STABLE SPLITTINGS OF THE COMPLEX CONNECTIVE K-THEORY OF $BSO(2n+1)$

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Abstract. We give the stable splittings of the complex connective K-theory of the classifying space $BSO(2n+1)$, $n \geq 1$.

1. Introduction

In [6], E. Ossa has showed that
$$bu \wedge RP^\infty \wedge RP^\infty \simeq \bigvee_{0 \leq i,j} \Sigma^{2i+2j-2} HZ/2 \vee [\Sigma^2 bu \wedge RP^\infty].$$

In [2], B. R. Burner and J. P. C. Greenless give some studies on $bu \wedge BG$ for some finite groups $G$. Also, W. Stephen Wilson and D. Y. Yan [7] split $bu \wedge BO(n)$ into the suspended copies of $HZ/2$, $bu$, and $bu \wedge RP^\infty$. Via these splittings, we are going to split $bu \wedge BSO(2n+1)$.

First let’s recall the notations we need. Let $bu$ be the complex connective K-theory, $HZ/2$ be the $\mathbb{Z}/2$ Eilenberg-Mac Lane spectrum, $RP^\infty = BO(1)$ be the infinite real projective space, $BO(n)$ be the classifying space of the $n$-th orthogonal group, $BSO(n)$ be the classifying space of the $n$-th special orthogonal group. To simplify the notations, let $H^*(X) = H^*(X, \mathbb{Z}/2)$, $\tilde{H}^*(X) = \tilde{H}^*(X, \mathbb{Z}/2)$, $H_*(X) = H_*(X, \mathbb{Z}/2)$, and $\tilde{H}_*(X) = \tilde{H}_*(X, \mathbb{Z}/2)$. We also write $\otimes$ instead of $\otimes_{\mathbb{Z}/2}$ and all the spaces, the spectra, and the homotopy equivalences are localized at prime 2.

Recall that $H^*(BO(n)) = \mathbb{Z}/2[w_1, w_2, \ldots, w_n]$, where $w_i$ is the $i$-th Stiefel-Whitney class. In particular, $H^*(RP^\infty) = H^*(BO(1)) = \mathbb{Z}/2[w_1]$. Then let $b_i \in H_i(RP^\infty)$ be the dual class of $w_i^2 \in H^*(RP^\infty)$, $i \geq 0$, hence $H_i(BO(n))$ is the $\mathbb{Z}/2$-module generated by the monomials $b_{i_1}b_{i_2} \cdots b_{i_n}$, deg$(b_{i_1}b_{i_2} \cdots b_{i_n}) = i_1 + i_2 + \cdots + i_n$, $b_{i_1}b_{i_2} \cdots b_{i_n} = f_s(b_{i_1} \otimes b_{i_2} \otimes \cdots \otimes b_{i_n})$, $0 \leq i_1 \leq i_2 \leq \cdots \leq i_n$, where $f : \times_{i=1}^n RP^\infty \longrightarrow BO(n)$ is the classifying map. Moreover, let $h_n : BSO(n) \longrightarrow BO(n)$ be the 2-folds map, then we have $H^*(BSO(n)) = \mathbb{Z}/2[\hat{w}_2, \hat{w}_3, \ldots, \hat{w}_n]$, where $\hat{w}_i = h_n^*(w_i)$, $2 \leq i \leq n$.

Also recall that $bu_* = \mathbb{Z}(2)[v_1]$, where deg$(v_1) = 2$, and $H^*(bu) \cong A/\langle A(Q_0, Q_1) \cong A \otimes_E \mathbb{Z}/2$, where $A$ is the mod 2 Steenrod algebra.
A (Q₀, Q₁) is the ideal of A generated by Q₀ = Sq¹ and Q₁ = Sq³ + Sq²Sq¹, and E = ℤ/2⟨Q₀, Q₁⟩, the exterior algebra on Q₀ and Q₁, is a subalgebra of A. Then by the Cartan formula $Sq^i(xy) = \sum_j Sq^j(x)Sq^{i-j}(y)$, we have $Q_k(xy) = Q_k(x)y + xQ_k(y)$, $k = 0$ or 1. Moreover, since for any space X, $H^*(X)$ is an E-module, we say an element $x$ in $H^*(X)$ is decomposable if $x = Q_0(y) + Q_1(z)$ for some $y, z \in H^*(X)$, and we say an element is indecomposable if it is not decomposable.

For $n \geq 1$, let $T_{2n+1} = \{ t_j \mid j \in \Lambda_{2n+1} \}$ be a largest E-linearly independent subset of $H^*(BSO(2n + 1))$ such that each $t_j$ is a monomial in $H^*(BSO(2n + 1))$.

Now we state the main result of this paper.

**Theorem A.** For each $n \geq 1$, $H^*(BSO(2n + 1))$ is isomorphic to $D_{2n+1} \oplus M_{2n+1}$ as an E-module, where $D_{2n+1}$ is an E-module with the $\mathbb{Z}/2$-generators $\tilde{w}_2^{2m_1} \tilde{w}_4^{2m_2} \cdots \tilde{w}_{2n}^{2m_n}$, $\sum_{i=1}^n m_i > 0$, $m_i \geq 0$, each $\tilde{w}_2^{2m_1} \tilde{w}_4^{2m_2} \cdots \tilde{w}_{2n}^{2m_n}$ has the trivial E-action, and $M_{2n+1}$ is a free E-module with the E-basis $T_{2n+1}$ described as above.

**Theorem B.** For each $n \geq 1$, there is a stable splitting

$$bu \wedge BSO(2n + 1) \cong [\bigvee_{\alpha} \Sigma^\alpha H\mathbb{Z}/2] \vee [\bigvee_{\beta} \Sigma^\beta bu],$$

where $\alpha = \deg t_j$, $t_j \in T_{2n+1}$, the generators of $M_{2n+1}$, and the $\beta$, and their degrees, correspond to the generators of $D_{2n+1}$.

To prove the stable splitting of $bu \wedge BSO(2n + 1)$ (Theorem B), we need to apply the stable splitting of $bu \wedge BO(n)$ [7] to decompose $\tilde{H}^*(BSO(2n + 1))$ as a direct sum of an E-module $D_{2n+1}$ and a free E-module $M_{2n+1}$ (Theorem A). Then we construct the map

$$g = g_0 \vee g_1 : bu \wedge BSO(2n + 1) \rightarrow [\bigvee_{\alpha} \Sigma^\alpha H\mathbb{Z}/2] \vee [\bigvee_{\beta} \Sigma^\beta bu]$$

and prove that $g$ induces an isomorphism on the mod 2 cohomology, hence $g$ is a homotopy equivalence and Theorem A follows.

