Convex compact sets in $\mathbb{R}^{N-1}$ give traveling fronts of cooperation-diffusion systems in $\mathbb{R}^N$

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Abstract

This paper studies traveling fronts to cooperation-diffusion systems in $\mathbb{R}^N$ for $N \geq 3$. We consider $(N-2)$-dimensional smooth surfaces as boundaries of strictly convex compact sets in $\mathbb{R}^{N-1}$, and define an equivalence relation between them. We prove that there exists a traveling front associated with a given surface and show its stability. The associated traveling fronts coincide up to phase transition if and only if the given surfaces satisfy the equivalence relation.

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Key words: traveling front, cooperation-diffusion system, non-symmetric

1 Introduction

In this paper we study the following cooperation-diffusion system.

$$\begin{aligned} &\frac{\partial \boldsymbol{u}}{\partial t} = A\Delta \boldsymbol{u} + \boldsymbol{f}(\boldsymbol{u}), \qquad && \boldsymbol{x} \in \mathbb{R}^N, t > 0, \\ &\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0, && \boldsymbol{x} \in \mathbb{R}^N. \end{aligned} \tag{1.1}$$

Here $N \geq 3$ is a given integer, and $\boldsymbol{u}_0$ is a given bounded and uniformly continuous function from $\mathbb{R}^N$ to $\mathbb{R}^2$. Now we put $\mathrm{D}_j = \partial/\partial x_j$ and $\mathrm{D}_{ij} = \partial^2/\partial x_i \partial x_j$ for $1 \leq i \leq N$ and $1 \leq j \leq N$, and have $\Delta = \sum_{j=1}^N \mathrm{D}_{jj}$. A matrix A and a function $\boldsymbol{f}$ are given by

$$A = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad \boldsymbol{f}(\boldsymbol{u}) = \begin{pmatrix} f_1(\boldsymbol{u}) \\ f_2(\boldsymbol{u}) \end{pmatrix}$$

for $d_j > 0$ for $j = 1, 2$ and for any $\boldsymbol{u} = (u_1, u_2)$, respectively. Now we put

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{n} = \begin{pmatrix} n \\ n \end{pmatrix}$$

for every $n \in \mathbb{Z}$. For any $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, v_2)$ we define $\boldsymbol{u} \leq \boldsymbol{v}$ if and only if $u_j \leq v_j$ for all $j = 1, 2$, and define $\boldsymbol{u} < \boldsymbol{v}$ if and only if $u_j < v_j$ for all $j = 1, 2$, respectively.

The following is the standing assumptions in this paper.

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(A1) $\boldsymbol{f}$ is of class C^2 from $[-1,2]^2$ to $\mathbb{R}^2$. One has $\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0}$, $\boldsymbol{f}(\boldsymbol{1}) = \boldsymbol{0}$ and

$$\mathrm{D}_j f_i(\boldsymbol{u}) > 0 \quad \text{if} \quad \boldsymbol{0} < \boldsymbol{u} < \boldsymbol{1} \text{ and } i \neq j.$$

(A2) There exist $k > 0$ and $\boldsymbol{\Phi} = (\Phi_1, \Phi_2)$ with

$$\begin{aligned} A\boldsymbol{\Phi}''(y) + k\boldsymbol{\Phi}'(y) + \boldsymbol{f}(\boldsymbol{\Phi}(y)) = \boldsymbol{0}, \qquad & y \in \mathbb{R}, \\ \boldsymbol{\Phi}(-\infty) = \boldsymbol{1},\ \boldsymbol{\Phi}(+\infty) = \boldsymbol{0}, & \\ -\boldsymbol{\Phi}'(y) \geq \boldsymbol{0} \qquad & y \in \mathbb{R}. \end{aligned} \tag{1.2}$$

(A3) All roots of $\det(\lambda^2 A + k\lambda I + \boldsymbol{f}'(\boldsymbol{0})) = 0$ have nonzero real parts. There exists $\boldsymbol{q}_0 \in \mathbb{R}^2$ with $\boldsymbol{q}_0 > \boldsymbol{0}$ and $\boldsymbol{f}'(\boldsymbol{0})\boldsymbol{q}_0 < \boldsymbol{0}$.

(A4) All roots of $\det(\lambda^2 A + k\lambda I + \boldsymbol{f}'(\boldsymbol{1})) = 0$ have nonzero real parts. There exists $\boldsymbol{p}_0 \in \mathbb{R}^2$ with $\boldsymbol{p}_0 > \boldsymbol{0}$ and $\boldsymbol{f}'(\boldsymbol{1})\boldsymbol{p}_0 < \boldsymbol{0}$.

(A5) There exists $\boldsymbol{a} = (a_1, a_2)$ with $\boldsymbol{0} < \boldsymbol{a} < \boldsymbol{1}$ and $\boldsymbol{f}(\boldsymbol{a}) = \boldsymbol{0}$, and

$$\boldsymbol{f}'(\boldsymbol{a}) = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$$

satisfies $\alpha_{12} > 0$, $\alpha_{21} > 0$ and $\alpha_{12}\alpha_{21} - \alpha_{11}\alpha_{22} > 0$. If $\boldsymbol{u}$ satisfies $\boldsymbol{0} \leq \boldsymbol{u} \leq \boldsymbol{1}$ and $\boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{0}$, one has $\boldsymbol{u} \in \{\boldsymbol{0}, \boldsymbol{1}, \boldsymbol{a}, (0,1), (1,0)\}$.

Here $\mathrm{D}_j f_i = \partial f_i / \partial u_j$ means the derivative of f_i by the j-the component u_j.

Example 1 (the Lotka–Volterra system) Kan-on [5] proved the existence and uniqueness of

$$\begin{aligned} &U''(y) + sU'(y) + U(y)(1 - U(y) - c_0 V(y)) = 0, \\ &dV''(y) + sV'(y) + V(y)(a_0 - b_0 U(y) - V(y)) = 0, \end{aligned} \qquad y \in \mathbb{R},$$
$$\begin{pmatrix} U(-\infty) \\ V(-\infty) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} U(+\infty) \\ V(+\infty) \end{pmatrix} = \begin{pmatrix} 0 \\ a_0 \end{pmatrix},$$
$$-U'(y) > 0, \quad V'(y) > 0 \qquad y \in \mathbb{R},$$

and showed

$$\frac{\partial s}{\partial a_0} < 0, \quad \frac{\partial s}{\partial b_0} > 0, \quad \frac{\partial s}{\partial c_0} < 0.$$

Here a_0, b_0, c_0 are positive constants with

$$\frac{1}{c_0} < a_0 < b_0,$$

which implies the *strong competition* between two species. If $s > 0$,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, \quad \boldsymbol{f}(\boldsymbol{u}) = \begin{pmatrix} u_1(1 - c_0 a_0 - u_1 + c_0 a_0 u_2) \\ (1 - u_2)(b_0 u_1 - a_0 u_2) \end{pmatrix}$$

satisfy (A1)–(A5) with

$$k = s, \quad \boldsymbol{\Phi}(y) = {}^t \left(U(y), 1 - \frac{V(y)}{a_0} \right).$$

Indeed, we have

$$\boldsymbol{a} = \begin{pmatrix} \dfrac{c_0 a_0 - 1}{b_0 c_0 - 1} \\ \dfrac{b_0(c_0 a_0 - 1)}{a_0(b_0 c_0 - 1)} \end{pmatrix}, \quad \boldsymbol{f}'(\boldsymbol{a}) = \begin{pmatrix} -\dfrac{c_0 a_0 - 1}{b_0 c_0 - 1} & \dfrac{c_0 a_0 (c_0 a_0 - 1)}{b_0 c_0 - 1} \\ \dfrac{b_0(b_0 - a_0)}{a_0(b_0 c_0 - 1)} & -\dfrac{b_0 - a_0}{b_0 c_0 - 1} \end{pmatrix}$$

and

$$\alpha_{12}\alpha_{21} - \alpha_{11}\alpha_{22} = \frac{(b_0 - a_0)(c_0 a_0 - 1)}{b_0 c_0 - 1} > 0.$$

If $s < 0$,

$$A = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{f}(\boldsymbol{u}) = \begin{pmatrix} u_1(a_0 - b_0 - a_0 u_1 + b_0 u_2) \\ (1 - u_2)(c_0 a_0 u_1 - u_2) \end{pmatrix}$$

satisfy (A1)–(A5) with

$$k = -s, \quad \boldsymbol{\Phi}(y) = {}^t \left(\frac{V(-y)}{a_0}, 1 - U(-y) \right).$$

Indeed, we have

$$\boldsymbol{a} = \begin{pmatrix} \dfrac{b_0 - a_0}{a_0(b_0 c_0 - 1)} \\ \dfrac{c_0(b_0 - a_0)}{b_0 c_0 - 1} \end{pmatrix}, \quad \boldsymbol{f}'(\boldsymbol{a}) = \begin{pmatrix} -\dfrac{b_0 - a_0}{b_0 c_0 - 1} & \dfrac{b_0(b_0 - a_0)}{a_0(b_0 c_0 - 1)} \\ \dfrac{c_0 a_0(c_0 a_0 - 1)}{b_0 c_0 - 1} & -\dfrac{c_0 a_0 - 1}{b_0 c_0 - 1} \end{pmatrix}$$

and

$$\alpha_{12}\alpha_{21} - \alpha_{11}\alpha_{22} = \frac{(b_0 - a_0)(c_0 a_0 - 1)}{b_0 c_0 - 1} > 0.$$

Example 2 (General Cases) In addition to (A1), (A3), (A4) and (A5), assume that all eigenvalues of $\boldsymbol{f}'(\boldsymbol{0})$ and all eigenvalues of $\boldsymbol{f}'(\boldsymbol{1})$ lie in $\{\lambda \in \mathbb{C} \,|\, \mathrm{Re} < 0\}$. Then there exists $k \in \mathbb{R}$ and $\boldsymbol{\Phi} = (\Phi_1, \Phi_2)$ that satisfy the one-dimensional profile equation (1.2). See [16] for the proof. Then assumptions (A1)–(A5) hold true if $k > 0$.

For a scalar bistable reaction-diffusion equations called the Nagumo equation or the unbalanced Allen–Cahn equation, multi-dimensional traveling fronts have been studied by [9, 2, 10, 3, 4, 12, 13, 7] and so on. Traveling fronts associated with strictly convex compact domain in $\mathbb{R}^{N-1}$ with a smooth boundary are studied for a scalar bistable reaction-diffusion equations in [14, 15]. The Lotka–Volterra system is a typical example of two-component cooperation-diffusion systems. For this system the existence and uniqueness of a one-dimensional traveling front is studied by Kan-on [5]. See also [6] and [1]. For general cooperation-diffusion systems, the existence and uniqueness of a one-dimensional traveling front is studied by [16]. Two-dimensional V-form fronts are studied by Wang [17]. Traveling fronts with pyramidal shapes are studied by Ni and myself [8]. The purpose of this paper is to show that a strictly convex compact set in $\mathbb{R}^{N-1}$ with a smooth boundary gives a traveling front to a competition-diffusion system in $\mathbb{R}^N$ by using a clear and concise argument. The argument in this paper might be useful for studies on other reaction-diffusion systems that admit comparison principles.

In this paper we assume $c > k$. By a moving coordinate system with speed c toward the x_N-direction we rewrite the cooperation-diffusion system. Let $\boldsymbol{x}' = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$. Let $z = x_N - ct$ and

$$\boldsymbol{u}(\boldsymbol{x}', x_N, t) = \boldsymbol{w}(\boldsymbol{x}', z, t).$$

Then we have

$$\begin{aligned} &(\mathrm{D}_t - A\Delta - c\mathrm{D}_N)\,\boldsymbol{w} - \boldsymbol{f}(\boldsymbol{w}) = \boldsymbol{0}, && \boldsymbol{x} \in \mathbb{R}^N, t > 0, \\ &\boldsymbol{w}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}), && \boldsymbol{x} \in \mathbb{R}^N. \end{aligned} \tag{1.3}$$

Rewriting z by x_N, we denote the solution of (1.3) by $\boldsymbol{w}(\boldsymbol{x}, t; \boldsymbol{u}_0)$. The profile equation of a traveling front in $\mathbb{R}^N$ is given by

$$(-A\Delta - c\mathrm{D}_N)\,\boldsymbol{v} - \boldsymbol{f}(\boldsymbol{v}) = \boldsymbol{0}, \qquad \boldsymbol{x} \in \mathbb{R}^N. \tag{1.4}$$

Now we define

$$m_* = \frac{\sqrt{c^2 - k^2}}{k}. \tag{1.5}$$

Then a planar traveling front

$$\boldsymbol{\Phi}\left(\frac{k}{c}(x_N - m_* x_1)\right)$$

satisfies (1.4).

In §4 we prove that there exists a cylindrically symmetric traveling front solution $\boldsymbol{U} = (U_1, U_2)$. Since it depends only on $|\boldsymbol{x}'|$ and x_N for $\boldsymbol{x} = (\boldsymbol{x}', x_N) \in \mathbb{R}^N$, we write $\boldsymbol{U}(|\boldsymbol{x}'|, x_N) = (U_1(|\boldsymbol{x}'|, x_N), U_2(|\boldsymbol{x}'|, x_N))$. Denoting $|\boldsymbol{x}'|$ and x_N by r and z, respectively, we have

$$\begin{gathered} A\left(-\mathrm{D}_{rr} - \frac{N-2}{r}\mathrm{D}_r - \mathrm{D}_{zz}\right)\boldsymbol{U} - c\mathrm{D}_z\boldsymbol{U} - \boldsymbol{f}(\boldsymbol{U}(r,z)) = \boldsymbol{0}, \qquad r > 0, z \in \mathbb{R}, \\ \mathrm{D}_r\boldsymbol{U}(0,z) = \boldsymbol{0}, \qquad z \in \mathbb{R}, \\ U_1(0,0) = a_1, \\ \boldsymbol{0} < \boldsymbol{U}(r,z) < \boldsymbol{1},\ \mathrm{D}_r\boldsymbol{U}(r,z) \geq \boldsymbol{0},\ -\mathrm{D}_z\boldsymbol{U}(r,z) > \boldsymbol{0}, \qquad r \geq 0, z \in \mathbb{R}, \\ \lim_{s\to\infty}\boldsymbol{U}(s+r, \phi(s)+z) = \boldsymbol{\Phi}\left(\frac{k}{c}(z - m_* r)\right) \qquad \text{in } C^2_{\mathrm{loc}}(\mathbb{R}^2). \end{gathered} \tag{1.6}$$

Here $\mathrm{D}_r\boldsymbol{U} = \partial\boldsymbol{U}/\partial r$, $\mathrm{D}_{rr}\boldsymbol{U} = \partial^2\boldsymbol{U}/\partial r^2$, $\mathrm{D}_z\boldsymbol{U} = \partial\boldsymbol{U}/\partial z$ and $\mathrm{D}_{zz}\boldsymbol{U} = \partial^2\boldsymbol{U}/\partial z^2$. Now ϕ is a nonnegative function with $\lim_{r\to\infty}\phi'(r) = m_*$. See §4 for the definition and properties of ϕ. See Lemma 8 and Lemma 10 in §4 for detailed properties of $\boldsymbol{U}$.

For any positive-valued function $g \in C^2(S^{N-2})$ let $D_g = \left\{ r\boldsymbol{\xi} \,|\, 0 \leq r < g(\boldsymbol{\xi}), \boldsymbol{\xi} \in S^{N-2} \right\}$ and let $C_g = \partial D_g = \{ g(\boldsymbol{\xi})\boldsymbol{\xi} \,|\, \boldsymbol{\xi} \in S^{N-2} \}$. Now we choose the signs of principal curvatures of C_g such that the principal curvatures of the boundary of S^{N-2} are $+1$ in this paper. Then, if all principal curvatures of C_g are positive at every point of C_g, $\overline{D_g}$ is a strictly convex compact set in $\mathbb{R}^{N-1}$. Let $\mathcal{G}$ be given by

$$\left\{ g \in C^2(S^{N-2}) \,|\, g > 0, \text{all principal curvature of } C_g \text{ are positive at every point of } C_g \right\}.$$

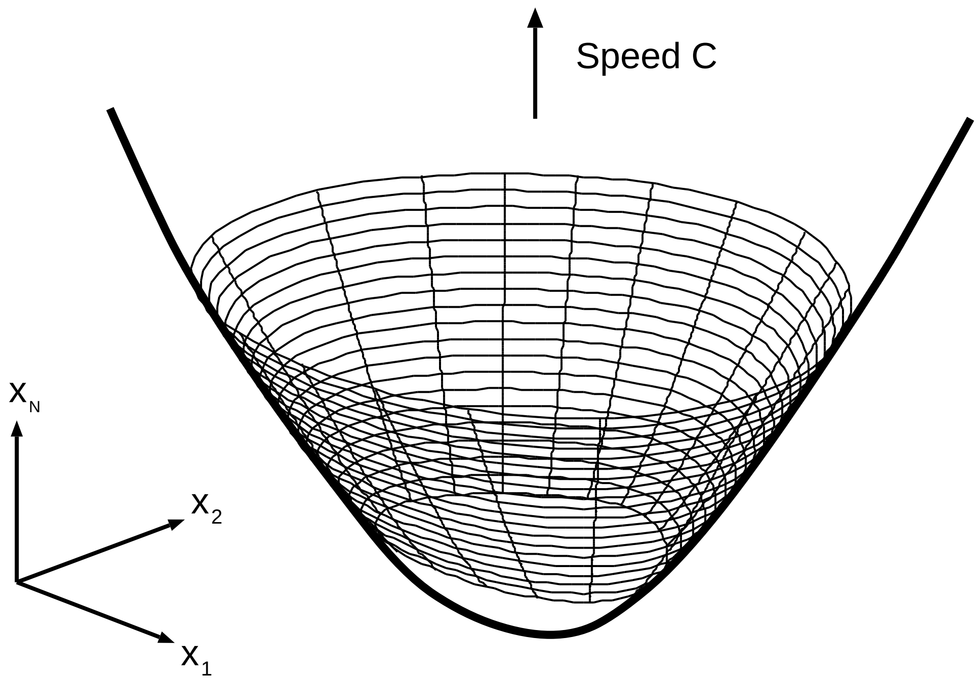

Figure 1: The graph of a level set of a component of $\widetilde{\boldsymbol{U}}$.

