Local NMR relaxation rates $T_1^{-1}$ and $T_2^{-1}$ depending on the $d$-vector symmetry in the vortex state of chiral and helical $p$-wave superconductors

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(Received 26 January 2018; published 10 April 2018)

Local NMR relaxation rates in the vortex state of chiral and helical $p$-wave superconductors are investigated by the quasiclassical Eilenberger theory. We calculate the spatial and resonance frequency dependences of the local NMR spin-lattice relaxation rate $T_1^{-1}$ and spin-spin relaxation rate $T_2^{-1}$. Depending on the relation between the NMR relaxation direction and the $d$-vector symmetry, the local $T_1^{-1}$ and $T_2^{-1}$ in the vortex core region show different behaviors. When the NMR relaxation direction is parallel to the $d$-vector component, the local NMR relaxation rate is anomalously suppressed by the negative coherence effect due to the spin dependence of the odd-frequency $s$-wave spin-triplet Cooper pairs. The difference between the local $T_1^{-1}$ and $T_2^{-1}$ in the site-selective NMR measurement is expected to be a method to examine the $d$-vector symmetry of candidate materials for spin-triplet superconductors.

DOI: 10.1103/PhysRevB.97.134507

I. INTRODUCTION

The spin-triplet superconductors have attracted much attention since exotic states such as odd-frequency Cooper pairs and Majorana states are expected to be induced at the vortex core and surface regions. Although ruthenate superconductor Sr$_2$RuO$_4$, heavy fermion superconductor UPt$_3$, and other materials have been suggested as spin-triplet superconductors by many experimental and theoretical studies [1–6], the $d$-vector symmetries have not been identified. The $d$-vector symmetry was discussed to explain experimental observations such as magnetic field orientation dependences of the Knight shift [4,5] and the Pauli-limit behavior of $H_{c2}$ [6]. In addition to these approaches, we need new methods to clarify the $d$-vector symmetry in the spin-triplet superconductors.

In the spin-triplet chiral $p$-wave superconductors, the theoretical studies for the local NMR spin-lattice relaxation rate $T_1^{-1}$ revealed that the local $T_1^{-1}$ in the vortex core region is anomalously suppressed [7–9]. The previous studies based on the Eilenberger theory found the site and resonance frequency dependences of $T_1^{-1}$, and the anomalous suppression of the local $T_1^{-1}$ is derived from the negative coherence effect related to the odd-frequency $s$-wave spin-triplet Cooper pairs [10]. Experimentally, the local $T_1^{-1}$ were detected by the site-selective NMR measurements for high-$T_c$ superconductors [11–13] and the conventional superconductor [14]. As shown in Fig. 1(a), the local $T_1^{-1}(r)$ as a function of internal field $B(r)$ at the same position $r$ can be observed by tuning the resonance frequency among the resonance line shape, since the internal field is in proportion to the resonance frequency. The spectrum of $B(r)$ in Sr$_2$RuO$_4$ is observed by $\mu$SR measurement [15].

In the uniform state of three-dimensional chiral superconductors with strong spin-orbit coupling and odd-parity pairing, significant suppression of the NMR relaxation rate is suggested for nuclear spins polarized along the nodal direction as a consequence of the spin-selective Majorana nature of nodal quasiparticles [16].

At the surface of the superfluid $^3$He B phase, the odd-frequency $s$-wave spin-triplet Cooper pairs were studied in relation to the static spin susceptibility [17], and Ising-type spin relaxation was discussed in relation to the Majorana state [18]. In addition, a strong relation between the Majorana zero-energy mode and the odd-frequency Cooper pair has been revealed [19–22]. These spin-dependent surface states have information of the pairing symmetry, and are expected to be studied also in the vortex core states in spin-triplet superconductors. Since the NMR relaxation rate can prove the direction of the conduction electrons’ spin, we expect that the $d$-vector structure in the vortex state is detected by site-selective NMR measurement with orientation control of the NMR relaxation direction. As shown in Fig. 1(b), for relaxation direction $\delta M$ parallel (perpendicular) to the static applied field $H$, we observe the NMR spin-lattice relaxation rate $T_1^{-1}$ (the NMR spin-spin relaxation rate $T_2^{-1}$) [23–25]. When applied fields $H$ are along the $z$ direction, $T_1^{-1}$ by $\delta M$||$z$ comes from the $xy$ component of dynamical spin susceptibility $\chi_{xx} + \chi_{yy}$. And $T_2^{-1}$ is from $\chi_{zz} + \chi_{xy}$ if $\delta M$||$x$. Therefore, the difference between $T_1^{-1}$ and $T_2^{-1}$ may reflect the orientation of the $d$ vector in spin-triplet superconductors.

In this paper, we study the local NMR relaxation rates $T_1^{-1}$ and $T_2^{-1}$ in the vortex state of chiral and helical $p$-wave superconductors. For chiral $p$-wave superconductors, we consider the two types of chiral $p$-wave states, $d$||$z$ and $d$||$x$, where the direction of the $d$ vector indicates the $z$ and $x$ axis, respectively. In particular, we discuss how the relaxation rates depend on the direction of the NMR relaxation directions, calculating the site $r$ and internal field $B$ dependences of $T_1^{-1}$.
and $T_2^{-1}$. These results help us to investigate the $d$-vector symmetry of chiral and helical $p$-wave superconductors by site-selective NMR measurement.

This paper is organized as follows. After the introduction, we describe our formulation of the quasiclassical Eilenberger equation in the vortex lattice state and the calculation method for the local NMR relaxation rates $T_1^{-1}$ and $T_2^{-1}$ in Sec. II. The derivations of $T_1^{-1}$ and $T_2^{-1}$ are explained in Appendix A based on the Eilenberger theory. In Sec. III, we investigate the site and $B$ dependences of local $T_1^{-1}$ and $T_2^{-1}$ in the chiral and helical $p$-wave superconductors to find the relation to the $d$-vector symmetry. In Sec. IV and Appendix B, to understand the $d$-vector dependence, we discuss the site dependence of the coherence terms and the odd-frequency $s$-wave spin-triplet Cooper pairs around a vortex. The last section is devoted to the summary.