In fact, there is an algebraic splitting of $\tilde{H}^*(BSO(2n))$ as Theorem A, that is, $\tilde{H}^*(BSO(2n))$ is isomorphic to $D_{2n} \oplus M_{2n} \oplus B_{2n}$ as an E-module, $n \geq 1$. Unfortunately, I cannot find a suitable space or spectrum corresponding to the $B_{2n}$ part.

The rest of paper is organized as follows: In Section 2, we will give some lemmas which link the Adams $E_2^{1,*}$ term of $\tilde{bu}_*(X)$ to the decomposition of $\tilde{H}^*(X)$. In Section 3, we will compute the Adams $E_2^{1,*}$ term of $\tilde{bu}_*(BO(n))$. 
In Section 4, we will study the map $Bg_{2n} : BO(2n) \rightarrow BSO(2n + 1)$. In Section 5, we will prove Theorem A. In Section 6, we will prove Theorem B.

2. The $E$-module structure of $\tilde{H}^*(BO(n))$ and the Adams spectral sequences for $\tilde{bu}_*(BSO(2n + 1))$

In this section, we will recall the Adams spectral sequence and give some lemmas which link some useful information of the decomposition of $\tilde{H}^*(X)$ to the Adams $E_2^{0,*}$ term of $\tilde{bu}_*(X)$ for any spaces $X$.

Let $A_* = \mathbb{Z}/2[\xi_1, \xi_2, \xi_3, \cdots]$, where $\xi_k$ are the Milnor’s generators with $\text{deg}(\xi_k) = 2^k - 1$, be the mod 2 dual Steenrod algebra with the coproduct $\Delta(\xi_k) = \sum_{i=0}^{k} \xi_{k-i} \otimes \xi_i$. Then recall that for any space or spectrum $Y$, the Adams spectral sequences [1]

$$\text{Ext}_{A_*}^* (H^*(X), \mathbb{Z}/2) \cong \text{Ext}_{A_*}^* (\mathbb{Z}/2, H_*(X)) \implies \pi_*(X(2))$$

can be used to compute $\tilde{bu}_*(Y)$ when $X = bu \wedge Y$. By a well-known change-of-rings isomorphism [3], we can replace

$$\text{Ext}_{E_*}^* (H^*(bu \wedge Y), \mathbb{Z}/2) \text{ with } \text{Ext}_{E_*}^* (H^*(Y), \mathbb{Z}/2),$$

$$\text{Ext}_{E_*}^* (\mathbb{Z}/2, H_*(bu \wedge Y)) \text{ with } \text{Ext}_{E_*}^* (\mathbb{Z}/2, \tilde{H}_*(Y)),$$

where $E_* = \mathbb{Z}/2(\xi_1, \xi_2)$ is the exterior algebra on $\xi_1$ and $\xi_2$. For simplicity of notations, let $E_2^*(Y)$ be $\text{Ext}_{E_*}^* (H^*(Y), \mathbb{Z}/2)$ and $E_2^*(Y)$ be $\text{Ext}_{E_*}^* (\mathbb{Z}/2, \tilde{H}_*(Y))$. Also recall that $E_2^*(Y)$ is isomorphic to the homology of the bar complex $\tilde{H}^*(Y)$.

$$\tilde{H}^*(Y) \xleftarrow{\bar{E}_*} \tilde{E}_* \otimes \tilde{H}^*(Y) \xrightarrow{\bar{E}_*} \tilde{E}_* \otimes \tilde{E}_* \otimes \tilde{H}^*(Y) \leftarrow \cdots$$

and $\bar{E}_2^*(Y)$ is isomorphic to the homology of the cobar complex

$$\tilde{H}_*(Y) \xrightarrow{\Delta_2} \bar{E}_* \otimes \tilde{H}_*(Y) \xrightarrow{\Delta_2} \bar{E}_* \otimes \bar{E}_* \otimes \tilde{H}_*(Y) \rightarrow \cdots,$$

where $\bar{E} = E \setminus \{1\}$ and $\bar{E}_* = E_* \setminus \{1\}$.

Moreover, we have the Adams spectral sequences

$$E_2^{0,*} \cong \text{Ext}_{E_*}^* (\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2[\overline{\nu_0}, \overline{\nu_1}] \cong \tilde{E}_2^{0,*} \cong \text{Ext}_{E_*}^* (\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2[\xi_1, \xi_2],$$

where $\overline{\nu_0} \in E_2^{1,1}$ and $\overline{\nu_1} \in E_2^{1,3}$ are detected by $Q_0$ and $Q_1$ respectively, $\overline{\nu_0}^2$ is detected by $Q_0 \otimes Q_0$, $\overline{\nu_1}^2$ is detected by $Q_1 \otimes Q_1$, $\overline{\nu_0} \overline{\nu_1}$ is detected by $Q_0 \otimes Q_1 + Q_1 \otimes Q_0$, $\xi_1 \in \overline{E}_2^{1,1}$, and $\xi_2 \in \overline{E}_2^{1,3}$ (here we use the ambiguous notations, that is, we use the same symbol $\xi_i$ in the chain level and the homology level).
Let $N^*$ be any $E$-module and $E_2^{1,*}(N^*)$ be the first line of the bar complex

$$N^* \mathcal{d}_1 \mathcal{E} \otimes N^* \mathcal{d}_2 \mathcal{E} \otimes \mathcal{E} \otimes N^* \rightarrow \cdots.$$  

Similarly, let $N_*$ be any $E_*$-comodule and $E_2^{1,*}(N_*)$ be the first line of the cobar complex

$$N_* \mathcal{\Delta}_1 \mathcal{E}_* \otimes N_* \mathcal{\Delta}_2 \mathcal{E}_* \otimes \mathcal{E}_* \otimes N_* \rightarrow \cdots.$$  

Then we have the following lemmas.

**Lemma 2.1.** As $E$-modules, if $N^* \cong K^* \oplus L^*$, then $E_2^{1,*}(N^*) \cong E_2^{1,*}(K^*) \oplus E_2^{1,*}(L^*)$. As $E_*$-comodules, if $N_* \cong K_* \oplus L_*$, then $E_2^{1,*}(N_*) \cong E_2^{1,*}(K_*) \oplus E_2^{1,*}(L_*)$.

**Proof.** This follows immediately from the definition of the bar and cobar complexes.

**Lemma 2.2.** If $E_2^{1,*}(N^*) = 0$ and $Q_0(x) + Q_1(y) + Q_0Q_1(z) = 0$ for some $x, y, z \in N^*$, then $x = 0$ or $x$ is decomposable, and $y = 0$ or $y$ is decomposable.