For any $g \in \mathcal{G}$ and $a \geq 0$ we define $g_1 = \tau_a g$ by

$$C_{g_1} = \left\{ \boldsymbol{z} \in \mathbb{R}^{N-1} \backslash D_g \mid \operatorname{dist}(\boldsymbol{z}, C_g) = a \right\}.$$

Then τ_a becomes a mapping in $\mathcal{G}$ by Lemma 11 in §5. We define an equivalence relation $g_1 \sim g_2$ if and only if one has either $g_1 = \tau_a g_2$ or $g_2 = \tau_a g_1$ for some $a \geq 0$. Roughly speaking, we define $g_1 \sim g_2$ if and only if one can expand D_{g_1} with a constant width and the expanded one equals D_{g_2} or one can expand D_{g_2} with a constant width and the expanded one equals D_{g_1}. See §5 for the details.

The following is the main assertion in this paper.

Theorem 1 *Let $g \in C^2(S^{N-2})$ satisfy $g(\boldsymbol{\xi}) > 0$ for all $\boldsymbol{\xi} \in S^{N-2}$. Assume that $D_g = \{ r\boldsymbol{\xi} \mid 0 \leq r \leq g(\boldsymbol{\xi}), \boldsymbol{\xi} \in S^{N-2} \}$ is a convex compact set in $\mathbb{R}^{N-1}$ and all principal curvatures of $\partial D_g = \{ g(\boldsymbol{\xi})\boldsymbol{\xi} \mid \boldsymbol{\xi} \in S^{N-2} \}$ are positive at every point of ∂D_g. Then there exists a unique solution $\widetilde{\boldsymbol{U}} = (\widetilde{U}_1, \widetilde{U}_2)$ to*

$$\left(-A \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} - c \frac{\partial}{\partial x_N} \right) \widetilde{\boldsymbol{U}} - \boldsymbol{f}(\widetilde{\boldsymbol{U}}) = \boldsymbol{0} \quad \text{in } \mathbb{R}^N, \tag{1.7}$$

$$\lim_{s \to \infty} \sup_{|\boldsymbol{x}| \geq s} \left| \widetilde{\boldsymbol{U}}(\boldsymbol{x}) - \min_{\boldsymbol{\xi} \in S^{N-2}} \boldsymbol{U}(|\boldsymbol{x}' - g(\boldsymbol{\xi})\boldsymbol{\xi}|, x_N) \right| = 0. \tag{1.8}$$

Here $\boldsymbol{x} = (\boldsymbol{x}', x_N) \in \mathbb{R}^N$. Let g_j satisfy the assumption stated above and let $\widetilde{\boldsymbol{U}}_j$ be the associated solution for $j = 1, 2$, respectively. One has

$$\widetilde{\boldsymbol{U}}_2(\boldsymbol{x}', x_N) = \widetilde{\boldsymbol{U}}_1(\boldsymbol{x}', x_N - \zeta) \qquad \text{for all } (\boldsymbol{x}', x_N) \in \mathbb{R}^N \tag{1.9}$$

with some $\zeta \in \mathbb{R}$ if and only if $g_1 \sim g_2$.

Let $\mathcal{G}$ be the set of all g that satisfies the assumption of Theorem 1. Let D_g be as in Theorem 1 for $g \in \mathcal{G}$. We define an equivalence relation in $\mathcal{G}$. Roughly speaking, we define $g_1 \sim g_2$ if and only if one can expand D_{g_1} with a constant width and the expanded one equals D_{g_2} or one can expand D_{g_2} with a constant width and the expanded one equals D_{g_1}. See §5 for the details. Theorem 1 says that each element of a quotient set $\mathcal{G}/\sim$ gives an N-dimensional traveling front $\widetilde{\boldsymbol{U}}$ in a competition-diffusion system. Figure 1 shows the graph of a level set of a component of $\widetilde{\boldsymbol{U}}$.

This paper is organized as follows. We state preliminaries in §2, and give a uniform estimate on pyramidal traveling fronts in §3 with respect to the number of lateral faces. Using this estimate we show that pyramidal traveling fronts converge to a cylindrically symmetric traveling front $\boldsymbol{U}$ as the number of lateral faces go to infinity and we state properties of $\boldsymbol{U}$ in §4. In §5 we define an equivalence relation for surfaces in $\mathbb{R}^{N-1}$ with positive principal curvatures. In §6 we give a proof of Theorem 1. We construct a weak supersolution and a weak subsolution by using $\boldsymbol{U}$, and prove the existence of $\widetilde{\boldsymbol{U}}$ between them. Finally we study the stability of a cylindrically non-symmetric traveling front $\widetilde{\boldsymbol{U}}$.

2 Preliminaries

Without loss of generality we can assume $|\boldsymbol{p}_0| = 1$ and $|\boldsymbol{q}_0| = 1$. From assumptions (A3) and (A4) we have

$$\boldsymbol{f}'(\boldsymbol{1})\boldsymbol{p}_0 \le -3\beta\boldsymbol{1}, \quad \boldsymbol{f}'(\boldsymbol{0})\boldsymbol{q}_0 \le -3\beta\boldsymbol{1}$$

for a constant $\beta > 0$. There exits a constant $\delta_* \in (0, 1/4)$ with

$$\begin{aligned} -\boldsymbol{f}'(\boldsymbol{v})\boldsymbol{p}_0 &\ge 2\beta\boldsymbol{1} \qquad \text{if } |\boldsymbol{v} - \boldsymbol{1}| < 2\delta_*, \\ -\boldsymbol{f}'(\boldsymbol{v})\boldsymbol{q}_0 &\ge 2\beta\boldsymbol{1} \qquad \text{if } |\boldsymbol{v}| < 2\delta_*. \end{aligned}$$

For

$$K = \max_{i,j} \max_{-\boldsymbol{1}\le \boldsymbol{u} \le \boldsymbol{2}} |\mathrm{D}_{ij} f_j(\boldsymbol{u})| \ge 0,$$

we define

$$\begin{aligned} \widetilde{\boldsymbol{f}}(\boldsymbol{u}) &= \boldsymbol{f}(\boldsymbol{u}) + \frac{K}{2}\begin{pmatrix} 2(u_2-1)\max\{0, u_1-1\} + (\max\{0, u_2-1\})^2 \\ 2(u_1-1)\max\{0, u_2-1\} + (\max\{0, u_1-1\})^2 \end{pmatrix} \\ &\quad + \frac{K}{2}\begin{pmatrix} 2u_2\max\{0, -u_1\} - (\max\{0, -u_2\})^2 \\ 2u_1\max\{0, -u_2\} - (\max\{0, -u_1\})^2 \end{pmatrix} \end{aligned}$$

and have

$$\begin{gathered} \widetilde{\boldsymbol{f}}(\boldsymbol{u}) = \boldsymbol{f}(\boldsymbol{u}), \ \widetilde{\boldsymbol{f}}'(\boldsymbol{u}) = \boldsymbol{f}'(\boldsymbol{u}) \qquad \text{if } \boldsymbol{0} \le \boldsymbol{u} \le \boldsymbol{1}, \\ \mathrm{D}_j \widetilde{f}_i(\boldsymbol{u}) \ge 0 \qquad \text{if } -\boldsymbol{1} \le \boldsymbol{u} \le \boldsymbol{2} \text{ and } i \ne j. \end{gathered}$$

See also [17]. Replacing $\boldsymbol{f}$ by $\widetilde{\boldsymbol{f}}$ if necessary, we can assume that $\boldsymbol{f}$ satisfies (A1)–(A5) and

$$\mathrm{D}_j f_i(\boldsymbol{u}) \ge 0 \qquad \text{if } -\boldsymbol{1} \le \boldsymbol{u} \le \boldsymbol{2} \text{ and } i \ne j$$

except that $\boldsymbol{f}$ is of class C^1 in $[-1, 2]^2$ to $\mathbb{R}^2$. Then we have the comparison principle as follows. For the proof see [11] for instance.

Lemma 1 *If* $\overline{\boldsymbol{u}}(\boldsymbol{x},t)$ *and* $\underline{\boldsymbol{u}}(\boldsymbol{x},t)$ *satisfy*

$$
\begin{gathered}
(\mathrm{D}_t - A\Delta)\overline{\boldsymbol{u}} - \boldsymbol{f}(\overline{\boldsymbol{u}}) \geq \boldsymbol{0}, \quad (\mathrm{D}_t - A\Delta)\underline{\boldsymbol{u}} - \boldsymbol{f}(\underline{\boldsymbol{u}}) \leq \boldsymbol{0} \qquad \boldsymbol{x} \in \mathbb{R}^N,\ t>0,\\
-\boldsymbol{1} \leq \underline{\boldsymbol{u}}(\boldsymbol{x},t) \leq \boldsymbol{2}, \quad -\boldsymbol{1} \leq \overline{\boldsymbol{u}}(\boldsymbol{x},t) \leq \boldsymbol{2} \qquad \boldsymbol{x} \in \mathbb{R}^N,\ t>0,\\
-\boldsymbol{1} \leq \underline{\boldsymbol{u}}(\boldsymbol{x},0) \leq \overline{\boldsymbol{u}}(\boldsymbol{x},0) \leq \boldsymbol{2} \qquad \boldsymbol{x} \in \mathbb{R}^N,
\end{gathered}
$$

then one has

$$\underline{\boldsymbol{u}}(\boldsymbol{x},t) \leq \overline{\boldsymbol{u}}(\boldsymbol{x},t) \qquad \boldsymbol{x} \in \mathbb{R}^N,\ t>0.$$

Moreover, if $\underline{u}_j(\,\cdot\,,0) \not\equiv \overline{u}_j(\,\cdot\,,0)$ *for some* j *in addition, one has* $\underline{u}_j(\boldsymbol{x},t) < \overline{u}_j(\boldsymbol{x},t)$ *for* $\boldsymbol{x} \in \mathbb{R}^N$*,* $t>0$*.*

Using this comparison principle and (A2), we have $-\boldsymbol{\Phi}'(y) > \boldsymbol{0}$ for $y \in \mathbb{R}$. Let

$$M = \max_{i,j} \max_{-\boldsymbol{1}\leq \boldsymbol{s} \leq \boldsymbol{2}} |\mathrm{D}_j f_i(\boldsymbol{s})| > 0, \tag{2.1}$$

and define $\theta_* \in (0,\pi/2)$ by $\tan\theta_* = m_*$. Choosing $\chi_0 \in C^\infty(\mathbb{R})$ with

$$
\begin{array}{rll}
 & \chi_0(y) \equiv 1 & \text{if } \ 4y \geq 3,\\
0 < \chi_0(y) < 1, & 0 < \chi_0'(y) & \text{if } \ 1 < 4y < 3,\\
 & \chi_0(y) \equiv 0 & \text{if } \ 4y \leq 1,
\end{array}
$$

we define

$$\boldsymbol{p}(y) = \boldsymbol{q}_0 + \chi_0(y)(\boldsymbol{p}_0 - \boldsymbol{q}_0) \qquad \text{for } y \in \mathbb{R}. \tag{2.2}$$

Let $n \geq 2$ be a given integer and let$\{\boldsymbol{a}_j\}_{j=1}^n$ be a set of unit vectors in $\mathbb{R}^{N-1}$ with $\boldsymbol{a}_i \neq \boldsymbol{a}_j$ for $i \neq j$. Then $\boldsymbol{a}_j = (a_j^1, \ldots, a_j^{N-1})$ satisfies

$$|\boldsymbol{a}_j|^2 = \sum_{i=1}^{N-1} (a_j^i)^2 = 1 \qquad \text{for all } 1 \leq j \leq n.$$

Here we put $\boldsymbol{x}' = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$ and $\boldsymbol{x} = (\boldsymbol{x}', x_N) = (x_1, \ldots, x_N) \in \mathbb{R}^N$ with $|\boldsymbol{x}'| = \sqrt{\sum_{i=1}^{N-1} x_i^2}$ and $|\boldsymbol{x}| = \sqrt{\sum_{i=1}^{N} x_i^2}$, respectively. For $\boldsymbol{x}' \in \mathbb{R}^{N-1}$ we set

$$
\begin{aligned}
h_j(\boldsymbol{x}') &= m_*(\boldsymbol{a}_j, \boldsymbol{x}'), &(2.3)\\
h(\boldsymbol{x}') &= \max_{1\leq j\leq n} h_j(\boldsymbol{x}') = m_* \max_{1\leq j \leq n} (\boldsymbol{a}_j, \boldsymbol{x}'). &(2.4)
\end{aligned}
$$

Here $(\boldsymbol{a}_j, \boldsymbol{x}')$ denotes the inner product of vectors $\boldsymbol{a}_j$ and $\boldsymbol{x}'$. In this paper we call $\{(\boldsymbol{x}', x_N) \in \mathbb{R}^N \,\big|\, x_N \geq h(\boldsymbol{x}')\}$ a pyramid. Setting

$$\Omega_j = \{\boldsymbol{x}' \in \mathbb{R}^{N-1} \,|\, h(\boldsymbol{x}') = h_j(\boldsymbol{x}')\}$$

for $j = 1, \ldots, n$, we have

$$\mathbb{R}^{N-1} = \cup_{j=1}^n \Omega_j.$$

We denote the boundary of Ω_j by $\partial\Omega_j$. Now we put

$$S_j = \{\boldsymbol{x} \in \mathbb{R}^N \,|\, x_N = h_j(\boldsymbol{x}') \quad \text{for } \boldsymbol{x}' \in \Omega_j\}$$

for each j, and call $\cup_j^n S_j \subset \mathbb{R}^N$ the lateral faces of a pyramid. We put

$$\Gamma_j = \{\boldsymbol{x} \in \mathbb{R}^N \,|\, x_N = h_j(\boldsymbol{x}') \quad \text{for } \boldsymbol{x}' \in \partial\Omega_j\}$$

for $j = 1, \ldots, n$. Then $\cup_{j=1}^n \Gamma_j$ represents the set of all edges of a pyramid. For $\gamma > 0$ let

$$D(\gamma) = \left\{\boldsymbol{x} \in \mathbb{R}^N \mid \operatorname{dist}\left(\boldsymbol{x}, \cup_{j=1}^n \Gamma_j\right) > \gamma\right\}.$$

Now we define $\underline{\boldsymbol{v}}(\boldsymbol{x})$ by

$$\underline{\boldsymbol{v}}(\boldsymbol{x}) = \boldsymbol{\Phi}\left(\frac{k}{c}(x_N - h(\boldsymbol{x}'))\right) = \max_{1\le j\le n} \boldsymbol{\Phi}\left(\frac{k}{c}(x_N - h_j(\boldsymbol{x}'))\right).$$

Pyramidal traveling fronts are stated as follows.

Theorem 2 ([8]) *Assume (A1), (A2), (A3) and (A4). Let h be given in (2.4). Let $\boldsymbol{V} = (V_1, V_2)$ be defined by*

$$\boldsymbol{V}(\boldsymbol{x}) = \lim_{t\to\infty} \boldsymbol{w}(\boldsymbol{x}, t; \underline{\boldsymbol{v}}) \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^N,$$

where $\boldsymbol{w}(\boldsymbol{x}, t; \underline{\boldsymbol{v}})$ is the solution of (1.3) for $\boldsymbol{u}_0 = \underline{\boldsymbol{v}}$. Then one has

$$\begin{aligned} &(-A\Delta - c\mathrm{D}_N)\,\boldsymbol{V} - \boldsymbol{f}(\boldsymbol{V}) = \boldsymbol{0}, \qquad \boldsymbol{x} \in \mathbb{R}^N, \\ &\lim_{\gamma\to\infty} \sup_{\boldsymbol{x}\in D(\gamma)} |\boldsymbol{V}(\boldsymbol{x}) - \underline{\boldsymbol{v}}(\boldsymbol{x})| = 0, \\ &\boldsymbol{0} < \underline{\boldsymbol{v}}(\boldsymbol{x}) < \boldsymbol{V}(\boldsymbol{x}) < \boldsymbol{1} \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^N. \end{aligned} \tag{2.5}$$

Here we state lemmas that we will use later.

Lemma 2 *Let h be given in (2.4) and let $\boldsymbol{V} = (V_1, V_2)$ be as in Theorem 2. For any given $\boldsymbol{t} = (\boldsymbol{t}', t_N) \in \mathbb{R}^N$ with $t_N > 0$ and $m_*|\boldsymbol{t}'| \le t_N$, one has*

$$-\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{t}} \ge \boldsymbol{0} \qquad \text{in } \mathbb{R}^N. \tag{2.6}$$

Moreover, for $j = 1, 2$, one has

$$-\mathrm{D}_N V_j \ge \frac{k}{c}|\nabla V_j| \qquad \text{in } \mathbb{R}^N.$$

Proof. For any $\varepsilon > 0$, we have

$$\underline{\boldsymbol{v}}(\boldsymbol{x} + \varepsilon\boldsymbol{t}) \le \underline{\boldsymbol{v}}(\boldsymbol{x}) \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^N.$$

Then, from the definition of v, we get

$$\boldsymbol{V}(\boldsymbol{x} + \varepsilon\boldsymbol{t}) \le \boldsymbol{V}(\boldsymbol{x}) \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^N.$$

This gives

$$\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{t}} \leq \boldsymbol{0} \qquad \text{in } \mathbb{R}^N$$

for $j = 1, 2$. The latter inequality follows from

$$\left(-\frac{\nabla V_j}{|\nabla V_j|}, \boldsymbol{e}_N \right) \geq \cos\theta_* = \frac{k}{c},$$

where $\boldsymbol{e}_N = {}^t(0, \ldots, 0, 1) \in \mathbb{R}^N$. This completes the proof. □

Lemma 3 *Let $\boldsymbol{v}$ satisfy (1.4) with $\boldsymbol{0} \leq \boldsymbol{v} \leq \boldsymbol{1}$. Then one can choose a constant $m_0 > 0$ and has*

$$\max\left\{ \max_{1\leq j\leq 2} \sup_{\boldsymbol{x}\in\mathbb{R}^N} |\mathrm{D}_j \boldsymbol{v}(\boldsymbol{x})|, \max_{1\leq i,j\leq 2} \sup_{\boldsymbol{x}\in\mathbb{R}^N} |\mathrm{D}_i\mathrm{D}_j \boldsymbol{v}(\boldsymbol{x})| \right\} \leq m_0. \tag{2.7}$$

Here m_0 depends only on $(\boldsymbol{f}, A, c, N)$.