II. FORMULATION

We calculate the spatial structure of vortices in the vortex lattice state by quasiclassical Eilenberger theory. The quasiclassical theory is valid when the atomic scale is small enough compared to the superconducting coherence length. For many superconductors including Sr$_2$RuO$_4$, the quasiclassical condition is well satisfied.

For simplicity, we consider the chiral or the helical $p$-wave pairings on the two-dimensional cylindrical Fermi surface, $k = (k_x, k_y) = k_F (\cos \theta_k, \sin \theta_k)$, and the Fermi velocity $v_F = v_F k/k_F$. In the following, the hat symbol indicates the $2 \times 2$ matrix in spin space and the check symbol indicates the $4 \times 4$ matrix in particle-hole and spin spaces.

To obtain quasiclassical Green’s functions $\hat{g}(i\omega_n, r, \bf{k})$ in the vortex lattice state, we solve the Riccati equation derived from the Eilenberger equation [26,27]

$$-i v \cdot \nabla \hat{g}(i\omega_n, r, \bf{k}) = \frac{1}{2} [i\hat{\sigma}_3 \hat{\delta}(r, \bf{k}) - \hat{\Delta}(r, \bf{k}), \hat{g}(i\omega_n, r, \bf{k})]$$

(1)

in the clean limit, where $r$ is the center-of-mass coordinate of the pair, $v = v_F/v_0$, $\hat{\sigma}_3$ is the Pauli matrix defined in Eq. (A3), and $i\hat{\sigma}_3 = i\sigma_3 - v\bf{A}$ with Matsubara frequency $\omega_n$.

The quasiclassical Green’s functions and order parameter are described by

$$\hat{g}(i\omega_n, r, \bf{k}) = -i\pi \begin{bmatrix} \hat{g}(i\omega_n, r, \bf{k}) & i\hat{\dot{g}}(i\omega_n, r, \bf{k}) \\ -i\hat{\dot{g}}(i\omega_n, r, \bf{k}) & \hat{\dot{g}}(i\omega_n, r, \bf{k}) \end{bmatrix}$$

(2)

$$\hat{\Delta}(r, \bf{k}) = \begin{bmatrix} 0 \\ -\hat{\Delta}(r, \bf{k}) \end{bmatrix}$$

(3)

where $\hat{g}^2 = -\pi^2 \hat{I}$. The spin spaces of $\hat{g}$ and $\hat{\Delta}$ are, respectively, defined by the matrix elements

$$g_{\sigma\sigma'}(i\omega_n, r, \bf{k}) = \left[ g_0(i\omega_n, \bf{r}) \hat{1} + \sum_{\mu=x,y,z} g_{\mu}(i\omega_n, \bf{r}) \hat{\sigma}_{\mu} \right]_{\sigma\sigma'}$$

(4)

$$\Delta_{\sigma\sigma'}(r, \bf{k}) = \left[ \sum_{\mu=x,y,z} [d_{\mu}(r, \bf{k}) \hat{\sigma}_3] \right]_{\sigma\sigma'}$$

(5)

where $\sigma, \sigma' = \uparrow$ (up-spin) or $\downarrow$ (down-spin), and $d_{\mu}$ is the $\mu$ component of the $d$ vector. In addition, the matrix elements of the order parameter are defined by

$$\Delta_{\sigma\sigma'}(r, \bf{k}) = \Delta_{+,\sigma\sigma'}(r, \bf{k}) \phi_{p_+}(\bf{k}) + \Delta_{-,\sigma\sigma'}(r, \bf{k}) \phi_{p_-}(\bf{k})$$

(6)

with the order parameter $\Delta_{+,\sigma\sigma'}(r)$ and the pairing function $\phi_{p_\pm}(\bf{k}) = k_\pm i k_{\sigma}$ for the $p_\pm$ state.

Length, temperature, and magnetic field are, respectively, measured in units of $\xi_0, T_0$, and $B_0$. Here, $\xi_0 = \hbar v_F/2\pi k_B T_0$ and $B_0 = \phi_0/2\pi \xi_0^2$ with the flux quantum $\phi_0$. $T_0$ is the superconducting transition temperature at a zero magnetic field. The energy $E$, pair potential $\Delta$, and $\omega_n$ are in units of $\pi k_B T_0$. In the following, we set $\hbar = k_B = 1$. In this paper, our calculations are performed at $H/B_0 = 0.02$ and $T = 0.5 T_0$. In the chiral $p$-wave and helical $p$-wave states at $T = 0.5 T_0$, the upper critical fields $H_{c2}/B_0 \simeq 0.85$.