**Proof.** Since $E_2^{1,*}(N^*) = 0$ and $0 = Q_0(x) + Q_1(y) + Q_0Q_1(z) = \mathcal{\bar{d}}_1(Q_0 \otimes x + Q_1 \otimes y + Q_0Q_1 \otimes z)$, there exists $a_1, \ldots, a_9 \in N^*$ such that

$$Q_0 \otimes x + Q_1 \otimes y + Q_0Q_1 \otimes z = \mathcal{d}_2(Q_0 \otimes Q_0 \otimes a_1 + Q_0 \otimes Q_1 \otimes a_2 + Q_0 \otimes Q_0Q_1 \otimes a_3 + Q_1 \otimes Q_0 \otimes a_4 + Q_1 \otimes Q_1 \otimes a_5 + Q_1 \otimes Q_0Q_1 \otimes a_6 + Q_0Q_1 \otimes Q_0 \otimes a_7 + Q_0Q_1 \otimes Q_1 \otimes a_8 + Q_0Q_1 \otimes Q_0Q_1 \otimes a_9)$$

$$= Q_0 \otimes Q_0(a_1) + Q_0Q_1 \otimes a_2 + Q_0 \otimes Q_1(a_2) + Q_0 \otimes Q_0Q_1(a_3) + Q_1Q_0 \otimes a_4 + Q_1 \otimes Q_0Q_1(a_3) + Q_1 \otimes Q_1(a_5) + Q_1 \otimes Q_0Q_1(a_6) + Q_0Q_1 \otimes Q_0(a_7) + Q_0Q_1 \otimes Q_1(a_8) + Q_0Q_1 \otimes Q_0Q_1(a_9).$$

Then we get

$$x = Q_0(a_1) + Q_1(a_2) + Q_0Q_1(a_3),$$

and

$$y = Q_0(a_4) + Q_1(a_5) + Q_0Q_1(a_6).$$

This completes the proof.

**Lemma 2.3.** If $E_2^{1,*}(N^*) = 0$ and $Q_0Q_1(z) = 0$ for some $z \in N^*$, then $z = 0$ or $z$ is decomposable.

**Proof.** As the proof of Lemma 2.2, where $x = 0$ and $y = 0$, there exists $a_1, \ldots, a_9 \in N^*$ such that

$$0 = Q_0(a_1) + Q_1(a_2) + Q_0Q_1(a_3).$$
Theorem 3.1. Result in [7].

The Adams spectral sequence for $\tilde{bu}_*(BSO(2n + 1))$ is decomposable. As a result, $z$ is also decomposable or $z = 0$. This completes the proof. \hfill $\square$

Lemma 2.4. Let $T = \{ t_j \ | \ j \in \Lambda \}$ be a largest $E$-linearly independent subset of $N^*$. Then if $E_2^{1,*}(N^*) = 0$, $N^*$ is a free $E$-module with the $E$-basis $T$.

Proof. Let $M \subseteq N^*$ be the free $E$-submodule generated by $T$. We are going to show that $M = N^*$.

For any $u \in N^*$, since $T$ is a largest $E$-linearly independent subset of $N^*$, $Q_0Q_1(u)$ can be generated by $T$, hence there exists a finite sum $a$ ( $a$ could be 0 ) of some $t_j \in T$ such that $Q_0Q_1(u) = Q_0Q_1(a)$. Therefore, by Lemma 2.3, $Q_0Q_1(u + a) = 0$ implies $u + a = Q_0(v) + Q_1(w)$ for some $v, w \in N^*$. As above $u$ and $a$, there exists finite sums $b, c$ of some $t_j \in T$ such that $Q_0Q_1(v) = Q_0Q_1(b)$ and $Q_0Q_1(w) = Q_0Q_1(c)$. Thus we have

$$Q_1(u + a) = Q_1Q_0(v) = Q_1Q_0(b)$$
and
$$Q_0(u + a) = Q_0Q_1(w) = Q_0Q_1(c),$$

which means

$$Q_1(u) = Q_1(a) + Q_1Q_0(b) \in M$$

and

$$Q_0(u) = Q_0(a) + Q_0Q_1(c) \in M.$$ 

These also apply to $v$ and $w$, that is, both $Q_0(v)$ and $Q_1(w)$ are in $M$, hence $u = a + Q_0(v) + Q_1(w)$ follows. This completes the proof. \hfill $\square$

3. The $E_2^{1,*}$ term of the Adams spectral sequences for $\tilde{bu}_*(BO(n))$

To study the Adams $E_2^{1,*}$ term of $\tilde{bu}_*(BSO(2n + 1))$, we have to know the Adams $E_2^{1,*}$ term and $\tilde{E}_2^{1,*}$ term of $\tilde{bu}_*(BO(n))$. So first we recall the result in [7].

Theorem 3.1. (Theorem 1.1 of [7]) As an $E$-module, $\tilde{H}^*(BO(n))$ is isomorphic to $D_1^* \oplus D_2^* \oplus M$, where $D_1^*$ is a trivial $E$-module with $E$-generators

$$w_2^{2m_1}w_4^{2m_2} \cdots w_{2k}^{2m_k}$$
such that $\sum_{i=1}^{k} m_i > 0$, $2k \leq n$,
$D_2^*$ is an $E$-module, free over the exterior algebra on $Q_0$, with $E$-generators
\[ w_1^{2j+1} w_2^{2m_1} w_4^{2m_2} \cdots w_{2t}^{2m_t} \text{ such that } \sum_{i=1}^{t} m_i \geq 0, \ j \geq 0, \ 2t \leq n - 1, \]
and
\[ Q_1(w_1^{2j+1} w_2^{2m_1} w_4^{2m_2} \cdots w_{2t}^{2m_t}) = Q_0(w_1^{2j+3} w_2^{2m_1} w_4^{2m_2} \cdots w_{2t}^{2m_t}), \]
and $M$ is a free $E$-module.

Thus we can compute the Adams $E_2^{1,*}$ term and $E_2^{1,*}$ term of $\widetilde{bu}_*(BO(n))$.