Proof. Let $p \in (1, \infty)$ and γ_0 satisfy

$$0 < \gamma_0 < 1 - \frac{N}{p}.$$

For any $\boldsymbol{x}_0 \in \mathbb{R}^N$ and $j = 1, 2$, we have

$$\|v_j\|_{W^{2,p}(B(\boldsymbol{x}_0;1))} \leq C_0 \left(\|v_j\|_{L^p(B(\boldsymbol{x}_0;1))} + \left\| \max_{\boldsymbol{0}\leq \boldsymbol{s}\leq \boldsymbol{1}} |f_j(\boldsymbol{s})| \right\|_{L^p(B(\boldsymbol{x}_0;1))} \right)$$

by applying the Schauder interior estimate to (1.4). Here a constant C_0 depends only on c and $\boldsymbol{f}$, and $B(\boldsymbol{x}_0; 1) = \{\boldsymbol{x} \in \mathbb{R}^N \mid |\boldsymbol{x} - \boldsymbol{x}_0| < 1\}$. Now we have

$$\max_{1\leq j\leq 2} \sup_{\boldsymbol{x}\in\mathbb{R}^N} |\mathrm{D}_j \boldsymbol{v}(\boldsymbol{x})| < \infty$$

by using the Sobolev imbedding $W^{2,p}(B(\boldsymbol{x}_0; 2)) \subset C^{1,\gamma_0}(\overline{B(\boldsymbol{x}_0; 1)})$. Differentiating (1.4) by x_i and applying the argument state above to $\mathrm{D}_i \boldsymbol{v}$, we complete the proof. □

Lemma 4 *Let h be given by (2.4), let $\boldsymbol{V}$ be as in Theorem 2 and let $1 \leq j \leq N - 1$. Assume*

$$h(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_{N-1}) = h(x_1, \ldots, x_{j-1}, |x_j|, x_{j+1}, \ldots, x_{N-1})$$

for $(x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$. Then one has

$$\boldsymbol{V}(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_N) = \boldsymbol{V}(x_1, \ldots, x_{j-1}, |x_j|, x_{j+1}, \ldots, x_N)$$

for $(x_1, \ldots, x_N) \in \mathbb{R}^N$, and

$$\begin{aligned} \mathrm{D}_j \boldsymbol{V}(x_1, x_2, \ldots, x_N) \geq \boldsymbol{0} \qquad & \textit{for} \qquad x_j > 0,\ (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N) \in \mathbb{R}^{N-1}, \\ \mathrm{D}_j \boldsymbol{V}(x_1, \ldots, x_{j-1}, 0, x_{j+1}, x_N) = \boldsymbol{0} \qquad & \textit{for} \qquad (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N) \in \mathbb{R}^{N-1}. \end{aligned}$$

Proof. Without loss of generality, we can assume $j = 1$. The former statement follows from the definition of V in Theorem 2 and

$$\underline{\boldsymbol{v}}(x_1, x_2, \ldots, x_{N-1}) = \underline{\boldsymbol{v}}(|x_1|, x_2, \ldots, x_{N-1})$$

for $(x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$.

For the latter statement we have

$$\begin{aligned} \mathrm{D}_1 h(x_1, x_2, \ldots, x_{N-1}) \geq 0 \quad &\text{for} \quad x_1 > 0, (x_2, \ldots, x_{N-1}) \in \mathbb{R}^{N-2}, \\ \mathrm{D}_1 h(0, x_2, \ldots, x_{N-1}) = 0 \quad &\text{for} \quad (x_2, \ldots, x_{N-1}) \in \mathbb{R}^{N-2}. \end{aligned}$$

Then we get

$$\begin{aligned} \mathrm{D}_1 \underline{\boldsymbol{v}}(x_1, x_2, \ldots, x_{N-1}) \geq \mathbf{0} \quad &\text{for} \quad x_1 > 0, (x_2, \ldots, x_{N-1}) \in \mathbb{R}^{N-2}, \\ \mathrm{D}_1 \underline{\boldsymbol{v}}(0, x_2, \ldots, x_{N-1}) = \mathbf{0} \quad &\text{for} \quad (x_2, \ldots, x_{N-1}) \in \mathbb{R}^{N-2}. \end{aligned}$$

Now $\boldsymbol{w}_1(\boldsymbol{x}, t) = \mathrm{D}_1 \boldsymbol{w}(\boldsymbol{x}, t; \underline{\boldsymbol{v}})$ satisfies

$$\begin{aligned} (\mathrm{D}_t - A\Delta - c\mathrm{D}_N - \boldsymbol{f}'(\boldsymbol{w}(\boldsymbol{x}, t; \underline{\boldsymbol{v}})))\, \boldsymbol{w}_1 = \mathbf{0} \quad &\text{for} \quad x_1 > 0, (x_2, \ldots, x_N) \in \mathbb{R}^{N-1}, t > 0, \\ \mathrm{D}_1 \boldsymbol{w}_1(0, x_2, \ldots, x_N) = \mathbf{0} \quad &\text{for} \quad (x_2, \ldots, x_N) \in \mathbb{R}^{N-1}. \\ \boldsymbol{w}_1(\boldsymbol{x}, 0) \geq \mathbf{0} \quad &\text{for} \quad x_1 > 0, (x_2, \ldots, x_N) \in \mathbb{R}^{N-1}. \end{aligned}$$

Then we get

$$\mathrm{D}_1 \boldsymbol{w}(\boldsymbol{x}, t; \underline{\boldsymbol{v}}) \geq \mathbf{0} \qquad \text{for } x_1 > 0,\ (x_2, \ldots, x_N) \in \mathbb{R}^{N-1},\ t > 0.$$

Sending $t \to \infty$, we obtain

$$\begin{aligned} \mathrm{D}_1 \boldsymbol{V}(\boldsymbol{x}) \geq \mathbf{0} \qquad &\text{for } x_1 > 0,\ (x_2, \ldots, x_N) \in \mathbb{R}^{N-1}, \\ \mathrm{D}_1 \boldsymbol{V}(0, x_2, \ldots, x_N) = \mathbf{0} \qquad &\text{for } (x_2, \ldots, x_N) \in \mathbb{R}^{N-1}. \end{aligned}$$

This completes the proof. ∎

Let

$$\boldsymbol{K}(\boldsymbol{x}, t) = \begin{pmatrix} K_1(\boldsymbol{x}, t) & 0 \\ 0 & K_2(\boldsymbol{x}, t) \end{pmatrix},$$

where

$$K_i(\boldsymbol{x}, t) = \frac{1}{(4\pi d_i t)^{\frac{N}{2}}} \exp\left(-\frac{|\boldsymbol{x}'|^2 + (x_N + ct)^2}{4 d_i t} \right) \qquad \text{for } i = 1, 2.$$

The following lemma plays an important role.

Lemma 5 *Let $\boldsymbol{t}$ be as in Lemma 2 and let M be given by (2.1). Let $\boldsymbol{v}$ satisfy (1.4) with*

$$\mathbf{0} \leq -\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{t}}(\boldsymbol{x}) \qquad \textit{for all } \boldsymbol{x} \in \mathbb{R}^N.$$

Then one has

$$\begin{aligned} e^{-M}\int_{\mathbb{R}^N} \boldsymbol{K}(\boldsymbol{x}-\boldsymbol{y},1)\left(-\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{t}}(\boldsymbol{y})\right)\,\mathrm{d}\boldsymbol{y} &\\ +\int_0^1 e^{-M(1-s)}\left(\int_{\mathbb{R}^N}\boldsymbol{K}(\boldsymbol{x}-\boldsymbol{y},1-s)\begin{pmatrix}\mathrm{D}_2 f_1(\boldsymbol{v}(\boldsymbol{y}))(-\frac{\partial v_2}{\partial \boldsymbol{t}}(\boldsymbol{y}))\\ \mathrm{D}_1 f_2(\boldsymbol{v}(\boldsymbol{y}))(-\frac{\partial v_1}{\partial \boldsymbol{t}}(\boldsymbol{y}))\end{pmatrix}\mathrm{d}\boldsymbol{y}\right)\mathrm{d}s &\\ \le -\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{t}}(\boldsymbol{x}) & \quad (2.8)\end{aligned}$$

for all $\boldsymbol{x}\in\mathbb{R}^N$. *Especially, for every* $\boldsymbol{x}_0\in\mathbb{R}^N$ *and* $R>0$, *one has*

$$\left(\max_{B(\boldsymbol{x}_0;R)}\left(-\frac{\partial v_j}{\partial \boldsymbol{t}}\right)\right)^{N+1} \le K_R \min_{B(\boldsymbol{x}_0;R)}\left(-\frac{\partial v_j}{\partial \boldsymbol{t}}\right) \quad \text{for } 1\le i\le 2,$$

$$\left(\min_{B(\boldsymbol{x}_0;R)}\mathrm{D}_j f_i(\boldsymbol{v})\right)\left(\max_{B(\boldsymbol{x}_0;R)}\left(-\frac{\partial v_j}{\partial \boldsymbol{t}}\right)\right)^{N+1} \le K_R \min_{B(\boldsymbol{x}_0;R)}\left(-\frac{\partial v_i}{\partial \boldsymbol{t}}\right) \quad \text{if } i\ne j.$$

Here a positive constant K_R *depends only on* $(R,\boldsymbol{f},A,c,N)$.

Proof. Now $\boldsymbol{w}(\boldsymbol{x},t;\boldsymbol{v})=\boldsymbol{v}(\boldsymbol{x})$ satisfies

$$\begin{aligned}(\mathrm{D}_t - A\Delta - c\mathrm{D}_N)\boldsymbol{w} - \boldsymbol{f}(\boldsymbol{w}) = \boldsymbol{0} \qquad & \boldsymbol{x}\in\mathbb{R}^N, t>0,\\ \boldsymbol{w}(\boldsymbol{x},0)=\boldsymbol{v}(\boldsymbol{x}) \qquad & \boldsymbol{x}\in\mathbb{R}^N.\end{aligned}$$

Then $-\frac{\partial \boldsymbol{w}}{\partial \boldsymbol{t}}$ satisfies

$$\begin{aligned}(\mathrm{D}_t - d_1\Delta - c\mathrm{D}_N - \mathrm{D}_1 f_1(\boldsymbol{v}))\left(-\frac{\partial w_1}{\partial \boldsymbol{t}}\right) &= \mathrm{D}_2 f_1(\boldsymbol{v})\left(-\frac{\partial v_2}{\partial \boldsymbol{t}}\right)\ge 0,\\ (\mathrm{D}_t - d_2\Delta - c\mathrm{D}_N - \mathrm{D}_2 f_2(\boldsymbol{v}))\left(-\frac{\partial w_2}{\partial \boldsymbol{t}}\right) &= \mathrm{D}_1 f_2(\boldsymbol{v})\left(-\frac{\partial v_1}{\partial \boldsymbol{t}}\right)\ge 0\end{aligned}$$

for $\boldsymbol{x}\in\mathbb{R}^N$ and $t>0$ with

$$-\frac{\partial \boldsymbol{w}}{\partial \boldsymbol{t}}(\boldsymbol{x},0) = -\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{t}}(\boldsymbol{x}) \qquad \text{for } \boldsymbol{x}\in\mathbb{R}^N.$$

Let $\boldsymbol{W}(\boldsymbol{x},t)=(W_1(\boldsymbol{x},t),W_2(\boldsymbol{x},t))$ be given by

$$\begin{aligned}(\mathrm{D}_t - A\Delta - c\mathrm{D}_N + M)\boldsymbol{W} = \begin{pmatrix}\mathrm{D}_2 f_1(\boldsymbol{v})\left(-\frac{\partial v_2}{\partial \boldsymbol{t}}\right)\\ \mathrm{D}_1 f_2(\boldsymbol{v})\left(-\frac{\partial v_1}{\partial \boldsymbol{t}}\right)\end{pmatrix} \qquad & \boldsymbol{x}\in\mathbb{R}^N, t>0,\\ \boldsymbol{W}(\boldsymbol{x},0) = -\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{t}}(\boldsymbol{x}) \qquad & \boldsymbol{x}\in\mathbb{R}^N.\end{aligned}$$

Then we have

$$(\mathrm{D}_t - A\Delta - c\mathrm{D}_N + M)\begin{pmatrix}W_1+\frac{\partial w_1}{\partial \boldsymbol{t}}\\ W_2+\frac{\partial w_2}{\partial \boldsymbol{t}}\end{pmatrix} = -\begin{pmatrix}(M+\mathrm{D}_1 f_1(\boldsymbol{v}))\left(-\frac{\partial w_1}{\partial \boldsymbol{t}}\right)\\ (M+\mathrm{D}_2 f_2(\boldsymbol{v}))\left(-\frac{\partial w_2}{\partial \boldsymbol{t}}\right)\end{pmatrix}\le \boldsymbol{0}.$$

The maximum principle yields

$$\boldsymbol{W}(\boldsymbol{x},t) \leq -\frac{\partial \boldsymbol{w}}{\partial \boldsymbol{t}}(\boldsymbol{x},t) = -\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{t}}(\boldsymbol{x}).$$

Now we have

$$\begin{aligned}\boldsymbol{W}(\boldsymbol{x},t) &= e^{-Mt}\int_{\mathbb{R}^N} \boldsymbol{K}(\boldsymbol{x}-\boldsymbol{y},t)\left(-\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{t}}(\boldsymbol{y})\right)\mathrm{d}\boldsymbol{y} \\ &\quad + \int_0^t e^{-M(t-s)}\left(\int_{\mathbb{R}^N} \boldsymbol{K}(\boldsymbol{x}-\boldsymbol{y},t-s)\begin{pmatrix}\mathrm{D}_2 f_1(\boldsymbol{v}(\boldsymbol{y}))\left(-\frac{\partial v_2}{\partial \boldsymbol{t}}(\boldsymbol{y})\right)\\ \mathrm{D}_1 f_2(\boldsymbol{v}(\boldsymbol{y}))\left(-\frac{\partial v_1}{\partial \boldsymbol{t}}(\boldsymbol{y})\right)\end{pmatrix}\mathrm{d}\boldsymbol{y}\right)\mathrm{d}s > \boldsymbol{0}.\end{aligned}$$

Putting $t=1$, we obtain (2.8). Fix $i \in \{1,2\}$, and let $j \neq i$. Let $\boldsymbol{y}_0 \in \overline{B(\boldsymbol{x}_0;R)}$ satisfy

$$-\frac{\partial v_j}{\partial \boldsymbol{t}}(\boldsymbol{y}_0) = \max_{B(\boldsymbol{x}_0;R)}\left(-\frac{\partial v_j}{\partial \boldsymbol{t}}\right).$$

Then we have

$$-\frac{\partial v_j}{\partial \boldsymbol{t}}(\boldsymbol{x}) \geq \frac{1}{2}\max_{B(\boldsymbol{x}_0;R)}\left(-\frac{\partial v_j}{\partial \boldsymbol{t}}\right) \qquad \text{if } \boldsymbol{x} \in B(\boldsymbol{y}_0;r_0),$$

where $r_0 = \delta_0 \max_{\overline{B(\boldsymbol{x}_0;R)}}(-\frac{\partial v_j}{\partial \boldsymbol{t}})$ and we can fix $\delta_0 > 0$ depending only on $(R,\boldsymbol{f},A,c,N)$ due to Lemma 3. From (2.8) we find

$$\frac{1}{2}\max_{B(\boldsymbol{x}_0;R)}\left(-\frac{\partial v_j}{\partial \boldsymbol{t}}\right)e^{-M}\int_{B(\boldsymbol{y}_0;r_0)} \boldsymbol{K}(\boldsymbol{x}-\boldsymbol{y},1)\,\mathrm{d}\boldsymbol{y} \leq -\frac{\partial v_j}{\partial \boldsymbol{t}}(\boldsymbol{x})$$

for all $\boldsymbol{x} \in \mathbb{R}^N$. Then we obtain

$$\begin{aligned}&\left(\min_{B(\boldsymbol{x}_0;R)} \mathrm{D}_j f_i(\boldsymbol{v})\right)\max_{B(\boldsymbol{x}_0;R)}\left(-\frac{\partial v_j}{\partial \boldsymbol{t}}\right)\int_0^1 e^{-M(1-s)}\left(\int_{B(\boldsymbol{y}_0;r_0)\cap B(\boldsymbol{x}_0;R)} K_i(\boldsymbol{x}-\boldsymbol{y},1-s)\mathrm{d}\boldsymbol{y}\right)\mathrm{d}s \\ &\qquad \leq -2\frac{\partial v_i}{\partial \boldsymbol{t}}(\boldsymbol{x})\end{aligned}$$

for all $\boldsymbol{x} \in \mathbb{R}^N$. Using these inequalities, we obtain the last two inequalities of the lemma. This completes the proof. □

3 A uniform estimate on pyramidal traveling fronts

In this section we give an estimate of the widths of transition layers of pyramidal traveling fronts with n lateral faces uniformly in n. This uniform estimate enables us to take the limit of $n \to \infty$ in §4.

Lemma 6 *Let $\boldsymbol{v}^{(n)} = (v_1^{(n)}, v_2^{(n)})$ be as in Theorem 2 associated with (2.4). Then, for any $\delta \in (0,\delta_*)$, there exists $\varepsilon_0 > 0$ such that one has*

$$-\mathrm{D}_N v_j^{(n)}(\boldsymbol{x}) \geq \varepsilon_0 \quad \text{if} \quad |f_j(\boldsymbol{v}^{(n)}(\boldsymbol{x}))| \geq \delta$$

for $j=1,2$. Here ε_0 depends only on $(\delta,\boldsymbol{f},A,c,N)$, and is independent of (2.4).