We set the magnetic field along the $z$ axis, where the vector potential $\bf{A}(r) = \frac{1}{2} \bf{H} \times \bf{r} + a(r)$ in the symmetric gauge. $\bf{H} = (0, 0, H)$ is a uniform flux density, and $a(r)$ is related to the internal field $\bf{B}(r) = (0, 0, B(r)) = \bf{H} + \nabla \times a(r)$. The unit cell of the vortex lattice is set as a square lattice [1]. From the distribution of $\bf{B}(r)$, we calculate the resonance line shape called the Redfield pattern given by $P(B) = \int [\delta(B - B(r))] d\bf{r}$ in Fig. 1.
To determine the pair potential $\Delta(r)$ and the quasiequilibrium Green’s functions self-consistently, we calculate the order parameter $\Delta_n(r)$ by the gap equation
\begin{equation}
\dot{\Delta}_n(r) = g N_0 T \sum_{|\alpha|,|\alpha'|} \langle \phi^*_{\alpha}(k) \hat{f}(i \omega_n, r, k) \rangle_k,
\end{equation}
where $\langle \ldots \rangle_k$ indicates Fermi-surface average, $(g N_0)^{-1} = \ln T + 2 T \sum_{0 < \omega_n < \omega_{\text{cut}}} \omega_n^{-1}$, and we use $\omega_{\text{cut}} = 20 k_B T_0$. In Eq. (7), $p$-wave pairing interaction is isotropic in spin space. For the self-consistent calculation of the vector potential for the internal field $B(r)$, we use the current equation
\begin{equation}
\nabla \times (\nabla \times A) = -\frac{2T}{\kappa^2} \sum_{0 < \omega_n} \langle \psi \text{Im}(g_0) \rangle_k
\end{equation}
with the Ginzburg-Landau parameter $\kappa = B_0/\pi k_B T_c \sqrt{8 \pi N_0}$. In our calculations, we use $\kappa = 2.7$ appropriate to Sr$_2$RuO$_4$ as a candidate material for the chiral or helical $p$-wave superconductor. The results of this paper are not changed qualitatively by the choice of the Ginzburg-Landau parameter as long as a type-II superconducting vortex state is maintained.

We iterate calculations of Eqs. (1), (7), and (8) for $\omega_n$ until we obtain the self-consistent results of $A(r)$, $\Delta(r,k)$, and the quasiequilibrium Green’s functions in the vortex lattice state.

In the chiral $p$-wave superconductors, we only consider the $p_{\,-}$ state, i.e., antiparallel vortex state, since the antiparallel vortex state is stable compared to the parallel vortex state [28,29]. For variety of $d$-vector orientation, we calculate two types of chiral $p$-wave states, $d_{\parallel}$ and $d_{\perp}$, which are, respectively, defined by $d_{\parallel}(k) \propto (k_x - i k_y) \xi$ and $d_{\perp}(k) \propto (k_x - i k_y) \bar{\xi}$. In the helical $p$-wave superconductors, we set the $d$ vector as $d(k) \propto k_x \xi + k_y \bar{\xi} = \phi_p(k) d_{\parallel} + \phi_{p_\perp}(k) d_{\perp}$ in the uniform state at zero field, with $\phi_{p}(k) = \frac{1}{2} \left(1 + i \delta k_x \right)$. Thus, when we iterate calculations of Eqs. (1), (7), and (8), as an initial value, the $d$ vector is set to be $d(r,k) = \Psi(r) d(k)$ where $\Psi(r)$ is the Abrikosov vortex lattice solution.

Next, using the self-consistently obtained $A(r)$ and $\Delta(r)$, we calculate $\hat{g}(i \omega_n, r, k)|_{\omega_n=+i \eta}$ for real energy $E$ by solving Eilenberger Eq. (1) with $i \omega_n \rightarrow E \pm i \eta$. In this paper, we define the retarded and advanced Green’s functions $\hat{g}^R(E, r, k) = \hat{g}(i \omega_n, r, k)|_{\omega_n=+i \eta}$ and $\hat{g}^A(E, r, k) = \hat{g}(i \omega_n, r, k)|_{\omega_n=-i \eta}$, respectively. $\eta$ is a small parameter, and we use $\eta = 0.01$. We use the components of $\hat{g}^R(E, r, k)$ and $\hat{g}^A(E, r, k)$ to numerically calculate local $(T_1 T)^{-1}$ and $(T_2 T)^{-1}$ in the vortex lattice state.

A. NMR spin-lattice relaxation rate $T_{1\,-}^{-1}$

We consider the conventional form of the local hyperfine fields from the conduction electrons [24]. Therefore, relaxation rates $T_{1\,-}^{-1}$ and $T_{2\,-}^{-1}$ of the nuclear magnetization are affected by the dynamical spin susceptibility of the local electronic state, which reflects spin states of the odd-frequency $s$-wave spin-triplet Cooper pairs at the atomic site in the vortex core region, as discussed later.

The NMR spin-lattice relaxation rate $T_{1\,-}^{-1}$ by $\delta M \parallel z$ is calculated from $xy$ components of dynamical spin susceptibility, $x_{xx} + x_{yy} \propto x_{-+}$. Therefore, as described in Appendix A following Refs. [8,10], $T_{1\,-}^{-1}$ is given by
\begin{equation}
\frac{[T_1(T)/T]^{-1}}{[T_1(T_c)/T_c]^{-1}} = \frac{[T_{1\,E}(T)/T]^{-1} + [T_{1\,f}(T)/T]^{-1}}{[T_1(T_c)/T_c]^{-1}} = \frac{\int -\infty^\infty W_{\sigma\sigma'}^{(E,E)}(E) + W_{\sigma\sigma'}^{(E,E)}(E)}{4 T \cosh^2(E/2T)} dE,
\end{equation}
where
\begin{equation}
W_{\sigma\sigma'}^{(E,E)}(E) = \langle a_{\sigma\sigma'}^{(E,E)}(r,k) \rangle_k \langle a_{\sigma\sigma'}^{(E,E)}(r+k,k) \rangle_k,
\end{equation}

\begin{align*}
W_{\sigma\sigma'}^{(E,E)}(E) &= -\langle a_{\sigma\sigma'}^{(E,E)}(r,k) \rangle_k \langle a_{\sigma\sigma'}^{(E,E)}(r+k,k) \rangle_k, \\
W_{\sigma\sigma'}^{(E,E)}(E) &= \frac{1}{2} (a_{\sigma\sigma'}^{(E,E)}(r,k) - a_{\sigma\sigma'}^{(E,E)}(r,k)).
\end{align*}

B. NMR spin-spin relaxation rate $T_{2\,-}^{-1}$

From the field theory of NMR relaxation rate [24], we derive the NMR spin-spin relaxation rate $T_{2\,-}^{-1}$ by $\delta M \parallel x$ from the dynamical spin susceptibility $\chi_{xx} + \chi_{yy}$ based on the Eilenberger theory. Since $\chi_{yy}$ gives $\frac{1}{2} T_{2\,-}^{-1}$, $T_{2\,-}^{-1}$ is given in the form
\begin{equation}
T_{2\,-}^{-1} = \frac{1}{2} T_{2\,-}^{-1} + \frac{1}{2} r_{zz\,-}^{-1},
\end{equation}
where $\frac{1}{2} r_{zz\,-}^{-1}$ is the contribution from $\chi_{zz}$. In the following, we focus on $T_{2\,-}^{-1}$ instead of $T_{2\,-}^{-1}$.

