1}(x_\varepsilon)$

is $\mathbf{c}_{n+1}^0(0)$ or $\mathbf{c}_{n+1}^1(0)$ in (3.18). Since $t_0 \equiv p-4 \pmod{q}$ by (3.16), we see $c'(x_\varepsilon) = p-4$. Therefore,

$$x_\varepsilon \in \begin{cases} \tilde{a}_3^{p-4} \tilde{l}_{1,n} \tilde{l}_{1,1}^2 \tilde{l}_{3,0}^3 & \mathbf{c}_{n+1}(x_\varepsilon) = \mathbf{c}_{n+1}^0(0), \\ \tilde{a}_n^{p-4} \tilde{l}_{1,3} \tilde{l}_{1,1}^2 \tilde{l}_{n-1,0}^3 & \mathbf{c}_{n+1}(x_\varepsilon) = \mathbf{c}_{n+1}^1(0). \end{cases}$$

Since $\dim x_\varepsilon = p+4+\varepsilon$ and $\underline{\dim} \left(\tilde{a}_3^{p-4} \tilde{l}_{1,n} \tilde{l}_{1,1}^2 \tilde{l}_{3,0}^3 \right) = p+5 = \underline{\dim} \left(\tilde{a}_n^{p-4} \tilde{l}_{1,3} \tilde{l}_{1,1}^2 \tilde{l}_{n-1,0}^3 \right)$, we have $\varepsilon = 1$. It follows that there is no monomial for x_0 , and so ${}^M E_1^{p+3, t_0, *}$ = 0. Therefore, \bar{y}_0 survives to $y_0 = m_{p-1}(h_n)$.

We consider the case $\varepsilon = 1$. If $\mathbf{c}_{n+1}(x_1) = \mathbf{c}_{n+1}^1(0)$, then

$$x_1 \in a_n^{p-4} h_{1,3} h_{1,1} b_{1,0} h_{n,0} (\tilde{l}_{n-1,0}^{(2)})^2$$

by (3.15). Put $w_{i,j} = h_{n-1-i,i} h_{i,0} h_{n-1-j,j} h_{j,0}$. Then, we see that $(\tilde{l}_{n-1,0}^{(2)})^2 = \{w_{i,j} : 1 \leq i < j \leq n-2\}$. It follows that the monomial x_1 is of the form $x_{1,i,j} = a_n^{p-4} h_{1,3} h_{1,1} b_{1,0} h_{n,0} w_{i,j}$. Since $n > 4$, we have

$$d_1^M(x_{1,i,j}) = -4a_n^{p-5} a_4 h_{n-4,4} h_{1,3} h_{1,1} b_{1,0} h_{n,0} w_{i,j} + \cdots \neq 0.$$

The images $d_1^M(x_{1,i,j})$ are linearly independent, since so are $w_{i,j}$'s. Therefore, any linear combination of $x_{1,i,j}$'s doesn't survive to the May E_2 -term.

For the case $\mathbf{c}_{n+1}(x_1) = \mathbf{c}_{n+1}^0(0)$, we have

$$x_1 \in a_3^{p-4} h_{1,n} h_{1,1} b_{1,0} h_{3,0} (\tilde{l}_{3,0}^{(2)})^2$$

by (3.15). Since $(\tilde{l}_{3,0}^{(2)})^2 = \{h_{1,0} h_{2,0} h_{1,2} h_{2,1}\}$,

$$x_1 = a_3^{p-4} h_{1,n} h_{1,1} b_{1,0} h_{3,0} h_{1,0} h_{2,0} h_{1,2} h_{2,1},$$

which converges to $\bar{\gamma}_{p-1} h_1 b_0 k_0 h_n$ in the Adams E_2 -term by Lemma 3.1.

Therefore $d_r^M(x_1) = 0$ for $r \geq 1$, and so ${}^M E_r^{s+5, t_0, *}$ = 0 for $r \geq 2$.