Proof. Assume the contrary. Then there exist $\boldsymbol{y}^{(n)} \in \mathbb{R}^N$ such that we have

$$(-A\Delta - c\mathrm{D}_N)\,\boldsymbol{v}^{(n)} - \boldsymbol{f}(\boldsymbol{v}^{(n)}) = \boldsymbol{0} \qquad \text{in } \mathbb{R}^N,$$
$$|f_j(\boldsymbol{v}^{(n)}(\boldsymbol{y}^{(n)}))| \geq \delta, \quad \lim_{n\to\infty} |\mathrm{D}_N v_j^{(n)}(\boldsymbol{y}^{(n)})| = 0.$$

Let $\mathcal{V}_N$ and $\mathcal{A}_{N-1}$ be the volume and the surface area of a unit ball in $\mathbb{R}^N$. We choose $R > 0$ large enough such that we have

$$\delta R \mathcal{V}_N \geq 2\max\{1, m_0\}\mathcal{A}_{N-1} \max\{d_1, d_2, |c|\}\,.$$

Lemma 2 and Lemma 5 give

$$\lim_{n\to\infty} \sup_{B(\boldsymbol{y}^{(n)};R)} |\nabla \boldsymbol{v}^{(n)}| = 0$$

or

$$\lim_{n\to\infty} \sup_{B(\boldsymbol{y}^{(n)};R)} |\mathrm{D}_i f_j(\boldsymbol{v}^{(n)})| = 0 \qquad \text{for } i \neq j.$$

Then we have

$$\lim_{n\to\infty} \sup_{B(\boldsymbol{y}^{(n)};R)} |f_j(\boldsymbol{v}^{(n)}) - \alpha_0| = 0$$

for a constant $\alpha_0 \in [0, 1]$ with $|\alpha_0| \geq \delta$. Integrating (1.4) over $B(\boldsymbol{y}^{(n)}; R)$, we have

$$\int_{\partial B(\boldsymbol{y}^{(n)};R)} \left(A\frac{\partial v_j^{(n)}}{\partial \boldsymbol{\nu}} + c\nu_N v_j^{(n)} \right) \mathrm{d}s = -\int_{B(\boldsymbol{y}^{(n)};R)} f_j(\boldsymbol{v}^{(n)}(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}. \tag{3.1}$$

Now we estimate the left-hand side as

$$\left| \int_{\partial B(\boldsymbol{y}^{(n)};R)} \left(A\frac{\partial v_j^{(n)}}{\partial \boldsymbol{\nu}} + c\nu_N v_j^{(n)} \right) \mathrm{d}s \right| \leq \sqrt{2}\max\{1, m_0\}\max\{d_1, d_2, |c|\}\mathcal{A}_{N-1}R^{N-1}.$$

Next we estimate the right-hand side as

$$\lim_{j\to\infty} \left| \int_{B(\boldsymbol{y}^{(n)};R)} f_j(\boldsymbol{v}^{(n)})\,\mathrm{d}\boldsymbol{x} \right| \geq \delta R^N \mathcal{V}_N.$$

Now we have a contradiction in (3.1) taking j large enough. This completes the proof. □

Proposition 1 *Let $\boldsymbol{v} = (v_1, v_2)$ be as in Theorem 2 associated with (2.4). Then there exists positive constants p_0 and δ_0 such that one has*

$$-\mathrm{D}_N \boldsymbol{v}(\boldsymbol{x}) \geq p_0 \boldsymbol{1} \quad \text{if } |\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{a}| < \delta_0. \tag{3.2}$$

Here p_0 and δ_0 depend only on $(\boldsymbol{f}, A, c, N)$ and are independent of (2.4).

For a while we make preparation to prove this proposition. We write (1.4) as

$$\begin{aligned} -d_1\Delta v_1 - c\mathrm{D}_N v_1 - f_1(v_1, v_2) = 0, \\ -d_2\Delta v_2 - c\mathrm{D}_N v_2 - f_2(v_1, v_2) = 0, \end{aligned} \quad \boldsymbol{x} \in \mathbb{R}^N. \tag{3.3}$$

Without loss of generality we can assume

$$\frac{\alpha_{11}}{d_1} \geq \frac{\alpha_{22}}{d_2}. \tag{3.4}$$

Indeed, otherwise we exchange v_1 and v_2, and study (v_2, v_1) to

$$\begin{array}{l} -d_2\Delta v_2 - c\mathrm{D}_{N} v_2 - f_2(v_1, v_2) = 0, \\ -d_1\Delta v_1 - c\mathrm{D}_{N} v_1 - f_1(v_1, v_2) = 0, \end{array} \quad \boldsymbol{x} \in \mathbb{R}^N. \tag{3.5}$$

Then we have (3.4) and assumptions (A1)–(A5) for this new system. Next we assume $d_1 = 1$ without loss of generality. Denoting d_2 simply by d, we have

$$-\Delta v_1 - c\mathrm{D}_{N} v_1 - f_1(v_1, v_2) \;=\; 0, \tag{3.6}$$

$$-\Delta v_2 - \frac{c}{d}\mathrm{D}_{N} v_2 - \frac{1}{d} f_2(v_1, v_2) \;=\; 0 \tag{3.7}$$

for all $\boldsymbol{x} \in \mathbb{R}^N$. Now we have

$$d\alpha_{11} \geq \alpha_{22}. \tag{3.8}$$

Multiplying (3.6) by $\mathrm{D}_{N} v_2$ and (3.7) by $\mathrm{D}_{N} v_1$ respectively and using

$$(\mathrm{D}_{N} v_2)\Delta v_1 + (\mathrm{D}_{N} v_1)\Delta v_2 = \operatorname{div}\left((\mathrm{D}_{N} v_2)\nabla v_1\right) + \operatorname{div}\left((\mathrm{D}_{N} v_1)\nabla v_2\right) - \mathrm{D}_{N}\left(\nabla v_1, \nabla v_2\right),$$

we obtain

$$\begin{aligned} &-\operatorname{div}\left((\mathrm{D}_{N} v_2)\nabla v_1\right) - \operatorname{div}\left((\mathrm{D}_{N} v_1)\nabla v_2\right) + \mathrm{D}_{N}(\nabla v_1, \nabla v_2) \\ &-c\left(1 + \frac{1}{d}\right)(\mathrm{D}_{N} v_1)(\mathrm{D}_{N} v_2) - f_1(v_1, v_2)\mathrm{D}_{N} v_2 - \frac{1}{d} f_2(v_1, v_2)\mathrm{D}_{N} v_1 = 0. \end{aligned} \tag{3.9}$$

Multiplying (3.6) by $\mathrm{D}_{N} v_1$ and and using

$$(\mathrm{D}_{N} v_1)\Delta v_1 = \operatorname{div}\left((\mathrm{D}_{N} v_1)\nabla v_1\right) - \frac{1}{2}\mathrm{D}_{N}\left(|\nabla v_1|^2\right),$$

we get

$$-\operatorname{div}\left((\mathrm{D}_{N} v_1)\nabla v_1\right) + \frac{1}{2}\mathrm{D}_{N}\left(|\nabla v_1|^2\right) - c(\mathrm{D}_{N} v_1)^2 - f_1(v_1, v_2)\mathrm{D}_{N} v_1 = 0.$$

For $\alpha \geq 0$ that will be fixed later, we obtain

$$\begin{aligned} &-\operatorname{div}\left((\mathrm{D}_{N} v_2)\nabla v_1\right) - \operatorname{div}\left((\mathrm{D}_{N} v_1)\nabla v_2\right) + \mathrm{D}_{N}(\nabla v_1, \nabla v_2) \\ &-\alpha \operatorname{div}\left((\mathrm{D}_{N} v_1)\nabla v_1\right) + \frac{\alpha}{2}\mathrm{D}_{N}\left(|\nabla v_1|^2\right) \\ &-c\left(1 + \frac{1}{d}\right)(\mathrm{D}_{N} v_1)(\mathrm{D}_{N} v_2) - \alpha c(\mathrm{D}_{N} v_1)^2 \\ &-f_1(v_1, v_2)\mathrm{D}_{N} v_2 - \frac{1}{d} f_2(v_1, v_2)\mathrm{D}_{N} v_1 - \alpha f_1(v_1, v_2)\mathrm{D}_{N} v_1 = 0. \end{aligned} \tag{3.10}$$

Now we choose

$$\alpha = \frac{d\alpha_{11} - \alpha_{22}}{d\alpha_{12}} \geq 0.$$

For $\varepsilon_* > 0$ we introduce a quadratic form

$$F(y,z) = \left(\frac{d\alpha_{11}^2 + \alpha_{12}\alpha_{21} - \alpha_{11}\alpha_{22}}{2d\alpha_{12}} - \varepsilon_* \right) y^2 + (|\alpha_{11}| + \varepsilon_*)\, yz + \left(\frac{\alpha_{12}}{2} - \varepsilon_* \right) z^2$$

for $(y,z) \in \mathbb{R}^2$. An assumption (A5) implies that F is a positive-definite quadratic form if $\varepsilon_* > 0$ is small enough. We fix $\varepsilon_* > 0$ such that F is a positive-definite quadratic form.

Now we prove Proposition 1.

Proof of Proposition 1. Let $\boldsymbol{v}^{(n)} = (v_1^{(n)}, v_2^{(n)})$ be as in Proposition 1. For $\rho_1 \in (0, \rho)$ we define

$$\begin{aligned} \Omega &= \left\{ (\boldsymbol{x}', x_N) \in \mathbb{R}^N \,\middle|\, |\boldsymbol{x}' - \boldsymbol{x}'_n| < \ell, \quad a_1 - \rho_1 < v_1^{(n)}(\boldsymbol{x}', x_N) < a_1 \right\}, \\ \mathit{\Gamma}_{\mathrm{c}} &= \left\{ (\boldsymbol{x}', x_N) \in \mathbb{R}^N \,\middle|\, |\boldsymbol{x}' - \boldsymbol{x}'_n| \le \ell, \quad v_1^{(n)}(\boldsymbol{x}', x_N) = a_1 \right\}, \\ \mathit{\Gamma}_{\mathrm{u}} &= \left\{ (\boldsymbol{x}', x_N) \in \mathbb{R}^N \,\middle|\, |\boldsymbol{x}' - \boldsymbol{x}'_n| \le \ell, \quad v_1^{(n)}(\boldsymbol{x}', x_N) = a_1 - \rho_1 \right\}, \\ \mathit{\Gamma}_{\mathrm{f}} &= \left\{ (\boldsymbol{x}', x_N) \in \mathbb{R}^N \,\middle|\, |\boldsymbol{x}' - \boldsymbol{x}'_n| = \ell, \quad a_1 - \rho_1 \le v_1^{(n)}(\boldsymbol{x}', x_N) \le a_1 \right\}. \end{aligned}$$

Here $\ell > 0$ is determined by ρ_1 as

$$\frac{k}{4c} F(\rho_1, 0) \mathcal{V}_{N-1} \ell = 1 + m_0 (2 + \alpha) \mathcal{A}_{N-2}. \tag{3.11}$$

Now $\mathcal{V}_{N-1}$ is the volume of the unit ball in $\mathbb{R}^{N-1}$ and $\mathcal{A}_{N-2}$ is the surface area of the unit ball in $\mathbb{R}^{N-1}$, respectively. Then we have $\partial\Omega = \mathit{\Gamma}_{\mathrm{c}} \cup \mathit{\Gamma}_{\mathrm{u}} \cup \mathit{\Gamma}_{\mathrm{f}}$.

We assume that (3.2) does not hold true and get a contradiction. Then, using Lemma 5 and Lemma 6, we can have

$$\begin{gathered} \lim_{n\to\infty} \left(\max_{\mathit{\Gamma}_{\mathrm{c}}} \min_{1\le j\le N} \left(-\mathrm{D}_N v_j^{(n)} \right) \right) = 0, \\ \lim_{n\to\infty} \max_{\boldsymbol{x}\in\mathit{\Gamma}_{\mathrm{c}}} |v_2^{(n)}(\boldsymbol{x}) - a_2| = 0. \end{gathered} \tag{3.12}$$

Defining

$$a_2^{(n)} = \max_{\boldsymbol{x}\in\mathit{\Gamma}_{\mathrm{c}}} v_2^{(n)}(\boldsymbol{x})$$

and $\boldsymbol{a}^{(n)} = (a_1, a_2^{(n)})$, we have $\lim_{n\to\infty} |\boldsymbol{a}^{(n)} - \boldsymbol{a}| = 0$. Choosing $\rho > 0$ small enough, we have

$$\left.\begin{aligned} \frac{1}{d} f_2(\boldsymbol{v}) + \alpha f_1(\boldsymbol{v}) &\le \frac{1}{d} f_2(\boldsymbol{a}^{(n)}) + \alpha f_1(\boldsymbol{a}^{(n)}) + \mathrm{D}_1 F(\boldsymbol{v} - \boldsymbol{a}^{(n)}) \\ f_1(\boldsymbol{v}) &\le f_1(\boldsymbol{a}^{(n)}) + \mathrm{D}_2 F(\boldsymbol{v} - \boldsymbol{a}^{(n)}) \end{aligned}\right. \quad \text{for } \boldsymbol{a}^{(n)} - \rho\mathbf{1} \le \boldsymbol{v} \le \boldsymbol{a}^{(n)},$$

if n is large enough. Here we put $\mathrm{D}_1 F(y,z) = F_y(y,z)$ and $\mathrm{D}_2 F(y,z) = F_z(y,z)$ for $(y,z) \in \mathbb{R}^2$. Let

$$\kappa_n = \max \left\{ \frac{1}{d} |f_2(\boldsymbol{a}^{(n)})| + \alpha |f_1(\boldsymbol{a}^{(n)})|, |f_1(\boldsymbol{a}^{(n)})| \right\}$$

and we have $\lim_{n\to\infty} \kappa_n = 0$.

Now we show

$$a_2^{(n)} - \rho \le v_2^{(n)}(\boldsymbol{x}) \le a_2^{(n)} \qquad \text{for all } \boldsymbol{x} \in \overline{\Omega}, \tag{3.13}$$

if $\rho_1 \in (0, \rho)$ is small enough. Indeed, otherwise we have

$$\inf_{\rho_1 \in (0,\rho)} \inf_{n \in \mathbb{N}} \max_{\overline{\Omega}} \left(-\mathrm{D}_N v_2^{(n)} \right) > 0$$

from Lemma 6. Then, from Lemma 5, we get

$$\inf_{\rho_1 \in (0,\rho)} \inf_{n \in \mathbb{N}} \max_{\overline{\Omega}} \left(-\mathrm{D}_N v_1^{(n)} \right) > 0.$$

Due to Lemma 2 and Lemma 3, this inequality contradicts $v_1^{(n)} \equiv a_1$ on $\mathit{\Gamma}_{\mathrm{c}}$ by sending $\rho_1 \to 0$. Thus we obtain (3.13) and fix ρ_1 hereafter.

From now on we denote $\boldsymbol{v}^{(n)}$ simply by $\boldsymbol{v}$. Let $\boldsymbol{\nu} = {}^t(\nu_1, \ldots, \nu_N)$ denote the unit outward normal vector on $\partial\Omega$. Integrating both sides of (3.10) over Ω, we obtain

$$\begin{aligned} &\int_{\partial\Omega} \left(-(\mathrm{D}_N v_2)(\nabla v_1, \boldsymbol{\nu}) - (\mathrm{D}_N v_1)(\nabla v_2, \boldsymbol{\nu}) + \nu_N (\nabla v_1, \nabla v_2) \right) \mathrm{d}s \\ &\qquad + \alpha \int_{\partial\Omega} \left(-(\mathrm{D}_N v_1)(\nabla v_1, \boldsymbol{\nu}) + \frac{1}{2} |\nabla v_1|^2 \nu_N \right) \mathrm{d}s \\ &\qquad - \int_{\Omega} \left(f_1(v_1, v_2) \mathrm{D}_N v_2 + \left(\frac{1}{d} f_2(v_1, v_2) + \alpha f_1(v_1, v_2) \right) \mathrm{D}_N v_1 \right) \mathrm{d}\boldsymbol{x} \ge 0. \end{aligned} \tag{3.14}$$

Now we put

$$\begin{aligned} I(\boldsymbol{x}) &= -(\mathrm{D}_N v_2)(\nabla v_1, \boldsymbol{\nu}) - (\mathrm{D}_N v_1)(\nabla v_2, \boldsymbol{\nu}) + \nu_N (\nabla v_1, \nabla v_2) \\ &\qquad + \alpha \left(-(\mathrm{D}_N v_1)(\nabla v_1, \boldsymbol{\nu}) + \frac{1}{2} |\nabla v_1|^2 \nu_N \right), \\ J(\boldsymbol{x}) &= f_1(v_1, v_2) \mathrm{D}_N v_2 + \left(\frac{1}{d} f_2(v_1, v_2) + \alpha f_1(v_1, v_2) \right) \mathrm{D}_N v_1. \end{aligned}$$

Now we estimate the third term of (3.14) . We begin with

$$\begin{aligned} J(\boldsymbol{x}) &\ge \left(\mathrm{D}_2 F(v_1 - a_1, v_2 - a_2^{(n)}) - \kappa_n \right) \mathrm{D}_N v_2 + \left(\mathrm{D}_1 F(v_1 - a_1, v_2 - a_2^{(n)}) - \kappa_n \right) \mathrm{D}_N v_1 \\ &\ge \mathrm{D}_N \left(F(v_1 - a_1, v_2 - a_2^{(n)}) \right) + \kappa_n \left(\mathrm{D}_N v_1 + \mathrm{D}_N v_2 \right) \end{aligned} \tag{3.15}$$

for $\boldsymbol{x} \in \Omega$. Consequently we find

$$\begin{aligned} \int_{\Omega} J(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} &\ge \int_{\partial\Omega} \nu_N F(\boldsymbol{v} - \boldsymbol{a}^{(n)}) \, \mathrm{d}s - \kappa_n \left| \int_{\partial\Omega} \nu_N (v_1 + v_2) \, \mathrm{d}s \right| \\ &\qquad = \frac{1}{2} \int_{\mathit{\Gamma}_{\mathrm{u}}} \nu_N F(\rho_1, 0) \, \mathrm{d}s > 0, \end{aligned}$$

if n is large enough. Lemma 2 gives

$$\min_{\mathit{\Gamma}_{\mathrm{u}}} \nu_N \ge \frac{k}{c}.$$

Let $|\mathit{\Gamma}_{\mathrm{u}}|$ be the measure of $\mathit{\Gamma}_{\mathrm{u}}$. The measure of

$$B^{(N-1)}(\mathbf{0};\ell)=\left\{\boldsymbol{x}'\in\mathbb{R}^{N-1}\;\middle|\;|\boldsymbol{x}'|<\ell\right\}$$

is given by $\mathcal{V}_{N-1}\ell^{N-1}$. Then we get

$$|\mathit{\Gamma}_{\mathrm{u}}|\geq\mathcal{V}_{N-1}\ell^{N-1},$$

and

$$\int_{\mathit{\Gamma}_{\mathrm{u}}}\nu_N F(\rho_1,0)\,\mathrm{d}s\geq\frac{k}{c}\mathcal{V}_{N-1}\ell^{N-1}F(\rho_1,0),$$

and thus

$$\int_{\Omega}J(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\geq\frac{k}{2c}\mathcal{V}_{N-1}\ell^{N-1}F(\rho_1,0). \tag{3.16}$$