As described in Appendix A, $T_{2\,-}^{-1}$ is given by
\begin{equation}
\frac{1}{2} \frac{[T_{2\,-}(T)/T_c]^{-1} - [T_{2\,-}(T_c)/T_c]^{-1}}{[T_{2\,-}(T_c)/T_c]^{-1}} = \frac{1}{2} \frac{[T_{2\,E\,-}(T)/T_c]^{-1} + [T_{2\,f\,-}(T)/T_c]^{-1}}{[T_{2\,-}(T_c)/T_c]^{-1}} = \sum_{\sigma\sigma'} \int_{-\infty}^\infty \frac{W_{\sigma\sigma'}^{(E,E)}(E) + W_{\sigma\sigma'}^{(E,E)}(E)}{16 T \cosh^2(E/2T)} dE,
\end{equation}
where
\begin{equation}
W_{\sigma\sigma'}^{(E,E)}(E) = \langle a_{\sigma\sigma'}^{(E,E)}(r,k) \rangle_k \langle a_{\sigma\sigma'}^{(E,E)}(r+k,k) \rangle_k,
\end{equation}

\begin{align*}
W_{\sigma\sigma'}^{(E,E)}(E) &= \langle a_{\sigma\sigma'}^{(E,E)}(r,k) \rangle_k \langle a_{\sigma\sigma'}^{(E,E)}(r+k,k) \rangle_k, \\
W_{\sigma\sigma'}^{(E,E)}(E) &= \frac{1}{2} (a_{\sigma\sigma'}^{(E,E)}(r,k) - a_{\sigma\sigma'}^{(E,E)}(r,k)).
\end{align*}
The dashed lines are for \((T_0)\) vortex state of chiral and helical \(p\) wave superconductors. The results are presented in Figs. 2(a) and 2(b), we realize that the suppression of the relaxation rate occurs when the \(d\) vector is parallel to the NMR relaxation direction \(\delta M\). Therefore, we can extract information about the \(d\)-vector orientation from comparative observation of \((T_1)\) and \((T_2)\) at the vortex core region in the site-selective NMR measurement.

In Fig. 2(c) for the helical \(p\) wave state, the \(r\) dependences of \((T_1)\) and \((T_2)\) in the vortex state of chiral and helical \(p\)-wave superconductors to discuss how the difference between \((T_1)\) and \((T_2)\) for each \(d\)-vector symmetry is detected in the site-selective NMR measurement. As presented in Fig. 3, the local \([T_1(r,r')|T\] is plotted as a function of \(B\) at the same position \(r\). The signal from the higher (lower) field comes from inside (outside) of the vortex core. In Fig. 3(a) for the chiral \(p\)-wave \(d\|z\) state, \((T_1)\) shows monotonically decreasing behavior as a function of \(B\). \((T_2)\) shows also decreasing behavior in the range \(B \lesssim 1\), but in high-field ranges \((T_2)\) shows increasing behavior towards a large value at the vortex center. In Fig. 3(b) for the chiral \(p\)-wave \(d\|x\) state, \((T_1)\) shows similar monotonic decreasing behavior to those of \((T_1)\) in Fig. 3(a). On the other hand, \((T_1)\) in Fig. 3(b) shows similar enhancement at the vortex core but the magnitude is smaller, compared to that of \((T_2)\) in Fig. 3(a). In Fig. 3(c) for the helical \(p\)-wave state, \((T_1)\) and \((T_2)\) show decreasing behavior as a function of \(B\) in the range \(B \lesssim 1\), but these relaxation rates show increasing behavior in high-field ranges. The magnitude of \((T_2)\) is about half of \((T_1)\).

Therefore, the NMR relaxation rates show different behavior between \((T_1)\) and \((T_2)\) at the vortex core region in chiral \(p\)-wave \(d\|z\), chiral \(p\)-wave \(d\|x\), and helical \(p\)-wave states. The reason for this difference is related to the negative coherence effect and the odd-frequency spin-triplet Cooper pairs around the vortex center, as discussed in the next section.

III. SITE AND INTERNAL FIELD DEPENDENCIES OF LOCAL NMR RELAXATION RATES

First, we study the local NMR relaxation rates \([T_1(r,r')|T\] as a function of radius \(r\) on a line between next-nearest-neighbor (NNN) vortices in the vortex state of chiral and helical \(p\)-wave superconductors. The results are presented in Fig. 2. At the outside of the vortex core region \(r/a_x > 0.25\), the difference between \((T_1)\) and \((T_2)\) is very small in all pairing cases in Fig. 2. Therefore, we cannot obtain information about the \(d\)-vector symmetry from the observation of \((T_1)\) and \((T_2)\) in the uniform superconductivity outside of vortex cores.