By the above argument, for $r \geq 2$, we obtain $d_r(x) = 0$ for any $x \in {}^M E_r^{p+5, t_0, *}$. Hence $y_1 = m_{p-1}(b_{n-1})$ survives to the Adams E_2 -term. \square

Corollary 4.2. *The elements $m_s(h_n)$ for $3 \leq s < p$ and $m_s(b_{n-1})$ for $3 \leq s < p-2$ in the E_2 -terms are non-zero.*

Proof. Since $a_3 \in {}^M E_1^{*,*,*}$ survives to ${}^A E_2^{*,*}$, the multiplication by a_3 induces a homomorphism

$$(4.4) \quad (a_3)_* : {}^A E_2^{*,*} \rightarrow {}^A E_2^{*,*}.$$

Since $a_3^{p-s-1} \hat{\gamma}_s = \hat{\gamma}_{p-1}$ in the May E_1 -term by Lemma 3.1, we have $(a_3)_*^{p-s-1}(\bar{\gamma}_s) = \bar{\gamma}_{p-1}$, and hence $(a_3)_*^{p-s-1}(m_s(h_n)) = m_{p-1}(h_n)$. Proposition 4.1 implies the non-triviality of the first element.

Since Lemma 3.1 also implies $(a_3)_*^{p-s-1}(b_{n-1}g_0\bar{\gamma}_s) = b_{n-1}g_0\bar{\gamma}_{p-1}$, we obtain the non-triviality of the second elements similarly by Proposition 4.1. \square

Remark. In the May spectral sequence converging to ${}^A E_2^{*,*}(S^0)$, the generator a_3 in the E_1 -term is not permanent, and therefore the map (4.4) is not defined. This is a reason why we consider the second Smith-Toda spectrum $V(2)$ in this paper.

Proof of Theorem 1.1. It suffices to show that

$$(4.5) \quad {}^A E_2^{p+5+\varepsilon-s'-r, t_0-u_{s'}-r+1} = 0$$

for $\varepsilon \in \{0, 1\}$, $r \geq 2$ and $s' \geq \varepsilon$. Indeed, if it holds, then the elements $m_{p-1-s'}(h_n)$ and $m_{p-1-s'}(b_{n-1})$ in (4.1) we concern are not in the image of the Adams differential

$$(4.6) \quad d_r^A: {}^A E_r^{p+5+\varepsilon-s'-r, t_0-u_{s'}-r+1} \rightarrow {}^A E_r^{p+5+\varepsilon-s', t_0-u_{s'}},$$

and the theorem follows from (4.2) and Corollary 4.2. We show (4.5) by verifying

$$M E_2^{p+5+\varepsilon-s'-r, t_0-u_{s'}-r+1, *} = 0.$$

For a monomial $x \in M E_1^{p+5+\varepsilon-s'-r, t_0-u_{s'}-r+1, *}$ with $r \geq 2$, if $c_3(x) = 0$, then $\dim h_0(x) \leq 3$ by Lemma 3.2 2), which contradicts to (3.22). It follows that $\mathbf{c}_{n+1}(x) = \mathbf{c}_{n+1}^1(s')$ by Lemma 3.5, and so $s' + r \leq 2$ by Lemma 3.6. This implies

$$(s', r) = (0, 2).$$

Therefore, (4.5) holds except for this case.

We will show $M E_2^{p+3, t_0-1, *} = 0$. By Lemma 3.6, a monomial x in $M E_1^{p+3, t_0-1, *}$ is factorized into

$$x = lz$$

for $l \in \tilde{a}_n^e \tilde{l}_{n-3,3}^{e_3} \tilde{l}_{n-1,1}^{e_1} \tilde{l}_{n,0}^{e_0}$ and a monomial z with $\mathbf{c}_4(z) = (1, e_3, 2+e_3, e_3+e_1)$, $e_3 \in \{0, 1\}$ and $e_1 \in \{0, 1, 2\}$. We notice that we can tell the least dimension of z from $\mathbf{c}_4(z)$. Since $e = p - 5 - c'(z)$ by (3.7) and (3.16), we have

$$(4.7) \quad e_3 + e_1 + e_0 = p - 1 - e = 4 + c'(z)$$

by (3.24). These give rise to a table:

(e_3, e_1)	$(0, 0)$	$(0, 1)$	$(0, 2)$	$(1, 0)$	$(1, 1)$	$(1, 2)$
$\mathbf{c}_4(z)$	$(1, 0, 2, 0)$	$(1, 0, 2, 1)$	$(1, 0, 2, 2)$	$(1, 1, 3, 1)$	$(1, 1, 3, 2)$	$(1, 1, 3, 3)$
$\dim z \geq$	3	3	4	3	3	4
w	1	2	2	2	3	3

Here, w is the integer given in Lemma 3.6. We also see that $w - c'(z) - \dim z \in \{0, 1\}$ by the inequality of Lemma 3.6, and hence $w - \dim z \geq 0$.