Using

$$\boldsymbol{\nu}=\frac{\nabla v_1}{|\nabla v_1|}\qquad\text{on }\mathit{\Gamma}_{\mathrm{c}},$$

we have

$$I(\boldsymbol{x})=-(\mathrm{D}_N v_2)|\nabla v_1|+\frac{1}{2}\alpha(-\mathrm{D}_N v_1)|\nabla v_1|>0\qquad\text{on }\mathit{\Gamma}_{\mathrm{c}}.$$

and

$$\int_{\mathit{\Gamma}_{\mathrm{c}}}I(\boldsymbol{x})\,\mathrm{d}s=\int_{\mathit{\Gamma}_{\mathrm{c}}}\left(-(\mathrm{D}_N v_2)|\nabla v_1|+\frac{1}{2}\alpha(-\mathrm{D}_N v_1)|\nabla v_1|\right)\,\mathrm{d}s>0. \tag{3.17}$$

Using

$$\boldsymbol{\nu}=-\frac{\nabla v_1}{|\nabla v_1|}\qquad\text{on }\mathit{\Gamma}_{\mathrm{u}},$$

we get

$$I(\boldsymbol{x})=(\mathrm{D}_N v_2)|\nabla v_1|+\frac{1}{2}\alpha(\mathrm{D}_N v_1)|\nabla v_1|<0\qquad\text{on }\mathit{\Gamma}_{\mathrm{u}}$$

and

$$\int_{\mathit{\Gamma}_{\mathrm{u}}}I(\boldsymbol{x})\,\mathrm{d}s=\int_{\mathit{\Gamma}_{\mathrm{u}}}\left((\mathrm{D}_N v_2)|\nabla v_1|+\frac{1}{2}\alpha(\mathrm{D}_N v_1)|\nabla v_1|\right)\,\mathrm{d}s<0.$$

Now we continue the calculation as

$$\left|\int_{\mathit{\Gamma}_{\mathrm{f}}}I(\boldsymbol{x})\,\mathrm{d}s\right|\leq m_0\int_{\mathit{\Gamma}_{\mathrm{f}}}\left(-\mathrm{D}_N v_2-(1+\alpha)\mathrm{D}_N v_1\right)\,\mathrm{d}s\leq m_0(2+\alpha)\left|\partial B^{(N-1)}(\mathbf{0};\ell)\right|.$$

where

$$\partial B^{(N-1)}(\mathbf{0};\ell)=\left\{\boldsymbol{x}'\in\mathbb{R}^{N-1}\;\middle|\;|\boldsymbol{x}'|=\ell\right\}.$$

The measure of $\partial B^{(N-1)}(\mathbf{0};\ell)$ is given by $\mathcal{A}_{N-2}\ell^{N-2}$. Thus we obtain

$$\left|\int_{\mathit{\Gamma}_{\mathrm{f}}}I(\boldsymbol{x})\,\mathrm{d}s\right|\leq 2m_0(2+\alpha)\mathcal{A}_{N-2}\ell^{N-2}. \tag{3.18}$$

Using (3.14), (3.16), (3.17) and (3.18), we obtain

$$\left(\int_{\mathit{\Gamma}_{\mathrm{c}}}(-\mathrm{D}_N v_2)|\nabla v_1|+\frac{1}{2}\alpha(-DNv_1)|\nabla v_1|\right)\,\mathrm{d}s\geq\frac{k}{2c}F(\rho_1,0)\mathcal{V}_{N-1}\ell^{N-1}-m_0(2+\alpha)\mathcal{A}_{N-2}\ell^{N-2}.$$

Using (3.11), we obtain

$$\int_{\Gamma_{\mathrm{c}}} \left((-\mathrm{D}_{N} v_2)|\nabla v_1| + \frac{1}{2}\alpha(-\mathrm{D}_{N} v_1)|\nabla v_1| \right) \mathrm{d}s \geq \frac{k}{4c} F(\rho_1, 0)\mathcal{V}_{N-1}\ell^{N-1}. \tag{3.19}$$

Let ψ_{c} be defined by

$$\Gamma_{\mathrm{c}} = \left\{ (\boldsymbol{x}', \psi_{\mathrm{c}}(\boldsymbol{x}')) \;\middle|\; |\boldsymbol{x}' - \boldsymbol{x}_0'| \leq \ell \right\}.$$

Now Lemma 2 gives

$$|\nabla \psi_{\mathrm{c}}| \leq m_*.$$

Then we have

$$|\Gamma_{\mathrm{c}}| = \int_{B^{(N-1)}(\mathbf{0};\ell)} \sqrt{1 + |\nabla \psi_1|^2}\, \mathrm{d}\boldsymbol{x}' \leq \frac{c}{k}\mathcal{V}_{N-1}.$$

Then (3.19) contradicts (3.12) by using Lemma 2, Lemma 3 and Lemma 5. This completes the proof. □

Proposition 2 *Let $\boldsymbol{v} = (v_1, v_2)$ be as in Theorem 2 associated with (2.4). For any $\delta \in (0, \delta_*)$ one has*

$$-\mathrm{D}_{N}\boldsymbol{v}(\boldsymbol{x}) \geq B_\delta \mathbf{1} \qquad \text{if} \quad |\boldsymbol{v}(\boldsymbol{x})| \geq \delta \text{ or } |\boldsymbol{v}(\boldsymbol{x}) - \mathbf{1}| \geq \delta. \tag{3.20}$$

Here a positive constant B_δ depends only on $(\boldsymbol{f}, A, c, N)$, and is independent of (2.4).

Proof. Let $\boldsymbol{v}^{(n)} = (v_1^{(n)}, v_2^{(n)})$ be as in Theorem 2 associated with (2.4). In view of Lemma 6 and Proposition 1, it suffices to prove

$$\inf \left\{ v_1^{(n)}(\boldsymbol{x}) \,\middle|\, n \in \mathbb{N},\, \boldsymbol{x} \in \mathbb{R}^N,\, v_2^{(n)}(\boldsymbol{x}) = \mu_0 \right\} > 0,$$
$$\inf \left\{ v_2^{(n)}(\boldsymbol{x}) \,\middle|\, n \in \mathbb{N},\, \boldsymbol{x} \in \mathbb{R}^N,\, v_1^{(n)}(\boldsymbol{x}) = \mu_0 \right\} > 0$$

for every $\mu_0 \in (0, 1)$. It suffices to prove only the former inequality without loss of generality. Assume the contrary. Then, replacing $(\boldsymbol{v}^{(n)})_{n\in\mathbb{N}}$ by its subsequece and choosing $\boldsymbol{x}_n \in \mathbb{R}^N$, we have

$$\lim_{n\to\infty} v_1^{(n)}(\boldsymbol{x}_n) = 0, \quad v_2^{(n)}(\boldsymbol{x}_n) = \mu_0.$$

Then, using Lemma 2 and Lemma 5, we have

$$\lim_{n\to\infty} \sup_{B(\boldsymbol{x}_n;R)} v_1^{(n)} = 0 \qquad \text{for every } R > 0. \tag{3.21}$$

Then, from (1.4), we have

$$f_1(0, u_2) = 0 \qquad \text{for all } u_2 \in [0, \mu_0].$$

Now (A5) gives

$$f_2(0, u_2) \neq 0 \qquad \text{for all } u_2 \in (0, \mu_0].$$

Using $f_2(0, \mu_0) \neq 0$ and Lemma 6, we have, for some $\varepsilon_0 > 0$,

$$\inf_n \left(-\mathrm{D}_{N} v_2^{(n)}(\boldsymbol{x}) \right) \geq \varepsilon_0 > 0, \tag{3.22}$$

if $(v_1^{(n)}(\boldsymbol{x}), v_2^{(n)}(\boldsymbol{x}))$ lies in a neighborhood of $(0, \mu_0) \in [0,1]^2$, say, $B((0,\mu_0); r_0)$ for $r_0 > 0$. From (3.21) and Lemma 5, we have

$$\lim_{n\to\infty} \max\left\{ -\mathrm{D}_N v_1^{(n)}(\boldsymbol{x}) \,\middle|\, \boldsymbol{v}^{(n)}(\boldsymbol{x}) \in \overline{B((0,\mu_0); r_0)},\ \boldsymbol{x} \in \overline{B(\boldsymbol{x}_n; R)} \right\} = 0$$

for every $R > 0$. Combining this equality and (3.22), we have

$$\lim_{n\to\infty} \min\left\{ v_1^{(n)}(\boldsymbol{x}) \,\middle|\, \boldsymbol{x} \in \overline{B(\boldsymbol{x}_n; R)},\ v_2^{(n)}(\boldsymbol{x}) = \mu_0 + \frac{r_0}{2} \right\} = 0$$

for every $R > 0$. From this equality we have

$$f_1(0, u_2) = 0 \qquad \text{for all } u_2 \in [0, \mu_0 + r_0/2]$$

and

$$f_2(0, u_2) \neq 0 \qquad \text{for all } u_2 \in (0, \mu_0 + r_0/2].$$

Repeating this argument, we finally obtain

$$f_1(0, u_2) = 0 \qquad \text{for all } 0 \le u_2 \le 1,$$

which combined with (A5) gives

$$f_2(0, u_2) \neq 0 \qquad \text{for all } 0 < u_2 < 1.$$

Since we have $f_2(a_1, a_2) = 0$ and $\mathrm{D}_1 f_2 \ge 0$, we get $f_2(0, a_2) < 0$. Then we find

$$f_2(0, u_2) < 0 \qquad \text{for all } 0 < u_2 < 1. \tag{3.23}$$

Taking a subsequence if necessary, we define

$$V = \lim_{n\to\infty} v_2^{(n)}(\,\cdot\, + \boldsymbol{x}_n) \qquad \text{in } C^2_{\mathrm{loc}}(\mathbb{R}^N).$$

and have

$$-d_2 \Delta V - c\mathrm{D}_N V - f_2(0, V) = 0 \quad \text{in } \mathbb{R}^N, \tag{3.24}$$

$$-\frac{\partial V}{\partial \boldsymbol{t}} \ge 0 \quad \text{in } \mathbb{R}^N,$$

$$0 < V(\boldsymbol{x}) < 1 \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^N,$$

$$V(\boldsymbol{0}) = \mu_0, \tag{3.25}$$

$$\inf\{-\mathrm{D}_N V(\boldsymbol{x}) \,|\, \boldsymbol{x} \in \mathbb{R}^N, \delta \le V(\boldsymbol{x}) \le 1 - \delta\} > 0 \quad \text{for any } \delta \in (0, \delta_*). \tag{3.26}$$

Here $\boldsymbol{t}$ is as in Lemma 2, and the last inequality follows from Lemma 6 and (3.23). We define

$$\Omega_0 = \{(\boldsymbol{x}', x_N) \in \mathbb{R}^N \,|\, |\boldsymbol{x}'| < R, \delta < V(\boldsymbol{x}', x_N) < 1 - \delta\}$$

for $R > 0$ and $\delta \in (0, \delta_*)$. Integrating (3.24) over Ω_0, we find

$$c \int_{\Omega_0} (-\mathrm{D}_N V)\, \mathrm{d}\boldsymbol{x} + \int_{\Omega_0} (-f_2(0, V)) \mathrm{d}\boldsymbol{x} = d_2 \int_{\partial\Omega_0} \frac{\partial V}{\partial \boldsymbol{\nu}}\, \mathrm{d}s.$$

Now we put $\partial\Omega_0 = \mathit{\Gamma}_0 \cup \mathit{\Gamma}_1$, where

$$\begin{aligned} \mathit{\Gamma}_0 &= \left\{ (\boldsymbol{x}', x_{\scriptscriptstyle N}) \in \mathbb{R}^{\scriptscriptstyle N} \,\middle|\, |\boldsymbol{x}'| = R, \delta < V(\boldsymbol{x}', x_{\scriptscriptstyle N}) < 1-\delta \right\}, \\ \mathit{\Gamma}_1 &= \left\{ (\boldsymbol{x}', x_{\scriptscriptstyle N}) \in \mathbb{R}^{\scriptscriptstyle N} \,\middle|\, |\boldsymbol{x}'| < R, V(\boldsymbol{x}', x_{\scriptscriptstyle N}) \in \{\delta, 1-\delta\} \right\}. \end{aligned}$$

Now we have

$$\int_{\Omega_0} (-f_2(0,V))\mathrm{d}\boldsymbol{x} \leq d_2 \int_{\mathit{\Gamma}_0 \cup \mathit{\Gamma}_1} \frac{\partial V}{\partial \boldsymbol{\nu}}\,\mathrm{d}s.$$

If δ is small enough, we have

$$d_2 \left| \int_{\mathit{\Gamma}_1} \frac{\partial V}{\partial \boldsymbol{\nu}}\,\mathrm{d}s \right| \leq \frac{1}{2} \int_{\Omega_0} (-f_2(0,V))\mathrm{d}\boldsymbol{x}$$

for all $R > 1$. Thus we get

$$\frac{1}{2} \int_{\Omega_0} (-f_2(0,V))\mathrm{d}\boldsymbol{x} \leq d_2 \int_{\mathit{\Gamma}_0} \frac{\partial V}{\partial \boldsymbol{\nu}}\,\mathrm{d}s.$$

The right-hand side remains bounded uniformly in $R > 1$, while the left-hand side goes to $+\infty$ as $R \to \infty$, which gives a contradiction. This completes the proof. □

Now we put

$$\mathcal{L}[\boldsymbol{w}] = (\mathrm{D}_t - A\Delta - c\mathrm{D}_{\scriptscriptstyle N})\,\boldsymbol{w} - \boldsymbol{f}(\boldsymbol{w}),$$

and prove the following lemma.

Lemma 7 *Let $\boldsymbol{p}(\,\cdot\,)$ be given by (2.2). For any $\boldsymbol{v}$ with (1.4) and (3.20), let*

$$0 < \delta < \min\left\{ \delta_*, \frac{B_{\frac{1}{4}}}{8m_0\|\chi_0'\|_{L^\infty(0,1)}} \right\},$$

and let $\sigma > 0$ satisfy (3.27). Then

$$\begin{aligned} \boldsymbol{w}^+(\boldsymbol{x},t) &= \boldsymbol{v}(\boldsymbol{x}', x_{\scriptscriptstyle N} - \sigma\delta(1-e^{-\beta t})) + \delta e^{-\beta t}\boldsymbol{p}\left(|\boldsymbol{v}(\boldsymbol{x}', x_{\scriptscriptstyle N} - \sigma\delta(1-e^{-\beta t}))|\right), \\ \boldsymbol{w}^-(\boldsymbol{x},t) &= \boldsymbol{v}(\boldsymbol{x}', x_{\scriptscriptstyle N} + \sigma\delta(1-e^{-\beta t})) - \delta e^{-\beta t}\boldsymbol{p}\left(|\boldsymbol{v}(\boldsymbol{x}', x_{\scriptscriptstyle N} + \sigma\delta(1-e^{-\beta t}))|\right) \end{aligned}$$

become a supersolution and a subsolution of (1.3), respectively, that is, one has

$$\mathcal{L}[\boldsymbol{w}^+] \geq \mathbf{0}, \quad \mathcal{L}[\boldsymbol{w}^-] \leq \mathbf{0} \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^N,\ t > 0.$$

Proof. We take σ large enough to satisfy

$$\begin{aligned} \frac{1}{2}\sigma\beta B_{\frac{1}{4}} &> \beta + 2M + 2cm_0\|\chi_0'\|_{L^\infty(0,1)} \\ &\quad + 16N^2\max\{d_1,d_2\}\max\{1,m_0^2\}\max\left\{\|\chi_0'\|_{L^\infty(0,1)}, \|\chi_0''\|_{L^\infty(0,1)}\right\}. \quad (3.27) \end{aligned}$$

We put

$$\xi = x_{\scriptscriptstyle N} - \sigma\delta(1-e^{-\beta t}).$$

By direct calculation we have

$$\mathcal{L}[\boldsymbol{w}^+] = \delta e^{-\beta t}\boldsymbol{m}(\boldsymbol{x},t),$$

where

$$\begin{aligned}\boldsymbol{m}(\boldsymbol{x},t) &= \sigma\beta\left(-\mathrm{D}_N\boldsymbol{v}(\boldsymbol{x}',\xi)\right) - \beta\boldsymbol{p}\left(|\boldsymbol{v}(\boldsymbol{x}',\xi)|\right)\\ &\quad -2\sigma\delta\beta e^{-\beta t}\boldsymbol{p}'\left(|\boldsymbol{v}(\boldsymbol{x}',\xi)|\right)\left(\boldsymbol{v}(\boldsymbol{x}',\xi),\mathrm{D}_N\boldsymbol{v}(\boldsymbol{x}',\xi)\right)/|\boldsymbol{v}(\boldsymbol{x}',\xi)|\\ &\quad -\left(A\Delta + c\mathrm{D}_N\right)\boldsymbol{p}\left(|\boldsymbol{v}(\boldsymbol{x}',\xi)|\right)\\ &\quad -\int_0^1 \boldsymbol{f}'(\boldsymbol{v}(\boldsymbol{x}',\xi)+\delta e^{-\beta t}\theta\boldsymbol{p}\left(|\boldsymbol{v}(\boldsymbol{x}',\xi)|\right))\,\mathrm{d}\theta\,\boldsymbol{p}\left(|\boldsymbol{v}(\boldsymbol{x}',\xi)|\right).\end{aligned}$$

If $4|\boldsymbol{v}(\boldsymbol{x}',\xi)|\le 1$, we have

$$\boldsymbol{m}(\boldsymbol{x},t) \ge -\beta\boldsymbol{q}_0 - \int_0^1 \boldsymbol{f}'(\boldsymbol{v}(\boldsymbol{x}',\xi)+\delta e^{-\beta t}\theta\boldsymbol{q}_0)\,\mathrm{d}\theta\,\boldsymbol{q}_0 \ge -\beta\boldsymbol{q}_0 + 2\beta\boldsymbol{1} \ge \boldsymbol{0}.$$

If $4|\boldsymbol{v}(\boldsymbol{x}',\xi)|\ge 3$, we have

$$\boldsymbol{m}(\boldsymbol{x},t) \ge -\beta\boldsymbol{p}_0 - \int_0^1 \boldsymbol{f}'(\boldsymbol{v}(\boldsymbol{x}',\xi)+\delta e^{-\beta t}\theta\boldsymbol{p}_0)\,\mathrm{d}\theta\,\boldsymbol{p}_0 \ge -\beta\boldsymbol{p}_0 + 2\beta\boldsymbol{1} \ge \boldsymbol{0}.$$