On the other hand, in the vortex core region \(r/a_x < 0.25\), we can see the obvious difference between \((T_1)\) and \((T_2)\). In Fig. 2(a) for the chiral \(p\)-wave \(d\|z\) state, \((T_1)\) shows the anomalous suppression of the relaxation rate around the vortex core [10], but \((T_2)\) shows enhancement at the vortex core. The enhancement reflects the accumulation of low-energy quasiparticles around the vortex core. In Fig. 2(b) for the chiral \(p\)-wave \(d\|x\) state, at the vortex core \((T_2)\) shows the anomalous suppression, and \((T_1)\) shows the enhancement. From the differences between Figs. 2(a) and 2(b), we can see the obvious difference between \((T_1)\) and \((T_2)\) in the vortex core region.
contributions of the DOS term and the coherence term so that in (a) the chiral \( p \)-wave \( d \| z \) state and \( d \| x \) states and helical \( p \)-wave states, respectively. These are almost on the same line.

toward the outside of the vortex core. The \( r \) dependences of \( T_{zz}^{ff} \) \(^{-1} \) and \( T_{zz}^{ff} \) \(^{-1} \) show similar behavior also for the chiral \( p \)-wave \( d \| x \) state and the helical \( p \)-wave state.

Compared to \( T_{zz}^{ff} \) \(^{-1} \) and \( T_{zz}^{ff} \) \(^{-1} \), contributions of the coherence terms \( T_{zz}^{ff} \) \(^{-1} \) and \( T_{zz}^{ff} \) \(^{-1} \) in Fig. 4(a) are negligible in the outside region of the vortex core, but become comparative contributions with approaching the vortex center. For the chiral \( p \)-wave \( d \| z \) state, \( T_{zz}^{ff} \) \(^{-1} \) is negative and \( T_{zz}^{ff} \) \(^{-1} \) is positive. The former negative contributions are the origin of the anomalous suppression of \( T_{zz} \) \(^{-1} \) at the vortex core [10], and the latter further enhances \( T_{zz} \) \(^{-1} \) at the vortex core. Their magnitudes satisfy \( |T_{zz}^{ff} \) \(^{-1} \| z \rangle | - |T_{zz}^{ff} \) \(^{-1} \| z \rangle | \) in the whole range of \( r \) in Fig. 4(a). At the vortex center, \( T_{zz}^{ff} \) \(^{-1} \| z \rangle | - |T_{zz}^{ff} \) \(^{-1} \| z \rangle | \sim 24.7 \).

For the chiral \( p \)-wave \( d \| x \) state, the \( r \) dependence of \( T_{zz}^{ff} \) \(^{-1} \) is the same as \( T_{zz}^{ff} \) \(^{-1} \) in the chiral \( p \)-wave \( d \| z \) state. At the vortex center, \( T_{zz}^{ff} \) \( T_{zz}^{ff} \) \(^{-1} \| z \rangle | - |T_{zz}^{ff} \) \(^{-1} \| z \rangle | \sim 24.7 \). Therefore, \( T_{zz} \) \(^{-1} \) in the \( d \| x \) state shows similar anomalous suppression at the vortex core to those of \( T_{zz} \) \(^{-1} \) in the \( d \| z \) state. The anomalous suppression at the vortex core occurs when the NMR relaxation direction \( \delta M \) \( | d \rangle \). On the other hand, \( T_{zz} \) \(^{-1} \) \( = 0 \) in the \( d \| x \) state. Since the enhancement by \( T_{zz} \) \(^{-1} \) does not work, \( T_{zz} \) \(^{-1} \) in Fig. 3(b) is smaller than \( T_{zz} \) \(^{-1} \) in Fig. 3(a).

For the helical \( p \)-wave state with \( d \| z \) state, there are small deviations between them in our numerical calculation at finite temperature. However, as discussed in Appendix B, in the limit \( T \rightarrow 0 \) we expect \( T_{zz} \) \(^{-1} \| z \rangle | - |T_{zz} \) \(^{-1} \| z \rangle | \) to 0 so that \( T_{zz} \) \( \rightarrow 0 \) in the helical \( p \)-wave state.

In the relations \( T_{zz} \) \(^{-1} \| z \rangle | - |T_{zz} \) \(^{-1} \| z \rangle | \) for the chiral \( p \)-wave \( d \| z \) state, there are small deviations between them in our numerical calculation at finite temperature. However, as discussed in Appendix B, in the limit \( T \rightarrow 0 \) we expect \( T_{zz} \) \(^{-1} \| z \rangle | - |T_{zz} \) \(^{-1} \| z \rangle | \) to 0 so that \( T_{zz} \) \( \rightarrow 0 \) in the helical \( p \)-wave state.

In the previous study for the local \( T_{zz} \) \(^{-1} \) in the chiral \( p \)-wave \( d \| z \) state [10], it is revealed that the negative coherence term \( T_{zz} \) \(^{-1} \), inducing the anomalous suppression of \( T_{zz} \) \(^{-1} \) around the vortex core, is related to the odd-frequency a-wave spin-triplet Cooper pair \( F_{g}(E = 0, \) \( r \) \( ) \) in the low-energy limit at low \( T \) and the limit of the isolated vortex at low fields \( H \), and the negative coherence term \( T_{zz} \) \(^{-1} \) tends to cancel the local DOS term \( T_{zz} \) \(^{-1} \) \( = N(E = 0, \) \( r \) \( ) \) \( \times |F_{g}(E = 0, \) \( r \) \( )|^{2} \), where \( N(E = 0, \) \( r \) \( ) \) is the local DOS. A previous study using the Bogoliubov–de Gennes theory revealed the relation \( N(E = 0, \) \( r \) \( ) \) \( \times |F_{g}(E = 0, \) \( r \) \( )|^{2} \) in the helical \( p \)-wave \( d \| z \) state for the vortex core quasiparticle states with Majorana zero-energy mode [30].