**Lemma 3.2.** In the Adams spectral sequence
\[ E_{\infty}^{1,*}(\widetilde{H}^*(BO(n)), \ Z/2) \Longrightarrow \widetilde{bu}_*(BO(n)), \]
as a $\mathbb{Z}/2$-module, $E_2^{1,*}(BO(n))$ is generated by
\[ v_0 \otimes w_2^{2m_1} w_4^{2m_2} \cdots w_{2k}^{2m_k}, \ \sum_{i=1}^{k} m_i > 0, \ 2k \leq n, \]
\[ v_1 \otimes w_2^{2m_1} w_4^{2m_2} \cdots w_{2k}^{2m_k}, \ \sum_{i=1}^{k} m_i > 0, \ 2k \leq n, \]
\[ v_0 \otimes w_1^{2j+3} w_2^{2m_1} w_4^{2m_2} \cdots w_{2t}^{2m_t} + v_1 \otimes w_1^{2j+1} w_2^{2m_1} w_4^{2m_2} \cdots w_{2t}^{2m_t}, \]
\[ \sum_{i=1}^{t} m_i \geq 0, \ j \geq 0, \ 2t \leq n - 1. \]

**Proof.** Since by Theorem 3.1, $\widetilde{H}^*(BO(n))$ is isomorphic to $D_2^* \oplus D_2^* \oplus M$, by Lemma 2.1, we can compute $E_2^{1,*}(D_1^*)$, $E_2^{1,*}(D_2^*)$ and $E_2^{1,*}(M)$ separately. Then since $D_1^*$ is a trivial $E$-module, it is clearly that $E_2^{1,*}(D_1^*)$ has the $\mathbb{Z}/2$-generators
\[ v_0 \otimes w_2^{2m_1} w_4^{2m_2} \cdots w_{2k}^{2m_k}, \ \sum_{i=1}^{k} m_i > 0, \ 2k \leq n, \]
\[ v_1 \otimes w_2^{2m_1} w_4^{2m_2} \cdots w_{2k}^{2m_k}, \ \sum_{i=1}^{k} m_i > 0, \ 2k \leq n. \]

Moreover, $E_2^{1,*}(M) = 0$ since $M$ is free. Therefore, it is only left $E_2^{1,*}(D_2^*)$.

Since we have
\[ Q_0(w_1^{2j+1} w_2^{2m_1} w_4^{2m_2} \cdots w_{2t}^{2m_t}) = w_1^{2j+2} w_2^{2m_1} w_4^{2m_2} \cdots w_{2t}^{2m_t}, \]
and
\[ Q_1(w_1^{2j+1} w_2^{2m_1} w_4^{2m_2} \cdots w_{2t}^{2m_t}) = w_1^{2j+4} w_2^{2m_1} w_4^{2m_2} \cdots w_{2t}^{2m_t}, \]
the $\mathbb{Z}/2$-generators of $\ker \overline{d}^t_2$ for the bar complex of $D_2^*$ are

\[ Q_0 \otimes w_1^{2j+2} w_2^{2m_1} \cdots w_2^{2m_t}, \quad \sum_{i=1}^t m_i \geq 0, \quad j \geq 0, \quad 2t \leq n - 1, \]
\[ Q_1 \otimes w_1^{2j+2} w_2^{2m_1} \cdots w_2^{2m_t}, \quad \sum_{i=1}^t m_i \geq 0, \quad j \geq 0, \quad 2t \leq n - 1, \]
\[ Q_0Q_1 \otimes w_1^{2m_1} \cdots w_2^{2m_t}, \quad \sum_{i=1}^t m_i \geq 0, \quad s \geq 1, \quad 2t \leq n - 1, \]
and $Q_0 \otimes w_1^{2j+3} w_2^{2m_1} \cdots w_2^{2m_t} + Q_1 \otimes w_1^{2j+1} w_2^{2m_1} \cdots w_2^{2m_t},$ 
\[ \sum_{i=1}^t m_i \geq 0, \quad j \geq 0, \quad 2t \leq n - 1. \]

However, we also have

\[ \overline{d}_2(Q_0 \otimes Q_0 \otimes w_1^{2j+1}) = Q_0 \otimes w_1^{2j+2}, \quad j \geq 0, \]
\[ \overline{d}_2(Q_1 \otimes Q_1 \otimes w_1^{2j-1}) = Q_1 \otimes w_1^{2j+2}, \quad j \geq 1, \]
\[ \overline{d}_2(Q_0 \otimes Q_1 \otimes w_1^{2j+2}) = Q_0Q_1 \otimes w_1^{2j+2}, \quad j \geq 0, \]
\[ \overline{d}_2(Q_1 \otimes Q_0 \otimes w_1 + Q_0 \otimes Q_1 \otimes w_1 + Q_0 \otimes Q_0 \otimes w_1^2) = Q_1 \otimes w_1^2 + Q_1Q_0 \otimes w_1^2 + Q_0Q_1 \otimes w_1^2 + Q_0 \otimes w_1^2 + Q_0 \otimes w_1^2 = Q_1 \otimes w_1^2. \]
\[ \overline{d}_2(Q_0 \otimes Q_1 \otimes w_1^{2j+1} + Q_0 \otimes Q_0 \otimes w_1^{2j+3}) = Q_0Q_1 \otimes w_1^{2j+1} + Q_0 \otimes w_1^{2j+4} + Q_0 \otimes w_1^{2j+4} = Q_0Q_1 \otimes w_1^{2j+1}, \quad j \geq 0, \]
and the fact that $Q_0 \otimes w_1^{2j+3} w_2^{2m_1} \cdots w_2^{2m_t} + Q_1 \otimes w_1^{2j+1} w_2^{2m_1} \cdots w_2^{2m_t}$ can not be an image of $\overline{d}_2.$ This completes the proof. \qed

**Lemma 3.3.** In the Adams spectral sequence

\[ \text{Ext}^*_{E^*}(\mathbb{Z}/2, \tilde{H}_*(BO(n))) \Rightarrow \text{bu}_*(BO(n)), \]
as a $\mathbb{Z}/2$-module, $E_{1,1}^* \otimes (BO(n))$ is generated by

\[ \xi_1 \otimes b_{2j_1}^2 b_{2j_2}^2 \cdots b_{2j_k}^2, \quad 1 \leq j_1 \leq j_2 \leq \cdots \leq j_k, \quad 2k \leq n, \]
\[ \xi_2 \otimes b_{2j_1}^2 b_{2j_2}^2 \cdots b_{2j_k}^2, \quad 1 \leq j_1 \leq j_2 \leq \cdots \leq j_k, \quad 2k \leq n, \]
\[ \xi_1 \otimes b_{2i+1}^2 b_{2j_1}^2 b_{2j_2}^2 \cdots b_{2j_t}^2, \quad 0 \leq j_1 \leq j_2 \leq \cdots \leq j_t, \quad 2t \leq n - 1, \quad i \geq 0, \]
\[ \xi_2 \otimes b_{2i+1}^2 b_{2j_1}^2 b_{2j_2}^2 \cdots b_{2j_t}^2, \quad 0 \leq j_1 \leq j_2 \leq \cdots \leq j_t, \quad 2t \leq n - 1, \quad i \geq 0, \]
and subjects to the relations

\[ \xi_1 \otimes b_{2i+3}^2 b_{2j_1}^2 b_{2j_2}^2 \cdots b_{2j_t}^2 = \xi_2 \otimes b_{2i+1}^2 b_{2j_1}^2 b_{2j_2}^2 \cdots b_{2j_t}^2, \]
and $\xi_1 \otimes b_1 b_{2j_1}^2 b_{2j_2}^2 \cdots b_{2j_t}^2 = 0$.