Thus it suffices to prove $\boldsymbol{m}(\boldsymbol{x},t)\ge\boldsymbol{0}$ if $1<4|\boldsymbol{v}(\boldsymbol{x}',\xi)|<3$. In this case we have

$$\begin{aligned}\boldsymbol{m}(\boldsymbol{x},t) &\ge \sigma\beta\left(B_{\frac14}\boldsymbol{1} - 2\delta\boldsymbol{p}'\left(|\boldsymbol{v}(\boldsymbol{x}',\xi)|\right)\left(\boldsymbol{v}(\boldsymbol{x}',\xi),\mathrm{D}_N\boldsymbol{v}(\boldsymbol{x}',\xi)\right)/|\boldsymbol{v}(\boldsymbol{x}',\xi)|\right)\\ &\quad -\beta\boldsymbol{1} - \left(A\Delta + c\mathrm{D}_N\right)\boldsymbol{p}\left(|\boldsymbol{v}(\boldsymbol{x}',\xi)|\right) - 2M\boldsymbol{1}.\end{aligned}$$

Using Lemma 3 and the definition of $\boldsymbol{p}$, we get

$$\begin{aligned}\boldsymbol{m}(\boldsymbol{x},t) &\ge \frac12\sigma\beta B_{\frac14}\boldsymbol{1} - (\beta+2M)\boldsymbol{1} - 2cm_0\|\chi_0'\|_{L^\infty(0,1)}\boldsymbol{1}\\ &\quad -16N^2\max\{d_1,d_2\}\max\{1,m_0^2\}\max\left\{\|\chi_0'\|_{L^\infty(0,1)},\|\chi_0''\|_{L^\infty(0,1)}\right\}\boldsymbol{1}\ge\boldsymbol{0}.\end{aligned}$$

Thus we obtain

$$\mathcal{L}[\boldsymbol{w}^+] = \delta e^{-\beta t}\boldsymbol{m}(\boldsymbol{x},t) \ge \boldsymbol{0} \qquad \text{for all } \boldsymbol{x}\in\mathbb{R}^N,\ t>0.$$

□

4 Cylindrically symmetric traveling fronts

Let $\mathbb{N}$ be the set of positive integers and and let $\bar{\mathbb{N}} = \mathbb{N}\cup\{0\}$. For $m\in\mathbb{N}$ with $m\ge 2$ we define J as

$$J = \left\{ j\in\bar{\mathbb{N}} \;\middle|\; 0\le j\le 2^m-1\right\} \quad \text{if } N=3,$$
$$J = \left\{ (j_1,\ldots,j_{N-2})\in\bar{\mathbb{N}}^{N-2} \;\middle|\; 0\le j_i\le 2^m\ (1\le i\le N-3),\ 0\le j_{N-2}\le 2^m-1\right\} \text{ if } N\ge 4.$$

For each $\boldsymbol{j}=(j_1,\ldots,j_{N-2})\in J$, we define

$$\boldsymbol{a}_{\boldsymbol{j}} = \left(\cos\left(\frac{2\pi j_1}{2^m}\right),\sin\left(\frac{2\pi j_1}{2^m}\right)\right) \qquad \text{for } N=3,$$

and

$$\boldsymbol{a_j} = \begin{pmatrix} \cos\left(\dfrac{\pi j_1}{2^m}\right) \\ \sin\left(\dfrac{\pi j_1}{2^m}\right)\cos\left(\dfrac{\pi j_2}{2^m}\right) \\ \sin\left(\dfrac{\pi j_1}{2^m}\right)\sin\left(\dfrac{\pi j_2}{2^m}\right) \\ \vdots \\ \sin\left(\dfrac{\pi j_1}{2^m}\right)\ldots\sin\left(\dfrac{\pi j_{N-3}}{2^m}\right)\cos\left(\dfrac{2\pi j_{N-2}}{2^m}\right) \\ \sin\left(\dfrac{\pi j_1}{2^m}\right)\ldots\sin\left(\dfrac{\pi j_{N-3}}{2^m}\right)\sin\left(\dfrac{2\pi j_{N-2}}{2^m}\right) \end{pmatrix} \qquad \text{for } N \geq 4.$$

Let $h^{(m)}$ be as in (2.4) associated with $\{\boldsymbol{a_j} \,|\, \boldsymbol{j} \in J\}$ and let $\boldsymbol{V}^{(m)} = (V_1^{(m)}, V_2^{(m)})$ be as in Theorem 2 for $h^{(m)}$. Since $h^{(m)}$ is symmetric with respect to a plane $(\boldsymbol{x}', \boldsymbol{a_j}) = 0$, $\boldsymbol{V}^{(m)}(\,\cdot\,, x_N)$ is symmetric with respect to the same plane for any fixed $x_N \in \mathbb{R}$ by the definition of $\boldsymbol{V}^{(m)}$ in Theorem 2. We choose $\zeta_m \in \mathbb{R}$ by $V_1^{(m)}(0 \ldots, 0, \zeta_m) = a_1$ and define

$$\boldsymbol{U}_\infty(\boldsymbol{x}', x_N) = \lim_{m\to\infty} \boldsymbol{V}^{(m)}(\boldsymbol{x}', x_N + \zeta_m) \qquad \text{in } C^2_{\mathrm{loc}}(\mathbb{R}^N).$$

Since $\boldsymbol{V}^{(m)}(\boldsymbol{x}', x_N + \zeta_m)$ satisfies Lemma 6 and Proposition 1, $\boldsymbol{U}_\infty(\boldsymbol{x}', x_N)$ also satisfies Lemma 6 and Proposition 1. Now $\boldsymbol{U}_\infty$ is a function of $(|\boldsymbol{x}'|, x_N)$. We denote $|\boldsymbol{x}'|$ and x_N by r and z, respectively, and we write $\boldsymbol{U}_\infty(\boldsymbol{x}', x_N)$ by $\boldsymbol{U}(r,z) = (U_1(r,z), U_2(r,z))$. Then we have (1.6) in §1.

The property of $\boldsymbol{U} = (U_1, U_2)$ is as follows.

Lemma 8 *For any given $\boldsymbol{s} = (s_1, s_2)$ with $s_2 > 0$ and $m_*|s_1| \leq s_2$, one has*

$$\begin{aligned} \boldsymbol{0} < \boldsymbol{U}(r,z) < \boldsymbol{1} \qquad & \text{for all } r \geq 0,\ z \in \mathbb{R}, \\ -\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{s}} \geq \boldsymbol{0} \qquad & \text{for all } r \geq 0,\ z \in \mathbb{R}, \\ \mathrm{D}_r \boldsymbol{U}(r,z) \geq \boldsymbol{0} \qquad & \text{for all } r \geq 0,\ z \in \mathbb{R}. \end{aligned}$$

For any $\delta \in (0, \delta_*)$ one has

$$-\mathrm{D}_z \boldsymbol{U}(r,z) \geq B_\delta \boldsymbol{1} \qquad \text{if} \quad |\boldsymbol{U}(r,z)| \geq \delta \text{ or } |\boldsymbol{U}(r,z) - \boldsymbol{1}| \geq \delta.$$

Here B_δ is a positive constant in Proposition 2.

Proof. The inequalities in this lemma follow from the definition of $\boldsymbol{U}$, Lemma 2, Lemma 4, Lemma 6 and Proposition 2. □

Defining $\phi(r)$ by $U_1(r, \phi(r)) = a_1$, we obtain $\phi(0) = 0$ and

$$0 \leq \phi'(r) \leq m_* \qquad \text{for all } r \geq 0.$$

Then we have

$$\begin{aligned} \lim\nolimits_{R\to\infty} \sup\left\{ |\boldsymbol{U}(r,z)| \,\middle|\, z - \phi(r) \geq R \right\} = 0, \\ \lim\nolimits_{R\to\infty} \sup\left\{ |\boldsymbol{U}(r,z) - \boldsymbol{1}| \,\middle|\, z - \phi(r) \leq -R \right\} = 0, \end{aligned} \tag{4.1}$$

and thus

$$\lim_{R\to\infty} \sup \left\{ |\mathrm{D}_z \boldsymbol{U}(r,z)| \;\middle|\; |z-\phi(r)| \geq R \right\} = 0 \tag{4.2}$$

by applying the Schauder estimate to (2.5).

First we show the following lemma.

Lemma 9 *One has*

$$\lim_{r\to\infty} \phi'(r) = m_*.$$

Proof. It suffices to prove $\liminf_{r\to\infty} \phi'(r) = m_*$. Assume the contrary. Then there exists $(s_i)_{i\in\mathbb{N}}$ with

$$0 < s_1 < s_2 < \cdots < s_i < \cdots \to +\infty,$$
$$\sup_i \phi'(s_i) < m_*.$$

Assume, in addition, that there exists $\{\beta_i\}_{i\in\mathbb{N}}$ with $\lim_i \beta_i = +\infty$ and $0 < 2\beta_i < s_i$ for all $i \in \mathbb{N}$ such that we have

$$0 \leq \max_{[s_i, s_i+2\beta_i]} \phi' < m' < m_* \qquad \text{for all } i \in \mathbb{N}$$

or

$$0 \leq \max_{[s_i-2\beta_i, s_i]} \phi' < m' < m_* \qquad \text{for all } i \in \mathbb{N}$$

for $m' \in (0, m_*)$. Then we set

$$\overline{\boldsymbol{U}}(r,z) = \lim_{i\to\infty} \boldsymbol{U}(r+s_i+\beta_i, z+\phi(s_i+\beta_i)) \qquad \text{for } (r,z) \in \mathbb{R}^2,$$

or

$$\overline{\boldsymbol{U}}(r,z) = \lim_{i\to\infty} \boldsymbol{U}(r+s_i-\beta_i, z+\phi(s_i-\beta_i)) \qquad \text{for } (r,z) \in \mathbb{R}^2,$$

respectively. Here we have $\overline{\boldsymbol{U}} = (\overline{U}_1, \overline{U}_2)$. Then we get

$$\begin{aligned} (-A\mathrm{D}_{rr} - A\mathrm{D}_{zz} - c\mathrm{D}_z)\, \overline{\boldsymbol{U}}(r,z) &= \boldsymbol{f}(\overline{\boldsymbol{U}}(r,z)) && \text{for } (r,z) \in \mathbb{R}^2, \\ \overline{U}_1(0,0) &= a_1 \\ \mathrm{D}_N \overline{\boldsymbol{U}}(r,z) &< \mathbf{0} && \text{for } (r,z) \in \mathbb{R}^2. \end{aligned}$$

Defining $\bar{\phi}(r)$ by $\overline{U}_1(r, \bar{\phi}(r)) = a_1$, we have $\bar{\phi}(0) = 0$ and, with some $m' \in (0, m_*)$,

$$0 \leq \bar{\phi}'(r) \leq m' < m_* \qquad \text{for all } r \in \mathbb{R}$$

Let $\boldsymbol{v}_*$ be the two-dimensional front V-form associated with $x_N = m'|x|$ in Theorem 2 for $N = 2$. Let σ and δ be as in Lemma 7. Then we have

$$\overline{\boldsymbol{U}}(r,z) \leq \boldsymbol{v}_*(r, z-\lambda) + \delta \boldsymbol{p}\left(|\boldsymbol{v}_*(r,z-\lambda)|\right),$$

by taking $\lambda > 0$ large enough. Using the sides of

$$\overline{\boldsymbol{U}}(r,z) \leq \boldsymbol{v}_*(r, z-\lambda-\sigma\delta(1-e^{-\beta t})) + \delta e^{-\beta t} \boldsymbol{p}\left(|\boldsymbol{v}_*(r, z-\sigma\delta(1-e^{-\beta t}))|\right)\Big|_{t=0}$$

as initial values of

$$\boldsymbol{w}_t = A\left(\mathrm{D}_{rr} + \mathrm{D}_{zz}\right)\boldsymbol{w} + \boldsymbol{f}(\boldsymbol{w}) \qquad (r,z) \in \mathbb{R}^2,\ t > 0,$$

we obtain

$$\overline{\boldsymbol{U}}(r, z - ct) \le \boldsymbol{v}_*(r, z - \lambda - c't - \sigma\delta(1 - e^{-\beta t})) + \delta e^{-\beta t}\boldsymbol{p}\left(|\boldsymbol{v}_*(r, z - \lambda - c't - \sigma\delta(1 - e^{-\beta t}))|\right)$$

for $(r, z) \in \mathbb{R}^2$ and $t \ge 0$, where

$$c' = k\sqrt{1 + (m')^2} < c.$$

Sending $t \to +\infty$, we have a contradiction from $\overline{U}_1(0,0) = a_1$.

Thus we can choose $b_* > 0$ that is independent of i such that we have

$$\lim_{i\to\infty} \phi'(\xi_i) = m_*$$

for some $\xi_i \in [s_i - b_*, s_i + b_*]$. Then we have

$$\lim_{i\to\infty}\left(-\frac{\partial U_1}{\partial \boldsymbol{s}_0}\right)\Bigg|_{(\xi_i, \phi(\xi_i))} = 0,$$

where

$$\boldsymbol{s}_0 = \frac{k}{c}\begin{pmatrix} 1 \\ m_* \end{pmatrix}.$$

Now we have

$$\begin{aligned} &\left(-A\mathrm{D}_{rr} - \frac{N-2}{r}A\mathrm{D}_r - A\mathrm{D}_{zz} - c\mathrm{D}_z - f'(\boldsymbol{U})\right)\left(-\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{s}_0}\right) = \mathbf{0} \quad &&\text{for } r > 0,\ z \in \mathbb{R}, \\ &-\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{s}_0} \ge \mathbf{0} &&\text{for } r > 0,\ z \in \mathbb{R}. \end{aligned} \tag{4.3}$$

For

$$R_* = 1 + \frac{c}{k}b_*.$$

we have $(s_i, \phi(s_i)) \in B((\xi_i, \phi(\xi_i)); R_*)$. Then Lemma 5 gives

$$\lim_{i\to\infty}\left(-\frac{\partial U_1}{\partial \boldsymbol{s}_0}\right)\Bigg|_{(s_i, \phi(s_i))} = 0.$$

Since the gradient of U_1 does not vanish at $(s_i, \phi(s_i))$ by Lemma 8, we obtain $\lim_{i\to\infty}\phi'(s_i) = m_*$, which gives a contradiction. This completes the proof. □

Now we prove the following property of $\boldsymbol{U}$.

Lemma 10 *One has*

$$\lim_{s\to\infty} \boldsymbol{U}(s + r, \phi(s) + z) = \mathbf{\Phi}\left(\frac{k}{c}(z - m_* r)\right) \qquad \text{in } C^2_{\mathrm{loc}}(\mathbb{R}^2).$$

Moreover, for every $\eta \ge 0$ one has

$$\begin{aligned} &\boldsymbol{U}(r + \eta, z + m_*\eta) \le \boldsymbol{U}(r, z) \qquad \text{for all } r \ge 0,\ z \in \mathbb{R}, \\ &\lim_{R\to\infty}\ \sup_{(r,z)\in[R,\infty)\times\mathbb{R}} |\boldsymbol{U}(r,z) - \boldsymbol{U}(r + \eta, z + m_*\eta)| = 0. \end{aligned}$$

Proof. Using (4.3), Lemma 5 and

$$\lim_{s\to\infty}\left(-\frac{\partial U_1}{\partial \boldsymbol{s}_0}\bigg|_{(s,\phi(s))}\right)=0,$$

we obtain

$$\lim_{s\to\infty}\sup_{B((s,\phi(s));R)}\left(-\frac{\partial U_1}{\partial \boldsymbol{s}_0}\right)=0$$

for every $R>0$. Then, from Lemma 5, we have

$$\lim_{s\to\infty}\sup_{B((s,\phi(s));R)}\left(-\frac{\partial \boldsymbol{U}}{\partial \boldsymbol{s}_0}\right)=\boldsymbol{0}.$$

From this equality and (1.6), $\boldsymbol{U}$ converges to a unique one-dimensional traveling front to the direction $(-m_*,1)$ with speed k, that is,

$$\lim_{s\to\infty}\sup_{|r|+|z|\le R}\left|\boldsymbol{U}(r+s,\phi(s)+z)-\boldsymbol{\Phi}\left(\frac{k}{c}(z-m_*r)\right)\right|=0$$

for every $R>0$. This shows the first equality of the lemma. The second inequality in the lemma follows from Lemma 8. Combining these inequalities and Lemma 8, we obtain the third equality. This completes the proof. □

5 Surfaces in $\mathbb{R}^{N-1}$ with positive principal curvatures

Let $g\in C^2(S^{N-2})$ satisfy $g(\boldsymbol{\xi})>0$ for all $\boldsymbol{\xi}\in S^{N-2}$. We set

$$\begin{aligned} C_g &= \{g(\boldsymbol{\xi})\boldsymbol{\xi}\,|\,\boldsymbol{\xi}\in S^{N-2}\},\\ D_g &= \{r\boldsymbol{\xi}\,|\,0\le r\le g(\boldsymbol{\xi}),\boldsymbol{\xi}\in S^{N-2}\},\end{aligned}$$

and have $C_g=\partial D_g\subset\mathbb{R}^{N-1}$. For some neighborhood of $g(\boldsymbol{\xi})\boldsymbol{\xi}\in C_g$ with $\boldsymbol{\xi}\in S^{N-2}$, we introduce a coordinate and write C_g as $(\boldsymbol{y},\psi(\boldsymbol{y}))$ with $\psi(\boldsymbol{y}^0)=0$ and $\nabla\psi(\boldsymbol{y}^0)=\boldsymbol{0}$, where $\boldsymbol{y}=(y_1,\dots,y_{N-2})$. Here $g(\boldsymbol{\xi})\boldsymbol{\xi}\in C_g$ corresponds to $(\boldsymbol{y}^0,0)$ with $\boldsymbol{y}^0\in\mathbb{R}^{N-2}$ in this coordinate.