The table shows us that the inequation holds only when $(e_3, e_1) = (1, 1)$, $\dim z = 3$ and $c'(z) = 0$. Then the monomial x is of the form

$$x_j = a_n^{p-5} h_{n-3,3} h_{n-1,1} h_{n,0} h_{n-j,j} h_{j,0} h_{4,0} h_{2,0} h_{1,1}$$

for $j \geq 5$. Since

$$d_1^M(x_j) = -5a_n^{p-6} a_4 h_{n-4,4} h_{n-3,3} h_{n-1,1} h_{n,0} h_{n-j,j} h_{j,0} h_{4,0} h_{2,0} h_{1,1} + \cdots \neq 0,$$

the images $d_1^M(x_j)$ are linearly independent. Thus, (4.5) also holds in this case. \square

REFERENCES

- [1] R. L. Cohen, Odd primary infinite families in stable homotopy theory, *Mem. Amer. Math. Soc.* **30** (1981), no. 242.
- [2] J. Hong, X. Liu, and D. Zheng, On a family involving R. L. Cohen's ζ -element (II), *Sci. Chin. Math.* **58** (2014), 1–8.
- [3] C.-N. Lee, Detection of some elements in the stable homotopy groups of spheres, *Math. Z.* **222** (1996), 231–245.
- [4] J. Lin, A new family of filtration three in the stable homotopy of spheres, *Hiroshima Math. J.* **31** (2001), 477–492.
- [5] J. Lin, New Families in the Stable Homotopy of Spheres Revisited, *Acta Mathematica Sinica, English Series* **18** (2002), 95–106.
- [6] J. Lin, Two new families in the stable homotopy groups of sphere and Moore spectrum, *Chinese Ann. Math. Ser. B* **27** (2006), 311–328.
- [7] J. Lin and Q. Zheng, A new family of filtration seven in the stable homotopy of spheres, *Hiroshima Math. J.* **28** (1998), 183–205.
- [8] X. Liu, A nontrivial product in the stable homotopy groups of spheres, *Sci. China Ser. A Math.* **47** (2004), 831–841.
- [9] X. Liu, A new family of filtration $s + 6$ in the stable homotopy groups of spheres, *Acta Math. Sci. Ser. B* **26** (2006), 193–201.
- [10] X. Liu, Non-triviality of some compositions $\alpha_1 \beta_1 \gamma_s$ in the stable homotopy of spheres, *Adv. Math. (China)* **35** (2006), 733–738.
- [11] X. Liu, Non-trivial of two homotopy elements in $\pi_*(S)$, *J. Korean Math. Soc.* **43** (2006), 783–801.
- [12] X. Liu, A new infinite family $\alpha_1 \beta_2 \gamma_s$ in $\pi_*(S)$, *JP J. Geom. Topol.* **7** (2007), 51–63.
- [13] X. Liu, Non-triviality of an element $\alpha_1 \beta_1 \beta_s$ in the stable homotopy of spheres, *Acta Math. Sci. Ser. A. Chin. Ed.* **27** (2007), 208–214.
- [14] X. Liu, A nontrivial product of filtration $s + 5$ in the stable homotopy of spheres, *Acta Math. Sin. (Engl. Ser.)* **23** (2007), 385–392.
- [15] X. Liu, Some notes on the May spectral sequence (Chinese). *Acta Math. Sci. Ser. A. Chin. Ed.* **27** (2007), 802–810.
- [16] X. Liu, A nontrivial product in the stable homotopy groups of spheres, *Sci. China Ser. A.* **47** (2007), 831–841.
- [17] X. Liu, On the convergence of products $\gamma_s h_1 h_n$ in the Adams spectral sequence, *Acta Math. Sin. (Engl. Ser.)* **23** (2007), 1025–1032.
- [18] X. Liu, Detection of a new non-trivial family in the stable homotopy of spheres $\pi_*(S)$, *Tamkang J. Math.* **39** (2008), 75–83.