Proof. First recall the coaction of $\tilde{H}_* (BO(n))$ over $A_*$ is

$$\Delta (b_i) = \sum_{j=1}^{i} (\xi^j)_{i-j} \otimes b_j,$$

where $\xi = 1 + \xi_1 + \xi_2 + \xi_3 + \cdots$ [8], and we have the coproduct $\Delta (\xi_k) = \sum_{i=0}^{k} \xi^k_{k-i} \otimes \xi_i$. Thus the comodule structure of $\tilde{H}_* (BO(n))$ over $E_* = E_* \setminus \{1\}$ is generated by

$$\Delta (b_{2i}) = \xi_1 \otimes b_{2i-1} + \xi_2 \otimes b_{2i-3},$$
$$\Delta (b_{2i-1}) = 0,$$
$$\Delta (b_{2i}^2) = 0,$$

where $i \geq 1$. Moreover, in $E_*$, we have $\Delta (\xi_1) = 0$ and $\Delta (\xi_2) = 0$. So under the cobar complex

$$\tilde{H}_* (BO(n)) \xrightarrow{\Delta_1} E_* \otimes \tilde{H}_* (BO(n)) \xrightarrow{\Delta^2} E_* \otimes E_* \otimes \tilde{H}_* (BO(n)) \rightarrow \cdots,$$

we have

$$\Delta_2 (\xi_1 \otimes b_{2j_1}^2 b_{2j_2}^2 \cdots b_{2j_k}^2) = 0, \quad 1 \leq j_1 \leq j_2 \leq \cdots \leq j_k, \quad 2k \leq n,$$
$$\Delta_2 (\xi_2 \otimes b_{2j_1}^2 b_{2j_2}^2 \cdots b_{2j_k}^2) = 0, \quad 1 \leq j_1 \leq j_2 \leq \cdots \leq j_k, \quad 2k \leq n,$$

and under $\Delta_1$, the only methods to produce the above elements are

$$\Delta_1 (b_{2i+1} b_{2j_1}^2 \cdots b_{2j_t}^2) = \xi_1 \otimes b_{2i+3} b_{2j_1}^2 \cdots b_{2j_t}^2 + \xi_2 \otimes b_{2i+1} b_{2j_1}^2 \cdots b_{2j_t}^2, \quad i \geq 0,$$
$$\Delta_1 (b_{2j_1}^2 \cdots b_{2j_t}^2) = \xi_1 \otimes b_{2j_1}^2 \cdots b_{2j_t}^2.$$

Therefore, $\tilde{E}^{1,*}_2 (BO(n))$ at least contains the generators described in the statement of this lemma. Then since as $\mathbb{Z}/2$-modules,

$$\tilde{E}^{1,*}_2 (BO(n)) \cong E^{1,*}_2 (BO(n)),$$

counting the generators of $\tilde{E}^{1,k}_2 (BO(n))$ we just found and the generators of $E^{1,k}_2 (BO(n))$ in Lemma 3.2 for each $k \geq 1$, we can see that all the generators of $\tilde{E}^{1,*}_2 (BO(n))$ are found. This completes the proof. $\square$
4. The map $B_{g_{2n}} : BO(2n) \to BSO(2n + 1)$

In this section, first we construct the map

$$B_{g_{2n}} : BO(2n) \to BSO(2n + 1),$$

which is the classifying map of $g_{2n} : O(2n) \to SO(2n + 1)$ defined by $g_{2n}(\alpha) = \det \alpha \oplus \alpha$. Then we will show that $(B_{g_{2n}})_*$ is surjective and compute its behavior.

**Lemma 4.1.** The map $(B_{g_{2n}})_* : E_2^{1,*}(BO(2n)) \to E_2^{1,*}(BSO(2n + 1))$ is surjective.

**Proof.** Since the fibre of $B_{g_{2n}} : BO(2n) \to BSO(2n + 1)$ is

$$SO(2n + 1)/O(2n) = RP^{2n}$$

and the Euler characteristic $\chi(RP^{2n}) \equiv 1 \mod 2$, there exists a Becker-Gottlieb stable transfer

$$t : BSO(2n + 1) \to BO(2n)$$

such that $B_{g_{2n}} \circ t \simeq id$ (localized at prime 2). Hence the composite map

$$E_2^{1,*}(BSO(2n + 1)) \xrightarrow{t_*} E_2^{1,*}(BO(2n)) \xrightarrow{(B_{g_{2n}})_*} E_2^{1,*}(BSO(2n + 1))$$

is an isomorphism. This completes the proof. □

Now we recall some results in [10]. We have the following commutative diagram

$$\begin{array}{ccc}
BO(2n) & \xrightarrow{B_{g_{2n}}} & BSO(2n + 1) \\
\downarrow f_{2n} \searrow & & \downarrow h_{2n + 1} \\
& & BO(2n + 1)
\end{array}$$

where $h_{2n+1}$ is the usual 2-fold map and $f_{2n}$ is constructed similarly as $B_{g_{2n}}$. Then we have the following lemma.

**Lemma 4.2.** (Lemma 2.2 in [10]) In

$$(f_{2n})_* : H_*(BO(2n)) \to H_*(BO(2n + 1)),$$

we have

$$(f_{2n})_*(b_{m_1} b_{m_2} \cdots b_{m_{2n}}) = \sum \frac{(\sum_{k=1}^{2n} i_k)!}{\prod_{k=1}^{2n} i_k!} b_{\sum_{k=1}^{2n} i_k} b_{m_{i_1} - i_1} b_{m_{i_2} - i_2} \cdots b_{m_{i_{2n}} - i_{2n}},$$

where the sum is taken over the sequence $(i_1, i_2, i_3, \cdots, i_{2n})$, $0 \leq i_k \leq m_k$, $m_k \geq 0$, $1 \leq k \leq 2n$. 

Thus we have the following important proposition of \((B_{2n})_*\).

**Proposition 4.3.** In 
\[(B_{2n})_* : H_*(BO(2n)) \rightarrow H_*(BSO(2n+1)),\]
we have 
\[(B_{2n})_* (b_{2i+1}b_{2m_1}^2 b_{2m_2}^2 \cdots b_{2m_n}^2) = 0,\]
where \(i \geq 0, m_k \geq 0, 1 \leq k \leq n - 1.\)

Before we prove Proposition 4.3, we need two lemmas.