The spin-resolved local DOS \( N_{\sigma}(E, \) \( r \) \( ) \) is given by

\[
N_{\sigma}(E, \) \( r \) \( ) = \langle \text{Re}[g_{\sigma\sigma}^{R}(E, \) \( r \) \( , \) \( k \) \( )]|k\rangle,
\]  

where \( \sigma = \uparrow \) or \( \downarrow \) [27]. The local DOS is defined as \( 2N(E, \) \( r \) \( ) = N_{\uparrow}(E, \) \( r \) \( ) + N_{\downarrow}(E, \) \( r \) \( ) \). And the spin-dependent
s-wave Cooper pair is given by

\[ F_{s,\sigma}(E, r) = \phi_{s}(k) f_{s,\sigma}^{R}(E, r, k) \]

with the s-wave pairing function \( \phi_{s}(k) = 1 \). In the chiral \( p_{-} \)-wave Cooper pair considered in this paper, the induced Cooper pair components around a vortex should satisfy the condition \( L_{z} + W = 0 \), where \( L_{z} \) is an angular momentum for the induced components of Cooper pairs, and \( W \) is a winding number of the component around the vortex. The s-wave Cooper pair \( F_{s,\sigma}(E, r) \) with \( L_{z} = 0 \) has finite amplitude at the vortex center without the phase winding around the vortex \((W = 0)\).

In Fig. 4(b), we present the \( r \) dependence of the odd-frequency s-wave spin-triplet Cooper pairs around the vortex core. In the chiral \( p_{-} \)-wave \( d\parallel z \) state, \((\uparrow, \downarrow)\) and \((\uparrow, \uparrow)\) components are dominant as explained above. On the other hand, in the chiral \( p_{-} \)-wave \( d\parallel x \) state, the dominant components are \( |F_{s,\uparrow\uparrow}(E = 0, r)| = |F_{s,\downarrow\downarrow}(E = 0, r)| \). In the helical \( p_{-} \)-wave state, the dominant component is \( F_{s,\uparrow\downarrow}(E = 0, r) \). These dominant components in the three states have the same \( r \) dependence as shown in Fig. 4(b). The amplitude of \( F_{s,\downarrow\downarrow}(E = 0, r) \) in the helical \( p_{-} \)-wave state is very small at the vortex core compared to \( |F_{s,\uparrow\uparrow}(E = 0, r)| \), since induced Cooper pair components in \( \Delta_{s,\downarrow} \) satisfy the different condition \( L_{z} + W = 2 \), and the amplitude of the induced s-wave component with \( L_{z} = 0 \) vanishes at the vortex center due to the phase winding \( W = 2 \). Therefore, in the vortex center, \( |(T_{1}^{\uparrow\uparrow})^{-1}| \) in the helical \( p_{-} \)-wave state indicates the half value \((\geq 12.3)\) to \( |(T_{1}^{\uparrow\uparrow})^{-1}| \) of the odd-frequency s-wave Cooper pair, as \( |F_{s,\uparrow\uparrow}(E = 0, r)| = |F_{s,\downarrow\downarrow}(E = 0, r)| \).

The finite odd-frequency s-wave spin-triplet Cooper pairs around a vortex core induce the coherence terms. The equal spin components \((\uparrow, \uparrow)\) and \((\downarrow, \downarrow)\) contribute only to \( T_{2\uparrow\downarrow}^{s} \). Therefore, \( (T_{1}^{\uparrow\downarrow})^{-1} \) in the two cases of the \( d\parallel z \) state and the helical state. The spin components \((\uparrow, \downarrow)\) and \((\downarrow, \uparrow)\) contribute to both \( T_{1}^{\uparrow\uparrow} \) and \( T_{2\uparrow\downarrow}^{s} \). In the low-temperature limit, from Eq. (B4) in Appendix B, the coherence terms of \( (T_{1}^{\uparrow\downarrow})^{-1} \) and \( (T_{2\uparrow\downarrow}^{s})^{-1} \) are described by the zero-energy amplitude \( |F_{s,\sigma}(E = 0, r)\rangle \) of the odd-frequency s-wave spin-triplet Cooper pair, as \( |(T_{1}^{\uparrow\downarrow})^{-1}(E = 0, r)| \) and \( |(T_{2\uparrow\downarrow}^{s})^{-1}(E = 0, r)| \).

### TABLE I

Relation of the coherence terms with the DOS terms, and existence of the zero-energy odd-frequency s-wave spin-triplet Cooper pairs at the vortex center in the chiral \( p_{-} \)-wave \( d\parallel z \) and \( d\parallel x \) states, and the helical \( p_{-} \)-wave state. The value of \(|(T_{1}^{\uparrow\downarrow})^{-1}| \) in the chiral \( p_{-} \)-wave \( d\parallel z \) state is defined as \( C \).

<table>
<thead>
<tr>
<th>Main component of</th>
<th>Coherence term</th>
<th>Odd-frequency s-wave spin-triplet Cooper pair</th>
</tr>
</thead>
<tbody>
<tr>
<td>order parameter</td>
<td>((T_{1}^{\uparrow\downarrow})^{-1})</td>
<td>((T_{2\uparrow\downarrow}^{s})^{-1})</td>
</tr>
<tr>
<td>Chiral ( d\parallel z )</td>
<td>(\simeq - (T_{1}^{\uparrow\downarrow})^{-1} )</td>
<td>(+ C)</td>
</tr>
<tr>
<td>Chiral ( d\parallel x )</td>
<td>(0)</td>
<td>(- C/2)</td>
</tr>
<tr>
<td>Helical ( (d\parallel xy) )</td>
<td>(0)</td>
<td>(- C)</td>
</tr>
</tbody>
</table>
\( |F_{z,1}(E = 0, r)|^2 + |F_{z,1}(E = 0, r)|^2. \) Therefore, \((T_{2z}^f/T)^{-1}\) is positive for \(d\|z\) with finite \((\uparrow, \downarrow)\) and \((\downarrow, \uparrow)\). And \((T_{2z}^f/T)^{-1}\) is negative in the other two states, where finite components are \((\uparrow, \uparrow)\) and \((\downarrow, \downarrow)\). In the cases of the \(d\|z\) state, \(F_{z,1}(E = 0, r) = F_{z,1}(E = 0, r) = 0. \) In the two cases of the \(d\|x\) state and the helical state, \(F_{z,1}(E = 0, r) = F_{z,1}(E = 0, r) = 0. \)