Let $\boldsymbol{\nu}(\boldsymbol{y})$ be the unit normal vector of C_g at $(\boldsymbol{y},\psi(\boldsymbol{y}))$ pointing from D_g to $\mathbb{R}^{N-1}\backslash D_g$. We have

$$\boldsymbol{\nu}(\boldsymbol{y})=\frac{1}{1+|\nabla\psi(\boldsymbol{y})|^2}\begin{pmatrix}-\nabla\psi(\boldsymbol{y})\\ 1\end{pmatrix},$$

where

$$\nabla\psi(\boldsymbol{y})={}^t(\mathrm{D}_1\psi(\boldsymbol{y}),\dots,\mathrm{D}_{N-2}\psi(\boldsymbol{y})).$$

The eigenvalues $\kappa_1(\boldsymbol{y}^0),\dots,\kappa_{N-2}(\boldsymbol{y}^0)$ of the Hessian matrix

$$-D^2\psi(\boldsymbol{y}^0)=-\left(\mathrm{D}_{ij}\psi(\boldsymbol{y}^0)\right)_{1\le i,j\le N-2}$$

are the principal curvatures of C_g at $g(\boldsymbol{\xi})\boldsymbol{\xi}$. We take the basis of $\mathbb{R}^{N-1}$ as the eigenvectors of the Hessian matrix. Using this principal coordinate system, we have

$$-D^2\psi(\boldsymbol{y}^0)=\mathrm{diag}\left(\kappa_1(\boldsymbol{y}^0),\dots,\kappa_{N-2}(\boldsymbol{y}^0)\right)$$

and

$$\mathrm{D}_j \nu_i(\boldsymbol{y}^0) = \kappa_i(\boldsymbol{y}^0)\delta_{ij} \qquad 1 \leq i, j \leq N-2.$$

We define $\mathcal{G}$ by

$$\left\{ g \in C^2(S^{N-2}) \,|\, g \geq 0, \text{all principal curvature of } C_g \text{ are positive at every point of } C_g \right\}.$$

For any $g \in \mathcal{G}$ and $a \geq 0$ we define $g_1 = \tau_a g$ by

$$C_{g_1} = \{x' \in C_g \cup (\mathbb{R}^{N-1} \backslash D_g) \mid \mathrm{dist}(x', C_g) = a\}.$$

See Figure 2.

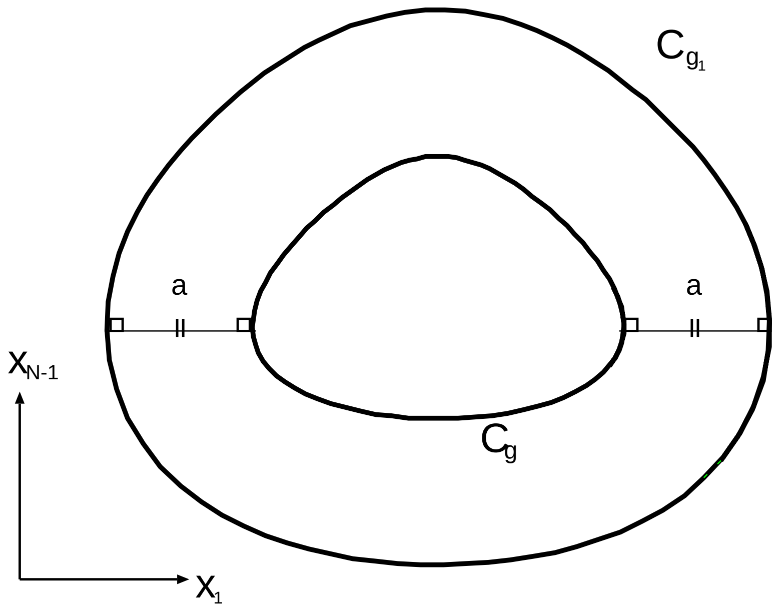

Figure 2: The graphs of C_g and C_{g_1}.

Then we have the following lemma.

Lemma 11 *For any $a \geq 0$, τ_a is a mapping in $\mathcal{G}$. Moreover one has*

$$\tau_b(\tau_a g) = \tau_{b+a} g \tag{5.1}$$

for any $a \geq 0$, $b \geq 0$ and $g \in \mathcal{G}$.

Proof. First we show $\tau_a g \in \mathcal{G}$ for $a \geq 0$ and $g \in \mathcal{G}$. In a neighborhood of $(\boldsymbol{y}^0, 0)$ we have

$$\frac{1}{2}\sum_{j=1}^{N-2}\left(\kappa_j(\boldsymbol{y}^0) - \varepsilon\right)(y_j - y_j^0)^2 \leq -\psi(\boldsymbol{y}) \leq \frac{1}{2}\sum_{j=1}^{N-2}\left(\kappa_j(\boldsymbol{y}^0) + \varepsilon\right)(y_j - y_j^0)^2,$$

where $\boldsymbol{y}^0 = {}^t(y_1^0, \ldots, y_{N-2}^0)$ and ε is any number with $0 < 2\varepsilon < \min\{\kappa_1(\boldsymbol{y}^0), \ldots, \kappa_{N-2}(\boldsymbol{y}^0)\}$. By putting

$$\begin{aligned} \kappa_{\min} &= \min\left\{\kappa_1(\boldsymbol{y}^0), \ldots, \kappa_{N-2}(\boldsymbol{y}^0)\right\}, \\ \kappa_{\max} &= \max\left\{\kappa_1(\boldsymbol{y}^0), \ldots, \kappa_{N-2}(\boldsymbol{y}^0)\right\}, \end{aligned}$$

we have

$$\frac{1}{2}\left(\kappa_{\min}-\varepsilon\right)\left|\boldsymbol{y}-\boldsymbol{y}^0\right|^2 \le -\psi(\boldsymbol{y}) \le \frac{1}{2}\left(\kappa_{\max}+\varepsilon\right)\left|\boldsymbol{y}-\boldsymbol{y}^0\right|^2$$

when $\boldsymbol{y}$ belongs to a neighborhood of $\boldsymbol{y}^0$.

Let r_0 and R_0 be the radii of the inscribed ball and the circumscribed ball of C_g at $g(\boldsymbol{\xi})\boldsymbol{\xi}$, respectively. Then we have

$$\left(\kappa_{\max}+\varepsilon\right)^{-1} \le r_0 \le R_0 \le \left(\kappa_{\min}-\varepsilon\right)^{-1}.$$

Next let r_1 and R_1 be the radii of the inscribed ball and the circumscribed ball of C_{g_1} at $(\boldsymbol{y}^0, a\nu(\boldsymbol{y}^0))$, respectively. Then we have

$$r_0 + a \le r_1 \le R_1 \le R_0 + a$$

and thus

$$a + \left(\kappa_{\max}+\varepsilon\right)^{-1} \le r_1 \le R_1 \le a + \left(\kappa_{\min}-\varepsilon\right)^{-1}.$$

Now the principal curvatures $(\tilde{\kappa}_j)_{1\le j\le N-2}$ of C_{g_1} at $(\boldsymbol{y}^0, \psi(\boldsymbol{y}^0) + a\nu(\boldsymbol{y}^0))$ satisfy

$$\left(a + \left(\kappa_{\min}-\varepsilon\right)^{-1}\right)^{-1} \le \tilde{\kappa}_j \le \left(a + \left(\kappa_{\max}+\varepsilon\right)^{-1}\right)^{-1}$$

for $1 \le j \le N-2$. Sending $\varepsilon \to 0$, we obtain

$$0 < \frac{\kappa_{\min}}{1 + a\kappa_{\min}} \le \tilde{\kappa}_j \le \frac{\kappa_{\max}}{1 + a\kappa_{\max}} \qquad (1 \le j \le N-2).$$

This shows that τ_a is a mapping in $\mathcal{G}$.

Next we prove (5.1). It suffices to prove that

$$\boldsymbol{\nu}(\boldsymbol{y}^0) = \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}$$

is orthogonal to the tangent space of C_{g_1} at $(\boldsymbol{y}^0, a\boldsymbol{\nu}(\boldsymbol{y}^0))$. Now C_{g_1} is parameterized by

$$\begin{pmatrix} \boldsymbol{y} \\ \psi(\boldsymbol{y}) \end{pmatrix} + \frac{a}{\sqrt{1 + |\nabla\psi(\boldsymbol{y})|^2}} \begin{pmatrix} -\nabla\psi(\boldsymbol{y}) \\ 1 \end{pmatrix}$$

when $\boldsymbol{y}$ belongs to a neighborhood of $\boldsymbol{y}^0$. Let $\{\boldsymbol{t}^{(j)}\}_{1\le j\le N-2}$ be the tangent vectors of C_{g_1} at $(\boldsymbol{y}^0, a\boldsymbol{\nu}(\boldsymbol{y}^0))$. We have

$$\begin{aligned}
\boldsymbol{t}^{(j)} &= \begin{pmatrix} \boldsymbol{e}_j \\ \mathrm{D}_j\psi(\boldsymbol{y}^0) \end{pmatrix} + \frac{a}{\sqrt{1 + |\nabla\psi(\boldsymbol{y}^0)|^2}} \begin{pmatrix} -\mathrm{D}_j\nabla\psi(\boldsymbol{y}^0) \\ 0 \end{pmatrix} \\
&\quad - a\left(1 + |\nabla\psi(\boldsymbol{y}^0)|^2\right)^{-\frac{3}{2}} \sum_{i=1}^{N-2} \mathrm{D}_i\psi(\boldsymbol{y}^0)\mathrm{D}_{ij}\psi(\boldsymbol{y}^0) \begin{pmatrix} -\nabla\psi(\boldsymbol{y}^0) \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{e}_j \\ 0 \end{pmatrix} + a\begin{pmatrix} -\mathrm{D}_j\nabla\psi(\boldsymbol{y}^0) \\ 0 \end{pmatrix}.
\end{aligned}$$

Here $\boldsymbol{e}_j \in \mathbb{R}^{N-2}$ has 1 for the j-th element and 0 for other elements. Then we find $(\boldsymbol{t}^j, \boldsymbol{\nu}(\boldsymbol{y}^0)) = 0$ for $1 \le j \le N-2$. This completes the proof of Lemma 11. □

Now we define an equivalence relation $g_1 \sim g_2$ for $g_1, g_2 \in \mathcal{G}$. We define $g_1 \sim g_2$ if and only if one has either $g_1 = \tau_a g_2$ or $g_2 = \tau_a g_1$ for some $a \ge 0$. We will show that $\mathcal{G}/\sim$ gives a traveling front of (1.1) in §6.

6 A traveling front associated with a surface C_g

In this section we give a proof to Theorem 1 in §1 as follows.

We construct a weak subsolution and a weak supersolution, and show the existence of $\widetilde{\boldsymbol{U}}$ between them. Let $\boldsymbol{U}$ be a cylindrically symmetric traveling front solution to (1.6). We define a weak supersolution $\overline{\boldsymbol{V}}(\boldsymbol{x})$ as

$$\overline{\boldsymbol{V}}(\boldsymbol{x}) = \min_{\boldsymbol{\xi}\in S^{N-2}} \boldsymbol{U}(|\boldsymbol{x}' - g(\boldsymbol{\xi})\boldsymbol{\xi}|, x_N) \qquad \text{for } (\boldsymbol{x}', x_N) \in \mathbb{R}^N.$$

Here $\boldsymbol{x} = (\boldsymbol{x}', x_N) \in \mathbb{R}^N$. Then, for each $\boldsymbol{\xi} \in S^{N-2}$, we have

$$\boldsymbol{w}(\boldsymbol{x}, t; \overline{\boldsymbol{V}}) \leq \boldsymbol{U}(|\boldsymbol{x}' - g(\boldsymbol{\xi})\boldsymbol{\xi})|, x_N) \qquad \text{for } (\boldsymbol{x}', x_N) \in \mathbb{R}^N,\ t > 0,$$

which gives

$$\boldsymbol{w}(\boldsymbol{x}, t; \overline{\boldsymbol{V}}) \leq \overline{V}(\boldsymbol{x}', x_N) = \min_{\boldsymbol{\xi}\in S^{N-2}} \boldsymbol{U}(|\boldsymbol{x}' - g(\boldsymbol{\xi})\boldsymbol{\xi})|, x_N) \qquad \text{for } (\boldsymbol{x}', x_N) \in \mathbb{R}^N,\ t > 0.$$

Here $\boldsymbol{w}(\boldsymbol{x}, t; \overline{\boldsymbol{V}})$ is the solution of (1.3) for $\boldsymbol{u}_0 = \overline{\boldsymbol{V}}$. Let $\{\kappa_j(\boldsymbol{\xi})\}_{1\leq j\leq N-2}$ denote the principal curvatures of C_g at $g(\boldsymbol{\xi})\boldsymbol{\xi}$ for $\boldsymbol{\xi} \in S^{N-2}$. By the assumption we have

$$\min_{1\leq j\leq N-2} \kappa_j(\boldsymbol{\xi}) > 0 \qquad \text{for all } \boldsymbol{\xi} \in S^{N-2}.$$

We choose $\eta > 0$ large enough such that we have

$$\eta > \max_{1\leq j\leq N-2} \max_{\boldsymbol{\xi}\in S^{N-2}} \frac{1}{\kappa_j(\boldsymbol{\xi})}$$

and D_g is included in the closure of a circumscribed ball of C_g at $g(\boldsymbol{\xi})\boldsymbol{\xi}$ with radius η for every $\boldsymbol{\xi} \in S^{N-2}$. See Figure 3. Let $\boldsymbol{\nu}(\boldsymbol{\xi})$ be the unit normal vector of C_g at $g(\boldsymbol{\xi})\boldsymbol{\xi}$ pointing from D_g to $\mathbb{R}^{N-1}\backslash D_g$ for $\boldsymbol{\xi} \in S^{N-2}$. A set $\{(g(\boldsymbol{\xi})\boldsymbol{\xi}, 0) \,|\, \boldsymbol{\xi} \in S^{N-2}\}$ lies in a cone

$$\{(\boldsymbol{x}', x_N) \in \mathbb{R}^N \,|\, x_N + m_*\eta \geq m_*|\boldsymbol{x}' - g(\boldsymbol{\xi}_1)\boldsymbol{\xi}_1 + \eta\boldsymbol{\nu}(\boldsymbol{\xi}_1)|\}$$

for all $\boldsymbol{\xi}_1 \in S^{N-2}$. Then, from Lemma 8, we have

$$\boldsymbol{U}(|\boldsymbol{x}' - g(\boldsymbol{\xi}_1)\boldsymbol{\xi}_1 + \eta\boldsymbol{\nu}(\boldsymbol{\xi}_1)|, x_N + m_*\eta) \leq \boldsymbol{U}(|\boldsymbol{x}' - g(\boldsymbol{\xi})\boldsymbol{\xi}|, x_N)$$

for all $(\boldsymbol{x}', x_N) \in \mathbb{R}^N$, $\boldsymbol{\xi} \in S^{N-2}$, $\boldsymbol{\xi}_1 \in S^{N-2}$. Thus we get

$$\max_{\boldsymbol{\xi}_1\in S^{N-2}} \boldsymbol{U}(|\boldsymbol{x}' - g(\boldsymbol{\xi}_1)\boldsymbol{\xi}_1 + \eta\boldsymbol{\nu}(\boldsymbol{\xi}_1)|, x_N + m_*\eta) \leq \min_{\boldsymbol{\xi}\in S^{N-2}} \boldsymbol{U}(|\boldsymbol{x}' - g(\boldsymbol{\xi})\boldsymbol{\xi}|, x_N)$$

for all $(\boldsymbol{x}', x_N) \in \mathbb{R}^N$. Now we define a weak subsolution $\underline{\boldsymbol{V}}(\boldsymbol{x})$ as

$$\underline{\boldsymbol{V}}(\boldsymbol{x}', x_N) = \max_{\boldsymbol{\xi}\in S^{N-2}} \boldsymbol{U}\left(|\boldsymbol{x}' - g(\boldsymbol{\xi})\boldsymbol{\xi} + \eta\boldsymbol{\nu}(\boldsymbol{\xi})|, x_N + m_*\eta\right) \qquad \text{for all } (\boldsymbol{x}', x_N) \in \mathbb{R}^N, \tag{6.1}$$

and have

$$\underline{\boldsymbol{V}}(\boldsymbol{x}', x_N) \leq \overline{\boldsymbol{V}}(\boldsymbol{x}', x_N) \qquad \text{for all } (\boldsymbol{x}', x_N) \in \mathbb{R}^N.$$

Then, for each $\boldsymbol{\xi} \in S^{N-2}$, we have

$$\boldsymbol{U}\left(|\boldsymbol{x}' - g(\boldsymbol{\xi})\boldsymbol{\xi} + \eta\boldsymbol{\nu}(\boldsymbol{\xi})|, x_N + m_*\eta\right) \leq \boldsymbol{w}(\boldsymbol{x}, t; \underline{\boldsymbol{V}}) \qquad \text{for } (\boldsymbol{x}', x_N) \in \mathbb{R}^N,\ t > 0,$$

which gives

$$\underline{V}(\boldsymbol{x}', x_N) = \max_{\boldsymbol{x} \in S^{N-2}} \boldsymbol{U}\left(|\boldsymbol{x}' - g(\boldsymbol{\xi})\boldsymbol{\xi} + \eta\boldsymbol{\nu}(\boldsymbol{\xi})|, x_N + m_*\eta\right) \leq \boldsymbol{w}(\boldsymbol{x}, t; \underline{\boldsymbol{V}})$$

for $(\boldsymbol{x}', x_N) \in \mathbb{R}^N$, $t > 0$. Here $\boldsymbol{w}(\boldsymbol{x}, t; \underline{\boldsymbol{V}})$ is the solution of (1.3) for $\boldsymbol{u}_0 = \underline{\boldsymbol{V}}$. Using $\underline{\boldsymbol{V}}(\boldsymbol{x}) \leq \overline{\boldsymbol{V}}(\boldsymbol{x})$, we obtain

$$\underline{\boldsymbol{V}}(\boldsymbol{x}) \leq \boldsymbol{w}(\boldsymbol{x}, t; \underline{\boldsymbol{V}}) \leq \boldsymbol{w}(\boldsymbol{x}, t; \overline{\boldsymbol{V}}) \leq \overline{\boldsymbol{V}}(\boldsymbol{x}) \qquad \boldsymbol{x} \in \mathbb{R}^N,\ t > 0.$$

Taking $\mu > 0$ large enough, we find

$$\boldsymbol{U}(|\boldsymbol{x}'|, x_N + \mu) < \underline{\boldsymbol{V}}(\boldsymbol{x}', x_N) \leq \overline{\boldsymbol{V}}(\boldsymbol{x}', x_N) < \boldsymbol{U}(|\boldsymbol{x}'|, x_N - \mu) \quad \text{for all } (\boldsymbol{x}', x_N) \in \mathbb{R}^N. \tag{6.2}$$

From Lemma 10, we have

$$\lim_{R\to\infty} \sup_{|\boldsymbol{x}'| \geq R, x_N \in \mathbb{R}} |\boldsymbol{U}(|\boldsymbol{x}' - g(\boldsymbol{\xi})\boldsymbol{\xi}|, x_N) - \boldsymbol{U}(|\boldsymbol{x}' - g(\boldsymbol{\xi})\boldsymbol{\xi} + \eta\boldsymbol{\nu}(\boldsymbol{\xi})|, x_N + m_*\eta)| = 0$$

for each $\boldsymbol{\xi} \in S^{N-2}$. Using this fact and (6.2), we have

$$\lim_{R\to\infty} \sup_{|\boldsymbol{x}'| + |x_N| \geq R} \left|\overline{\boldsymbol{V}}(\boldsymbol{x}', x_N) - \underline{\boldsymbol{V}}(\boldsymbol{x}', x_N)\right| = 0. \tag{6.3}$$

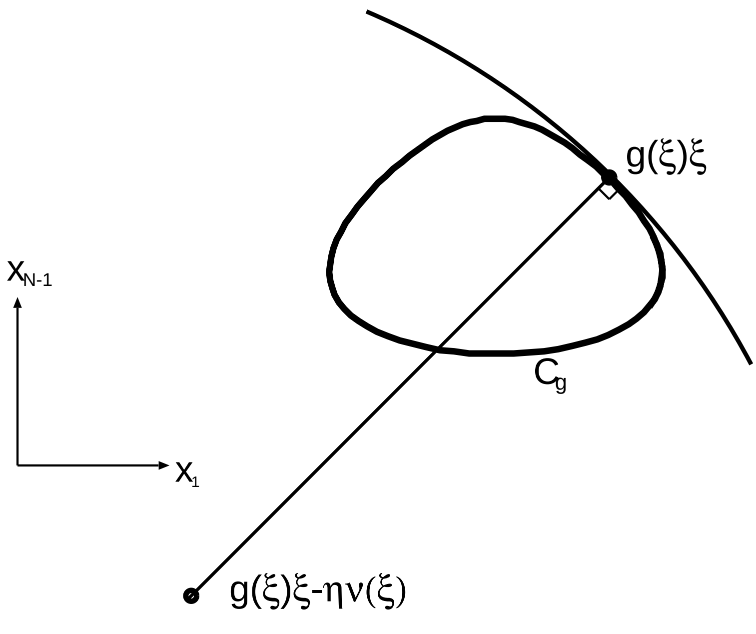

Figure 3: A circumscribed ball of C_g.