**Lemma 4.4.** \(\frac{(2n)!}{n!n!n!}\) is even for \(n \geq 1.\)

**Proof.** It follows immediately from the following equalities
\[
\frac{(2n)!}{n!n!n!} = 2^{n-n} n!(n-1)! = 2^n \binom{2n-1}{n}.
\]
This completes the proof. 

**Lemma 4.5.** \(\frac{(i_1 + \sum_{k=1}^n 2j_k)!}{i_1! \prod_{k=1}^n (j_k!)^2}\) is even for any \(i_1 \geq 0\) and at least one \(j_k \neq 0.\)

**Proof.** Assume \(j_1 \neq 0.\) Then it follows from the equality
\[
\frac{(i_1 + \sum_{k=1}^n 2j_k)!}{i_1! \prod_{k=1}^n (j_k!)^2} = \frac{(i_1 + \sum_{k=1}^n 2j_k)!}{i_1!(2j_1)! \prod_{k=2}^n (j_k!)^2} \cdot \frac{(2j_1)!}{(j_1!)^2}
\]
since \(\frac{(i_1 + \sum_{k=1}^n 2j_k)!}{i_1!(2j_1)! \prod_{k=2}^n (j_k!)^2}\) is an integer and \(\frac{(2j_1)!}{(j_1!)^2}\) is even. This completes the proof.

**Proof of Proposition 4.3.** By Lemma 4.2, we have the following formula
\[
(f_{2n})_* (b_{2i+1}b_{2m_1}^2 b_{2m_2}^2 \cdots b_{2m_n}^2) \\
= \sum \left( \frac{(i_1 + \sum_{k=1}^{n-1} (j_{k,1} + j_{k,2}))!}{i_1! \prod_{k=1}^{n-1} (j_{k,1}j_{k,2})!} b_{2i+1} \prod_{k=1}^{n-1} (b_{m_k-j_{k,1}} b_{m_k-j_{k,2}}) \right).
\]
Note that for a fixed sequence \((i_1, j_{1,1}, j_{1,2}, \cdots, j_{n-1,1}, j_{n-1,2})\) which contains exactly \(t\) couples \((j_{k,1}, j_{k,2})\) with \(j_{k,1} \neq j_{k,2},\) there exists \(2^t\) corresponding sequences which are got from interchanging \(j_{k,1}\) and \(j_{k,2}\) in some of those \(t\)
couples, hence there are $2^t$ identical terms in the above sum. Then since we are using the $\mathbb{Z}/2$-coefficient, we have

$$(f_{2n})_*(b_{2i+1}b_{m_1}^2b_{m_2}^2 \cdots b_{m_{n-1}}^2)$$

$$= \sum_{i_1=0}^{2i+1} \frac{(i_1 + 2 \sum_{k=1}^{n-1} j_k)}{!} b_{i_1} b_{2i+1-i_1} b_{m_1}^2 b_{m_2}^2 \cdots b_{m_{n-1}}^2$$

where $j_k = j_{k,1} = j_{k,2}$. So by Lemma 4.5,

$$(f_{2n})_*(b_{2i+1}b_{m_1}^2b_{m_2}^2 \cdots b_{m_{n-1}}^2)$$

$$= \sum_{i_1=0}^{2i+1} \frac{i_1}{!} b_{i_1} b_{2i+1-i_1} b_{m_1}^2 b_{m_2}^2 \cdots b_{m_{n-1}}^2$$

$$= b_{m_1}^2 b_{m_2}^2 \cdots b_{m_{n-1}}^2 \sum_{i_1=0}^{2i+1} b_{i_1} b_{2i+1-i_1}$$

$$= b_{m_1}^2 b_{m_2}^2 \cdots b_{m_{n-1}}^2 (b_0 b_{2i+1} + b_1 b_{2i} + \cdots + b_2 b_1 + b_{2i+1} b_0)$$

$$= 0.$$

Finally since we have the commutative diagram

$$\begin{array}{ccc}
BO(2n) & \xrightarrow{B_{2n}} & BSO(2n+1) \\
\downarrow f_{2n} & \Downarrow h_{2n+1} & \\
\downarrow & \downarrow & \\
BO(2n+1) & & \\
\end{array}$$

and since $(h_{2n+1})_*$ is injective, we also have $(B_{2n})_*(b_{2i+1}b_{m_1}^2b_{m_2}^2 \cdots b_{m_{n-1}}^2) = 0$. This completes the proof. □

5. Proof of Theorem A

In this section, we will use Lemma 3.5, Lemma 4.1, Proposition 4.3 and the Wu formula [9] to compute the Adams $E_1^{2,*}$ term of $\tilde{b}u_*(BSO(2n+1))$. Then we can prove Theorem B. First we recall the Wu formula.

**Proposition 5.1.** (Wu formula [9]) $Sq^k(w_m) = \sum_{t=0}^{k} \binom{m-k-t-1}{t} w_{k-t}w_{m+t}$, where the binomial coefficient $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ is taken mod 2.

Then let’s find the Adams $E_2^{1,*}$ term of $\tilde{b}u_*(BSO(2n+1))$. 

Theorem 5.2. As a $\mathbb{Z}/2$-module, $E_2^{1,*}(BSO(2n+1))$ is generated by $\overline{w}_0 \otimes \overline{w}_2^{-2m_1} \overline{w}_4^{-2m_2} \cdots \overline{w}_2^{-2m_n}$ and $\overline{v} \otimes \overline{w}_2^{-2m_1} \overline{w}_4^{-2m_2} \cdots \overline{w}_2^{-2m_n}$, where $\sum_{i=1}^{n} m_i > 0, m_i \geq 0$.

Proof. By the Wu formula, in $\tilde{H}^*(BSO(2n+1)) = \mathbb{Z}/2 [\overline{w}_2, \overline{w}_3, \ldots, \overline{w}_{2n+1}]$, we use the following diagrams of $E$-actions. It follows that the $E$-actions on the generators of $\tilde{H}^*(BSO(2n+1))$ must be the sum of $\overline{w}_{\text{odd}}w$, where $w = 1$ or $w$ is any monomial in $\tilde{H}^*(BSO(2n+1))$, hence the monomials $\overline{w}_2^{-2m_1} \overline{w}_4^{-2m_2} \cdots \overline{w}_2^{-2m_n}$ are all indecomposable, $\sum_{i=1}^{n} m_i > 0, m_i \geq 0$. So

$$\overline{v}_0 \otimes \overline{w}_2^{-2m_1} \overline{w}_4^{-2m_2} \cdots \overline{w}_2^{-2m_n}, \sum_{i=1}^{n} m_i > 0, m_i \geq 0,$$

and

$$\overline{v} \otimes \overline{w}_2^{-2m_1} \overline{w}_4^{-2m_2} \cdots \overline{w}_2^{-2m_n}, \sum_{i=1}^{n} m_i > 0, m_i \geq 0,$$

must be part of the $\mathbb{Z}/2$-generators of $E_2^{1,*}(BSO(2n+1))$.