Since our calculations are performed at the finite temperature \(T/T_\text{c} = 0.5\), the results in Fig. 4 deviate from the relations in the low-\(T\) limit in Eq. (B4) by the contribution of the relaxation process between up- and down-spin electrons. And detailed study of \(NMR\) relaxation rates in the contribution of the relaxation process is necessary in order to detect the spin dependence of the odd-frequency \(s\)-wave spin-triplet Cooper pairs and the anomalous suppression isothermal in the local \(T_{1}^{-1}\) \((T_{2}^{-1})\) occurs. In the helical \(p\)-wave state, since the odd-frequency \(s\)-wave spin-triplet Cooper pairs \(F_{s,1}(\uparrow \uparrow)\) and \(F_{s,1}(\downarrow \downarrow)\) are around the vortex core, the anomalous suppression in the local \(T_{1}^{-1}\) \((T_{2}^{-1})\) is small. We hope that these theoretical results of the local \(NMR\) relaxation rates will be examined by experiment in spin-triplet superconductors. This observation can be also a method to detect the spin dependence of the odd-frequency \(s\)-wave spin-triplet Cooper pairs.

**Acknowledgment**

This work was supported by Japan Society for the Promotion of Science KAKENHI Grant No. JP16J05824.

**Appendix A: NMR Relaxation Rates \(T_{1}^{-1}\) and \(T_{2}^{-1}\)**

We explain derivations of the \(NMR\) relaxation rates \(T_{1}^{-1}\) and \(T_{2}^{-1}\) in the quasiclassical Eilenberger theory. In the \(4 \times 4\) matrix form for the Green’s functions

\[
\tilde{G}(x,x') = \begin{bmatrix} \tilde{G}_1(x,x') & \tilde{F}(x,x') \\ \tilde{F}(x,x') & \tilde{G}_1(x,x') \end{bmatrix}
\]

in particle-hole and spin spaces, the spin components of \(\tilde{G}, \tilde{F},\) and \(\tilde{\sigma}\) are, respectively, defined as

\[
G_{\sigma\sigma'}(x,x') = -\langle T_{r} [\psi_{\sigma}(x)^{\dagger} \psi_{\sigma'}(x')] \rangle,
\]

\[
\tilde{G}_{\sigma\sigma'}(x,x') = -\langle T_{r} [\psi_{\sigma}(x)^{\dagger} \psi_{\sigma'}(x')] \rangle,
\]

\[
F_{\sigma\sigma'}(x,x') = -\langle T_{r} [\psi_{\sigma}(x) \psi_{\sigma'}(x')] \rangle,
\]

\[
\tilde{F}_{\sigma\sigma'}(x,x') = -\langle T_{r} [\psi_{\sigma}(x) \psi_{\sigma'}(x')] \rangle,
\]

where \(x = (r, \tau)\) with the coordinate \(r\) and imaginary time \(\tau\). The brackets \(\langle \cdots \rangle\) denote the thermal average, and \(T_{r}\) is the time-ordering operator. In the Eilenberger theory, the quasiclassical Green’s function is defined as \(\tilde{g} = \tilde{\tau}_{3} \int d\xi \hat{G}\), where \(\tilde{\tau}_{3} = \epsilon(k) - \mu \) is the energy variable in the \(k\) space. \(\epsilon(k)\) is the dispersion relation of electrons, and \(\mu\) is the chemical potential. \(\tilde{\tau}_{3}\) is the Pauli matrix defined as

\[
\tilde{\tau}_{3} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},
\]

where \(\hat{0}\) is the unit matrix. In Appendix A, we write the matrix components of \(\tilde{g}\) as

\[
\tilde{g} = \begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix},
\]

instead of the expression in Eq. (3).
Using the spectral representation

\[ T^{-1}_{1}(r,r') = (T_1[S_-(x')S_-(x)]) \]

\[ = \tilde{G}_1(x,x')G_{11}(x,x') - \tilde{F}_1(x,x')F_{11}(x,x'), \] (A5)

with Bose-Matsubara frequency \( \Omega_m \) and \( \beta \equiv (k_B T)^{-1} \).

For the quasilocal approximation as done in Refs. [8,9,34,35], in the limit \( r \rightarrow r' \) the Green’s functions are rewritten using the transformation

\[ A(r,r) = \lim_{r \rightarrow 0} \int \frac{d^3k}{(2\pi)^3} A(r,k)e^{ikr} \]

\[ \approx N_0 \sum \int d\xi_k A(r,kF,\xi_k), \] (A10)

with \( \xi_k \equiv r - r' \). In the following, for simplicity, we rewrite the Fermi velocity \( k_0 \rightarrow k \). Therefore, the quasilocal form of the local \( \chi_{--} \) is written as

\[ \chi_{--}(r,r';i\Omega_m) = \sum_{\sigma\sigma'} \{ \langle g_{zz}^{\sigma\sigma}(io\xi) \rangle \langle g_{11}^{\sigma\sigma}(io\xi - i\Omega_m, r,k) \rangle_k \]

\[ - \langle g_{zz}^{\sigma\sigma}(io\xi, r,k) \rangle \langle g_{11}^{\sigma\sigma}(io\xi - i\Omega_m, r,k) \rangle_k \}. \] (A12)