- [19] X. Liu, Detection of some elements in the stable homotopy groups of spheres, *Chin. Ann. Math. Ser. B* **29** (2008), 291–316.
- [20] X. Liu, On the ϖ_n -related elements in the stable homotopy group of spheres, *Arch. Math. (Basel)* **91** (2008), 471–480.
- [21] X. Liu, Some infinite elements in the Adams spectral sequence for the sphere spectrum, *J. Math. Kyoto Univ.* **48** (2008), 617–629.
- [22] X. Liu, On R. L. Cohen’s ζ -element, *Algebr. Geom. Topol.* **11** (2011), 1709–1735.
- [23] X. Liu, A composite map in the stable homotopy groups of spheres. *Forum Math.* **25** (2013), 241–253.
- [24] X. Liu and S. Jiang, Convergence of the products $b_0g_0\gamma_s$ in Adams spectral sequence, *Adv. Math. (Chin.)* **38** (2009), 319–326.
- [25] X. Liu and W. Li, A product involving the β -family in stable homotopy theory, *Bull. Malays. Math. Sci. Soc. (2)* **33** (2010), 411–420.
- [26] X. Liu and K. Ma, A new family in the homotopy groups of spheres, *Bull. Iranian Math. Soc.* **38** (2012), 313–322.
- [27] X. Liu and X. Wang, The convergence of $\gamma_s(b_0h_n - h_1b_{n-1})$, *Chinese Ann. Math. Ser. B.* **27** (2006), 329–340.
- [28] X. Liu and H. Zhao, On two non-trivial products in the stable homotopy groups of spheres, *Bol. Soc. Mat. Mexicana* **13** (2007), 367–380.
- [29] X. Liu, H. Zhao and Y. Jin, A non-trivial product of filtration $s + 6$ in the stable homotopy groups of spheres. *Acta Math. Sci. Ser. B. Engl. Ed.* **29** (2009), 276–284.
- [30] X. Liu and D. Zheng, On a family involving R. L. Cohen’s ζ -element. *Topol. Appl.* **160** (2013), 394–405.
- [31] D. C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, AMS Chelsea Publishing, Providence, 2004.
- [32] L. Smith, On realizing complex bordism modules, *Amer. J. Math.* **92** (1970), 793–856.
- [33] H. Toda, On spectra realizing exterior parts of the Steenrod algebra, *Topology*, **10** (1971), 53–65.
- [34] X. Wang and Q. Zheng, The convergence of $\tilde{\alpha}_s^{(n)}h_0h_k$, *Sci. China (ser. A)*, **41** (1998), 622–628.
- [35] H. Zhao, X. Liu and Y. Jin, A non-trivial product of filtration $s + 6$ in the stable homotopy groups of spheres, *Acta Math. Sci. Ser. B Engl. Ed.* **29** (2009), 276–284.
- [36] L. Zhong and X. Liu, On homotopy element $\alpha_1\beta_1\beta_2\gamma_s$, *Chin. Ann. Math., Ser. A* **34** (2013), 487–498.
- [37] L. Zhong and Y. Wang, Detection of a nontrivial product in the stable homotopy groups of spheres, *Algebr. Geom. Topol.* **13** (2013), 3009–3029.

RYO KATO
 FACULTY OF FUNDAMENTAL SCIENCE
 NATIONAL INSTITUTE OF TECHNOLOGY, NIIHAMA COLLEGE
 NIIHAMA, 792-8580, JAPAN

e-mail address: ryo_kato_1128@yahoo.co.jp

KATSUMI SHIMOMURA
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE AND TECHNOLOGY
KOCHI UNIVERSITY
KOCHI, 780-8520, JAPAN
e-mail address: katsumi@kochi-u.ac.jp

(Received March 15, 2019)

(Accepted July 21, 2019)