Then by Lemma 3.5, Lemma 4.1 and Proposition 4.3, $E_2^{1,*}(BSO(2n+1))$ contains at most the $\mathbb{Z}/2$-generators

$$(B_{2n})_*(\xi_1 \otimes b_{2j_1}^2 b_{2j_2}^2 \cdots b_{2j_k}^2), 1 \leq j_1 \leq j_2 \leq \cdots \leq j_k, 2k \leq 2n,$$

and

$$(B_{2n})_*(\xi_2 \otimes b_{2j_1}^2 b_{2j_2}^2 \cdots b_{2j_k}^2), 1 \leq j_1 \leq j_2 \leq \cdots \leq j_k, 2k \leq 2n,$$

Thus, counting the rank of $E_2^{1,k}(BSO(2n+1))$ and $E_2^{1,k}(BSO(2n+1))$ as $\mathbb{Z}/2$-modules for each $k \geq 1$, we can see that all the generators of $E_2^{1,*}(BSO(2n+1))$ are found. This completes the proof. $\Box$
Proof of Theorem A. For each $n \geq 1$, recall that $D_{2n+1}$ is the $E$-module with the $\mathbb{Z}/2$-generators $\overline{w}_2^{2m_1} \overline{w}_4^{2m_2} \cdots \overline{w}_{2n}^{2m_n}$, $\sum m_i > 0$, $m_i \geq 0$, and $M_{2n+1}$ is a free $E$-module with the $E$-basis $T_{2n+1} = \{t_j \mid j \in A_{2n+1}\}$ described in Section 1. Let $N$ be the $\mathbb{Z}/2$-submodule of $\tilde{H}^*(BSO(2n+1))$ generated by all but this kind of monomials $\overline{w}_2^{2m_1} \overline{w}_4^{2m_2} \cdots \overline{w}_{2n}^{2m_n}$, $\sum m_i > 0$, $m_i \geq 0$. Then since $\overline{w}_2^{2m_1} \overline{w}_4^{2m_2} \cdots \overline{w}_{2n}^{2m_n}$ are indecomposable, $N$ is an $E$-submodule and $\tilde{H}^*(BSO(2n+1)) \cong D_{2n+1} \oplus N$, as $E$-modules. Note that $T_{2n+1}$ is contained in $N$ since $\overline{w}_2^{2m_1} \overline{w}_4^{2m_2} \cdots \overline{w}_{2n}^{2m_n}$ can not be generated by $T_{2n+1}$.

Then by Theorem 5.2 and Lemma 2.1, the $\mathbb{Z}/2$-generators of $E_2^{1,k}(D_{2n+1})$ are

\[
\overline{w}_0 \otimes \overline{w}_2^{2m_1} \overline{w}_4^{2m_2} \cdots \overline{w}_{2n}^{2m_n}, \quad \sum_{i=1}^n m_i > 0, m_i \geq 0,
\]

and $E_2^{1,k}(N) = 0$. Thus by Lemma 2.4, $N$ is a free $E$-module with the $E$-basis $T_{2n+1}$, that is, $N = M_{2n+1}$. This completes the proof. \hfill \Box

6. Proof of Theorem B

In this section, we are going to prove Theorem A. First we recall what we need in [3]. Suppose $M$ and $N$ are left $A$-modules with the actions $\mu_M$ and $\mu_N$, then $M \otimes N$ is also a left $A$-module with the action defined by the composite map

$$A \otimes M \otimes N \xrightarrow{\psi \otimes \mu_M \otimes N} A \otimes A \otimes M \otimes N \xrightarrow{\psi \otimes \mu_N} A \otimes M \otimes A \otimes N \xrightarrow{\mu_M \otimes \mu_N} M \otimes N,$$

where $\psi$ is the diagonal map of $A$ and $T(a \otimes b) = (-1)^{\dim a \dim b} (b \otimes a)$ is the twist map. We write $D(M \otimes N)$ to indicate $M \otimes N$ with this left action. Similarly, $L(M \otimes N)$ indicates the extended $A$-action over $M$. Then we have the following proposition.

If $B$ is a Hopf subalgebra of $A$, then we know that $D(M \otimes N)$ is a left $B$-module and $A \otimes_B N$ is a left $A$-module with the extended action over $A$. Thus we have the following proposition.

**Proposition 6.1.** (Proposition 1.7 of [3]) If $B$ is a Hopf subalgebra of $A$, $M$ is a left $A$-module, and $N$ is a left $B$-module, then

$$D[M \otimes (A \otimes_B N)] \cong_L [A \otimes_B D(M \otimes N)]$$
as left $A$-modules.

**Remark 6.2.** Let $N$ be $\mathbb{Z}/2$ and $B$ be $E$ in Proposition 6.1. Since

$$D[M \otimes (A \otimes E \mathbb{Z}/2)] \cong_D [(A \otimes E \mathbb{Z}/2) \otimes M] \text{ and } D(M \otimes \mathbb{Z}/2) \cong M,$$

the isomorphism becomes

$$\theta : [A \otimes E M] \cong_D [(A \otimes E \mathbb{Z}/2) \otimes M]$$

and is given by $\theta(a \otimes x) = \sum a' \otimes 1 \otimes a'' x$, with the inverse $\theta^{-1}(a \otimes 1 \otimes x) = \sum a' \otimes \chi(a'') x$, where $\psi(a) = \sum a' \otimes a''$ and $\chi$ is the conjugation map. (See [1] and Proposition 1.1 of [3] for the details.)

Recall that $\pi^*(bu) \cong A/A(Q_0, Q_1) \cong A \otimes E \mathbb{Z}/2$ and the Künneth theorem gives the isomorphism

$$\phi : H^*(bu \wedge X) \cong H^*(bu) \otimes \tilde{H}^*(X) \cong A \otimes E \mathbb{Z}/2 \otimes \tilde{H}^*(X) \cong A \otimes E \tilde{H}^*(X)$$

for any space or spectrum $X$. Then by Theorem A, we have

$$H^*(bu \wedge BSO(2n+1)) \cong A \otimes E \tilde{H}^*(BSO(2n+1)) \cong A \otimes E D_{2n+1} \oplus A \otimes E M_{2n+1}$$

and

$$H^*(bu \wedge BO(n)) \cong A \otimes E \tilde{H}^*(BO(n)).$$

Next, to construct the homotopy equivalence we need, we have to recall the main result in [7]. Recall that in Theorem 3.1, we have

$$\tilde{H}^*(BO(n)) \cong D_1^* \oplus D_2^* \oplus M,$$

where $D_1^*$ is a trivial $E$-module with the generators $w_2^{2m_1}w_4^{2m_2} \cdots w_{2k}^{2m_k}$, $\sum m_i > 0$, $2k \leq n$. Note that in $H^*(BO(2n+1))$, $D_1^* \cong D_{2n+1}$.