Using $\boldsymbol{w}_t(\boldsymbol{x}, t; \underline{\boldsymbol{V}}) \geq \boldsymbol{0}$ for $t > 0$, we define $\widetilde{\boldsymbol{U}}$ as

$$\widetilde{\boldsymbol{U}}(\boldsymbol{x}) = \lim_{t\to\infty} \boldsymbol{w}(\boldsymbol{x}, t; \underline{\boldsymbol{V}}) \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^N.$$

Combining this convergence and (6.3), we get

$$\lim_{t\to\infty}\sup_{\boldsymbol{x}\in\mathbb{R}^N}|\boldsymbol{w}(\boldsymbol{x},t;\underline{\boldsymbol{V}})-\widetilde{\boldsymbol{U}}(\boldsymbol{x})|=0. \tag{6.4}$$

Then $\widetilde{\boldsymbol{U}}$ satisfies (1.7) with

$$\underline{\boldsymbol{V}}(\boldsymbol{x})<\widetilde{\boldsymbol{U}}(\boldsymbol{x})<\overline{\boldsymbol{V}}(\boldsymbol{x}) \qquad \text{for all } \boldsymbol{x}\in\mathbb{R}^N.$$

Now we state properties of $\widetilde{\boldsymbol{U}}$ as follows.

Lemma 12 *One has*

$$-\frac{\partial\widetilde{\boldsymbol{U}}}{\partial\boldsymbol{t}}\geq\mathbf{0} \quad \text{in } \mathbb{R}^N \tag{6.5}$$

for any $\boldsymbol{t}$ *in Lemma 2. There exists* $\mu>0$ *such that one has*

$$\boldsymbol{U}(|\boldsymbol{x}'|,x_N+\mu)<\widetilde{\boldsymbol{U}}(\boldsymbol{x}',x_N)<\boldsymbol{U}(|\boldsymbol{x}'|,x_N-\mu) \qquad \text{for all } (\boldsymbol{x}',x_N)\in\mathbb{R}^N. \tag{6.6}$$

For any $\delta\in(0,\delta_*)$ *one has*

$$-\mathrm{D}_N\widetilde{\boldsymbol{U}}(\boldsymbol{x})\geq B_\delta\mathbf{1} \qquad \text{if} \quad |\widetilde{\boldsymbol{U}}(\boldsymbol{x})|\geq\delta \text{ or } |\widetilde{\boldsymbol{U}}(\boldsymbol{x})-\mathbf{1}|\geq\delta.$$

Here B_δ *is a positive constant in Proposition 2.*

Proof. This lemma follows from the definition of $\widetilde{\boldsymbol{U}}$ and Lemma 8. □

By (6.3) we get (1.8). If $g_1\not\sim g_2$, we find

$$\widetilde{\boldsymbol{U}}_2(x_1,\ldots,x_{N-1},x_N)\not\equiv\widetilde{\boldsymbol{U}}_1(x_1,\ldots,x_{N-1},x_N-\zeta)$$

for any $\zeta\in\mathbb{R}$.

Proof of Theorem 1. It suffices to show the uniqueness. First assume that we have another $\widetilde{\boldsymbol{U}}_1$ satisfying (1.7) and (1.8) for the same g. Let $\delta>0$ and $\sigma>0$ be as in Lemma 7 for $\widetilde{\boldsymbol{U}}$. Then, we take $\lambda>0$ large enough and have

$$\widetilde{\boldsymbol{U}}_1(\boldsymbol{x}',x_N)\leq\widetilde{\boldsymbol{U}}(\boldsymbol{x}',x_N-\lambda)+\delta\boldsymbol{p}(|\widetilde{\boldsymbol{U}}(\boldsymbol{x}',x_N-\lambda)|).$$

Then we get

$$\widetilde{\boldsymbol{U}}_1(\boldsymbol{x}',x_N)\leq\widetilde{\boldsymbol{U}}(\boldsymbol{x}',x_N-\lambda-\sigma\delta(1-e^{-\beta t}))+\delta e^{-\beta t}\boldsymbol{p}(|\widetilde{\boldsymbol{U}}(\boldsymbol{x}',x_N-\lambda-\sigma\delta(1-e^{-\beta t})|).$$

Sending $t\to\infty$, we find

$$\widetilde{\boldsymbol{U}}_1(\boldsymbol{x}',x_N)\leq\widetilde{\boldsymbol{U}}(\boldsymbol{x}',x_N-\lambda-\sigma\delta).$$

We can replace $\widetilde{\boldsymbol{U}}$ and $\widetilde{\boldsymbol{U}}_1$, and carry on the argument stated above. Then we have

$$\widetilde{\boldsymbol{U}}(\boldsymbol{x}',x_N)\leq\widetilde{\boldsymbol{U}}_1(\boldsymbol{x}',x_N-\lambda'-\sigma'\delta')$$

for $\delta' > 0$, $\sigma' > 0$ and $\lambda' > 0$. Then, without loss of generality, we can assume

$$\widetilde{\boldsymbol{U}}(\boldsymbol{x}', x_N) \leq \widetilde{\boldsymbol{U}}_1(\boldsymbol{x}', x_N) \leq \widetilde{\boldsymbol{U}}(\boldsymbol{x}', x_N - \lambda)$$

for some $\lambda \geq 0$. Now we can define

$$\Lambda = \inf \left\{ \lambda \geq 0 \,\middle|\, \widetilde{\boldsymbol{U}}(\boldsymbol{x}', x_N) \leq \widetilde{\boldsymbol{U}}_1(\boldsymbol{x}', x_N) \leq \widetilde{\boldsymbol{U}}(\boldsymbol{x}', x_N - \lambda) \right\}.$$

Then we have $\Lambda \geq 0$ and

$$\widetilde{\boldsymbol{U}}(\boldsymbol{x}', x_N) \leq \widetilde{\boldsymbol{U}}_1(\boldsymbol{x}', x_N) \leq \widetilde{\boldsymbol{U}}(\boldsymbol{x}', x_N - \Lambda).$$

If $\Lambda = 0$, we have $\widetilde{\boldsymbol{U}} \equiv \widetilde{\boldsymbol{U}}_1$. Assume $\widetilde{\boldsymbol{U}} \not\equiv \widetilde{\boldsymbol{U}}_1$ and get a contradiction. Then we have $\Lambda > 0$ and the strong maximum principle gives

$$\widetilde{\boldsymbol{U}}(\boldsymbol{x}', x_N) < \widetilde{\boldsymbol{U}}_1(\boldsymbol{x}', x_N) < \widetilde{\boldsymbol{U}}(\boldsymbol{x}', x_N - \Lambda).$$

Let $\sigma > 0$ be as in Lemma 7 for $\widetilde{\boldsymbol{U}}$. Using (4.1), (6.6) and the Schauder estimate, we have

$$\lim_{R \to \infty} \sup \left\{ |\mathrm{D}_N \widetilde{\boldsymbol{U}}(\boldsymbol{x}', x_N)| \,\middle|\, |x_N - \phi(\boldsymbol{x}')| \geq R \right\} = 0.$$

Taking $R > 0$ large enough, we get

$$2\sigma \sup \left\{ |\mathrm{D}_N \widetilde{\boldsymbol{U}}(\boldsymbol{x}', x_N)| \,\middle|\, |x_N - \phi(\boldsymbol{x}')| \geq R - \Lambda - 1 \right\} < \inf_{0 \leq y \leq 1} |\boldsymbol{p}(y)|.$$

Using Lemma 10, we take $h \in (0, 1/(2\sigma))$ small enough and find

$$\widetilde{\boldsymbol{U}}(\boldsymbol{x}', x_N) \leq \widetilde{\boldsymbol{U}}(\boldsymbol{x}', x_N - \Lambda + 2h\sigma) \qquad \text{if } |x_N - \phi(\boldsymbol{x}')| \geq R - \Lambda - 1.$$

If $|x_N - \phi(\boldsymbol{x}')| \geq R - \Lambda - 1$, we have

$$\begin{aligned} \widetilde{\boldsymbol{U}}(\boldsymbol{x}', x_N - \Lambda + 2h\sigma) - \widetilde{\boldsymbol{U}}(\boldsymbol{x}', x_N) &> \widetilde{\boldsymbol{U}}(\boldsymbol{x}', x_N - \Lambda + 2h\sigma) - \widetilde{\boldsymbol{U}}(\boldsymbol{x}', x_N - \Lambda) \\ &= 2h\sigma \int_1^1 \mathrm{D}_N \widetilde{\boldsymbol{U}}(\boldsymbol{x}', x_N - \Lambda + 2h\sigma\theta)\, \mathrm{d}\theta \geq -h \left(\inf_{0 \leq y \leq 1} |\boldsymbol{p}(y)| \right) \mathbf{1}. \end{aligned}$$

Combining the two inequalities stated above, we find

$$\widetilde{\boldsymbol{U}}_1(\boldsymbol{x}', x_N) \leq \widetilde{\boldsymbol{U}}(\boldsymbol{x}', x_N - \Lambda + 2h\sigma) + h\boldsymbol{p}(|\widetilde{\boldsymbol{U}}(\boldsymbol{x}', x_N - \Lambda + 2h\sigma)|) \qquad \text{for all } (\boldsymbol{x}', x_N) \in \mathbb{R}^N,$$

which yields

$$\begin{aligned} &\widetilde{\boldsymbol{U}}_1(\boldsymbol{x}', x_N) \\ &\leq \widetilde{\boldsymbol{U}}(\boldsymbol{x}', x_N - \Lambda + 2h\sigma - h\sigma(1 - e^{-\beta t})) + he^{-\beta t}\boldsymbol{p}(|\widetilde{\boldsymbol{U}}(\boldsymbol{x}', x_N - \Lambda + 2h\sigma - h\sigma(1 - e^{-\beta t}))|). \end{aligned}$$

Sending $t \to \infty$, we get

$$\widetilde{\boldsymbol{U}}_1(\boldsymbol{x}', x_N) \leq \widetilde{\boldsymbol{U}}(\boldsymbol{x}', x_N - \Lambda + h\sigma).$$

This contradicts the definition of Λ. This gives $\Lambda = 0$ and the uniqueness of $\widetilde{\boldsymbol{U}}$. Finally, if $g_1 \sim g_2$, the definition of $\widetilde{\boldsymbol{U}}$ gives (1.9). This completes the proof of Theorem 1. □

Now we state the stability of $\widetilde{\boldsymbol{U}}$ as follows.

Corollary 3 (Stability) *Let $\widetilde{\boldsymbol{U}}$ and $\underline{\boldsymbol{V}}$ be as in Theorem 1 and (6.1), respectively. Let a bounded and uniformly continuous function $\boldsymbol{u}_0$ satisfy*

$$\lim_{R\to\infty}\sup_{|\boldsymbol{x}|\geq R}|\boldsymbol{u}_0(\boldsymbol{x})-\widetilde{\boldsymbol{U}}(\boldsymbol{x})|=0,$$
$$\underline{\boldsymbol{V}}(\boldsymbol{x})\leq \boldsymbol{u}_0(\boldsymbol{x})\leq \mathbf{1}\qquad \text{for all } \boldsymbol{x}\in\mathbb{R}^N.$$

Then one has

$$\lim_{t\to\infty}\sup_{\boldsymbol{x}\in\mathbb{R}^N}|\boldsymbol{w}(\boldsymbol{x},t;\boldsymbol{u}_0)-\widetilde{\boldsymbol{U}}(\boldsymbol{x})|=0.$$

Proof. For $a\geq 0$ we introduce

$$\boldsymbol{v}_1(\boldsymbol{x}',x_N)=\min_{\boldsymbol{\xi}\in S^{N-2}}\boldsymbol{U}(|\boldsymbol{x}'-\tau_a g(\boldsymbol{\xi})\boldsymbol{\xi}|,x_N-m_*a)$$

and have

$$\lim_{R\to\infty}\sup_{|\boldsymbol{x}|\geq R}|\boldsymbol{v}_1(\boldsymbol{x})-\underline{\boldsymbol{V}}(\boldsymbol{x})|=0,\quad \lim_{R\to\infty}\sup_{|\boldsymbol{x}|\geq R}|\boldsymbol{v}_1(\boldsymbol{x})-\widetilde{\boldsymbol{U}}(\boldsymbol{x})|=0$$

by using Lemma 10. Let $\delta\in(0,\delta_*)$ be as in Lemma 7. Using Lemma 8, we take $a>0$ large enough such that we get

$$\begin{aligned}\boldsymbol{U}(|\boldsymbol{x}'-g(\boldsymbol{\xi})\boldsymbol{\xi}+\eta\boldsymbol{\nu}(\boldsymbol{\xi})|,x_N+m_*\eta)-\delta\boldsymbol{p}(|\boldsymbol{U}(|\boldsymbol{x}'-g(\boldsymbol{\xi})\boldsymbol{\xi}+\eta\boldsymbol{\nu}(\boldsymbol{\xi})|,x_N+m_*\eta)|)&\\ \leq \boldsymbol{u}_0(\boldsymbol{x})&\\ \leq \boldsymbol{U}(|\boldsymbol{x}'-\tau_a g(\boldsymbol{\xi})\boldsymbol{\xi}|,x_N-m_*a)+\delta\boldsymbol{p}(|\boldsymbol{U}(|\boldsymbol{x}'-\tau_a g(\boldsymbol{\xi})\boldsymbol{\xi}|,x_N-m_*a)|)&\end{aligned}$$

for all $(\boldsymbol{x}',x_N)\in\mathbb{R}^N$ and $\boldsymbol{\xi}\in S^{N-2}$. Then we have

$$\begin{aligned}&\boldsymbol{U}(|\boldsymbol{x}'-g(\boldsymbol{\xi})\boldsymbol{\xi}+\eta\boldsymbol{\nu}(\boldsymbol{\xi})|,x_N+m_*\eta+\sigma\delta(1-e^{-\beta t}))\\ &\quad-\delta e^{-\beta t}\boldsymbol{p}(|\boldsymbol{U}(|\boldsymbol{x}'-g(\boldsymbol{\xi})\boldsymbol{\xi}+\eta\boldsymbol{\nu}(\boldsymbol{\xi})|,x_N+m_*\eta+\sigma\delta(1-e^{-\beta t}))|)\\ &\qquad\leq \boldsymbol{w}(\boldsymbol{x},t;\boldsymbol{u}_0)\\ &\qquad\leq \boldsymbol{U}(|\boldsymbol{x}'-\tau_a g(\boldsymbol{\xi})\boldsymbol{\xi}|,x_N-m_*a-\sigma\delta(1-e^{-\beta t}))\\ &\qquad\quad+\delta e^{-\beta t}\boldsymbol{p}(|\boldsymbol{U}(|\boldsymbol{x}'-\tau_a g(\boldsymbol{\xi})\boldsymbol{\xi}|,x_N-m_*a-\sigma\delta(1-e^{-\beta t}))|)\end{aligned}$$

Sending $t\to\infty$, we get

$$\begin{aligned}\max_{\boldsymbol{\xi}\in S^{N-2}}\boldsymbol{U}(|\boldsymbol{x}'-g(\boldsymbol{\xi})\boldsymbol{\xi}+\eta\boldsymbol{\nu}(\boldsymbol{\xi})|,x_N+m_*\eta+\sigma\delta)\leq\liminf_{t\to\infty}\boldsymbol{w}(\boldsymbol{x},t;\boldsymbol{u}_0)\leq\limsup_{t\to\infty}\boldsymbol{w}(\boldsymbol{x},t;\boldsymbol{u}_0)&\\ \leq\min_{\boldsymbol{\xi}\in S^{N-2}}\boldsymbol{U}(|\boldsymbol{x}'-\tau_a g(\boldsymbol{\xi})\boldsymbol{\xi}|,x_N-m_*a-\sigma\delta)&\end{aligned}$$

for all $(\boldsymbol{x}',x_N)\in\mathbb{R}^N$. Taking the left-hand side and the right-hand side as initial values of (1.3), we find

$$\widetilde{\boldsymbol{U}}(\boldsymbol{x}',x_N+\sigma\delta)\leq\liminf_{t\to\infty}\boldsymbol{w}(\boldsymbol{x},t;\boldsymbol{u}_0)\leq\limsup_{t\to\infty}\boldsymbol{w}(\boldsymbol{x},t;\boldsymbol{u}_0)\leq\widetilde{\boldsymbol{U}}(\boldsymbol{x}',x_N-\sigma\delta).$$

from Theorem 1. Since we can choose δ arbitrarily small, we complete the proof. □

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