Using the spectral representation

\[ g_{\sigma\sigma'}^\sigma(io\xi, r,k) = \int d\omega \frac{a_{\sigma\sigma'}^\sigma(\omega, r,k)}{io\Omega - \omega} \] (A13)

and an analytic continuation \( i\Omega_m \rightarrow \Omega + i\delta \), we obtain the local spin-lattice relaxation rate \( T^{-1}_{1}(r,r') \) as

\[ T^{-1}_{1}(r,r') = T \lim_{\Omega \rightarrow 0} \frac{\text{Im} \chi_{--}(r,r',i\Omega_m \rightarrow \Omega + i\delta)}{\Omega} \]

\[ = \pi N_0^2 \sum_\sigma \int dE \frac{1}{\cosh^2(E/2T)} \times \{ \langle a_{11}^{\sigma\sigma}(E) \rangle_k \langle a_{11}^{\sigma\sigma}(-E) \rangle_k \]

\[ - \langle a_{11}^{\sigma\sigma}(E) \rangle_k \langle a_{11}^{\sigma\sigma}(-E) \rangle_k \}. \] (A14)
where \( \delta_{\sigma\sigma'} = 1 \) when \( \sigma = \sigma' \) and \( \delta_{\sigma\sigma'} = 0 \) when \( \sigma \neq \sigma' \).

Therefore, in the low-energy limit, the DOS term and the coherence term of \((T_1 T)^{-1}\) and \((T_{zz} T)^{-1}\) in the vortex core region are, respectively, reduced to

\[
\begin{align*}
\frac{T_{1zz}(T)}{T_{zz}(T)} - 1 &= \frac{\langle \text{Re}[\hat{P}_{\uparrow\downarrow}^R(E = 0)] \rangle_k |\text{Re}[\hat{g}_{\uparrow\downarrow}^R(E = 0)]_k|^2}{N_s(E = 0)^2 + N_s(E = 0)^2}, \\
\frac{T_{1zz}(T)}{T_{zz}(T)} - 1 &= \frac{\sum_{\sigma'\sigma} \delta_{\sigma\sigma'} |\text{Re}[\hat{g}_{\sigma\sigma'}^R(E = 0)]_k|^2}{N_s(E = 0)^2 + N_s(E = 0)^2}, \\
2 \frac{T_{1zz}(T)}{T_{zz}(T)} - 1 &= \sum_{\sigma'\sigma} \delta_{\sigma\sigma'} |\langle f_{\sigma',\sigma}^R(E = 0) \rangle_k|^2.
\end{align*}
\]

where \( \hat{g}^R(E, r, k) = \hat{g}^R(-E, r, k) \), and Eqs. (17) and (18) are used.

Further, in the low-energy limit at low \( T \) and the limit of a single vortex at low \( H \), in the chiral \( p \)-wave state we have the relations \(|\langle g_{\sigma\sigma'}^R(E) \rangle_k| = |\langle f_{\sigma\sigma'}^R(E) \rangle_k| = |\langle f_{\sigma\sigma'}^R(E) \rangle_k|\) with \( \sigma \neq \sigma' \) by the Kramer-Pesch approximation for low-energy states [36,37]. Then, we obtain the relation for the coherence terms as

\[
\begin{align*}
\frac{T_{1zz}(T)}{T_{zz}(T)} - 1 &= \frac{T_{1zz}(T)}{T_{zz}(T)} - 1, \\
\frac{T_{1zz}(T)}{T_{zz}(T)} - 1 &= \frac{T_{1zz}(T)}{T_{zz}(T)} - 1, \\
2 \frac{T_{1zz}(T)}{T_{zz}(T)} - 1 &= \sum_{\sigma'\sigma} \delta_{\sigma\sigma'} |\langle f_{\sigma',\sigma}^R(E = 0) \rangle_k|^2.
\end{align*}
\]

at the vortex core region, since \(|\langle f_{\sigma}^R(E = 0) \rangle_k| = |\langle f_{\sigma}^R(E = 0) \rangle_k|\) shows a small amplitude in the vortex core region compared to the large amplitude \(|\langle f_{\sigma}^R(E = 0) \rangle_k|\).

From the definitions of the local DOS \(N_s(E, r) = \langle |\text{Re}[\hat{g}_{\sigma\sigma'}^R(E, r, k)]|^2 \rangle_k\) and the spin-dependent \(s\)-wave Cooper pair \(f_{\sigma,\sigma'}^R(E, r, k)\), the relations between \(|\langle g_{\sigma\sigma'}^R(E) \rangle_k| \) and \(|\langle f_{\sigma\sigma'}^R(E) \rangle_k| \) obtained from Kramer-Pesch approximation are rewritten by \(N_s(E, r)\) and \(|\mathcal{F}_{\sigma,\sigma'}^R(E, r, k)|\). As shown in Fig. 5(a), the relation \(|\langle g_{\sigma\sigma'}^R(E) \rangle_k| = |\langle f_{\sigma\sigma'}^R(E) \rangle_k|\), i.e., \( |\mathcal{F}_{\sigma,\sigma'}^R(E, r = 0)| = |N_s(E, r = 0)| \), for the helical \( p \)-wave state is confirmed at the low \( |E| \) region, not only \( E = 0 \), by our numerical calculations at a low field \( H/H_c = 0.023 \) and a finite temperature \( T/T_{10} = 0.5 \). The chiral \( p \)-wave cases are also confirmed. Therefore, the relations between the DOS
term and the coherence term for the chiral \( p \)-wave states in Eqs. (B5)–(B7) are consistent with our calculation results in Fig. 4 and Table I. Due to the contributions from finite energies \( N_r(E, r) \) and \( |J_{y,\sigma}(E, r)| \), the results in Fig. 4 at finite temperature deviate from Eq. (B8) from the relations Eq. (B4) in the low-\( T \) limit.

In addition, Fig. 5(b) shows that the condition \(|N_\uparrow(E = 0, r = 0)| = |J_{y,\uparrow}(E = 0, r = 0)| \) is sustained at finite \( H \), while small deviations from the relation appear with increasing \( H \). \( B \) dependences of the local \( T_0^{-1} \) in the chiral \( p \)-wave \( d\uparrow \uparrow \) states at various applied fields are investigated in a previous study [10].