**Theorem 6.3.** (Theorem 1.2 of [7]) For each $n \geq 1$, there is a stable splitting

$$bu \wedge BO(n) \simeq [\bigvee_{\alpha} HZ/2] \vee [\bigvee_{\beta} bu] \vee [\bigvee_{\gamma} bu \wedge RP^\infty],$$

where the $\alpha'$, and their degrees, correspond to the generators of $M$, the $\beta$, and their degrees, correspond to the generators of $D_1^*$, the $\gamma$, and their degrees, correspond to the generators of $D_2^*$.

**Remark 6.4.** Let $f$ be the above homotopy equivalence. Then

$$f^*(0 \oplus (1 \otimes \Sigma^\beta 1) \oplus 0) = 1 \otimes w_2^{2m_1}w_4^{2m_2} \cdots w_{2k}^{2m_k}.$$
for each generator $w_2^{2m_1}w_4^{2m_2} \cdots w_{2k}^{2m_k}$ of $D_1^*$ and the corresponding $\beta$, where
$1 \otimes \Sigma^\beta 1 \in A \otimes_E \overset{\phi}{\rightarrow} H^*(\Sigma^\beta bu) \cong H^*(\Sigma^\beta bu)$. For the details on the map $f$, see the proof of Theorem 1.2 and Section 3 of [7].

**Proof of Theorem B.** First we construct the stable map

$$g : bu \wedge BSO(2n + 1) \longrightarrow [\vee_{\alpha} \Sigma^\alpha \mathbb{H}Z/2] \vee [\vee_{\beta} \Sigma^\beta bu].$$

For each $E$-free generators $t_j \in \tilde{H}^E(\Sigma^\alpha (2n + 1), \deg t_j = \alpha$, $t_j \in T_{2n+1}$, let $g_{t_j} : BSO(2n + 1) \longrightarrow \Sigma^\alpha \mathbb{H}Z/2$ represent $t_j$, which means $g_{t_j}^*(\Sigma^\alpha 1) = t_j$. Let $i : bu \longrightarrow \mathbb{H}Z/2$ be the multiplicative canonical map and $\mu'$ be the ring structure map of $\mathbb{H}Z/2$. Then we define

$$g_0 : bu \wedge BSO(2n + 1) \longrightarrow bu \wedge [\vee_{\alpha} \Sigma^\alpha \mathbb{H}Z/2] \longrightarrow [\vee_{\alpha} \Sigma^\alpha \mathbb{H}Z/2],$$

where $\alpha = \deg t_j$ and $\nu : bu \wedge \mathbb{H}Z/2 \longrightarrow \mathbb{H}Z/2 \wedge \mathbb{H}Z/2 \longrightarrow \mathbb{H}Z/2$.

On the other hand, we define

$$g_1 : bu \wedge BSO(2n + 1) \longrightarrow (bu \wedge \mathbb{H}Z/2) \overset{\nu}{\longrightarrow} [\vee_{\alpha} \Sigma^\alpha \mathbb{H}Z/2] \vee [\vee_{\beta} \Sigma^\beta bu] \overset{p}{\longrightarrow} [\vee_{\beta} \Sigma^\beta bu],$$

where $p$ is the projection map. Then for each generator $w_2^{2m_1}w_4^{2m_2} \cdots w_{2n}^{2m_n}$ of $D_{2n+1}$, we have

$$g_{t_j}^*(\Sigma^\beta 1) = (bu \wedge \mathbb{H}Z/2)^{\alpha} (1 \otimes w_2^{2m_1}w_4^{2m_2} \cdots w_{2n}^{2m_n}) = 1 \otimes w_2^{2m_1}w_4^{2m_2} \cdots w_{2n}^{2m_n}.$$
Therefore, since the \( \mathbb{Z}/2 \)-basis of \( D_{2n+1} \) is consisted of the monomials \\
\[
\overline{w}_{2^{m_1}} \cdots \overline{w}_{2^{m_n}},
\]
which means \( D_{2n+1} \) is isomorphic to \( \tilde{H}^*(\vee S^3) \) as \( E \)-modules, and since the \( A \)-action on \( A \otimes E D_{2n+1} \) is just on \( A \), and so is \( A \otimes E \tilde{H}^*(\vee S^3) \), \( \Phi_1 \) is an isomorphism and this implies \( g_1^* \) takes \( H^*(\vee \Sigma^3 bu) \) isomorphically onto \( A \otimes E D_{2n+1} \).

Similarly, under the map \\
\[
\Phi_2 : H^*(\vee \Sigma^\alpha HZ/2) \xrightarrow{g_0^*} H^*(bu \wedge BSO(2n + 1)) \cong (A \otimes_E D_{2n+1}) \oplus (A \otimes_E M_{2n+1}) \xrightarrow{p_2} A \otimes_E M_{2n+1},
\]

where \( p_2 \) is the projection map, we have \\
\[
\Phi_2 : \Sigma^\alpha 1 \xrightarrow{g_0^*} 1 \otimes t_j \mapsto 0 \oplus (1 \otimes t_j) \xrightarrow{p_2} 1 \otimes t_j
\]
for each \( E \)-free generators \( t_j \) and the corresponding \( \Sigma^\alpha 1 \in H^*(\vee \Sigma^\alpha HZ/2) \).

Thus \( \Phi_2 \) is an isomorphism and this implies \( g_0^* \) takes the free \( A \)-module \( H^*(\vee \Sigma^\alpha HZ/2) \) isomorphically onto \( A \otimes_E M_{2n+1} \).

As a result, we see that the composite homomorphism \\
\[
H^*((\vee_{\alpha} \Sigma^\alpha HZ/2) \vee (\vee_{\beta} \Sigma^\beta bu)) \xrightarrow{g^* = g_0^* \circ g_1^*} H^*(bu \wedge BSO(2n + 1))
\]
\[
\cong (A \otimes E D_{2n+1}) \oplus (A \otimes E M_{2n+1})
\]
is an isomorphism, hence \( g \) is an equivalence at prime 2. This completes the proof. \( \square \)

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STABLE SPLITTINGS OF $bu \wedge BSO(2n + 1)$

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