ABSOLUTE CONTINUITY OF THE REPRESENTING MEASURES OF THE TRANSMUTATION OPERATORS ATTACHED TO THE ROOT SYSTEM OF TYPE $BC_2$

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Abstract. We prove in this paper the absolute continuity of the representing measures of the transmutation operators $V_k$, $V_k^W$ and $V_k^{W'}$, $V_k^{W'}$ associated respectively to the Cherednik operators and the Heckman-Opdam theory attached to the root system of type $BC_2$.

1. Introduction

We consider the differential-difference operators $T_j, j = 1, 2, ..., d$, associated with a root system $R$, a Weyl group $W$ and a multiplicity function $k$, introduced by I. Cherednik in [2], and called the Cherednik operators in the literature. These operators are helpful for the extension and simplification of the theory of Heckman-Opdam, which is a generalization of the harmonic analysis on the symmetric spaces $G|K$ (see [3, 4, 5, 7]).

The notion of transmutation operators called also the trigonometric Dunkl intertwining operators and their dual introduced in [8] are fundamental in the harmonic analysis associated to the Cherednik operators and the Heckman-Opdam theory. We have considered in [8, 9] the transmutation operators $V_k$, $V_k^W$ and $V_k^{W'}$, $V_k^{W'}$ associated respectively to the Cherednik operators and the Heckman-Opdam theory attached to the root system of type $BC_2$, and we have proved that these operators are integral transforms, more precisely, for all function $g$ in $\mathcal{E}(\mathbb{R}^2)$ (the space of $C^\infty$-functions on $\mathbb{R}^2$) we have

$$\forall \, x \in \mathbb{R}^2, \quad V_k(g)(x) = \int_{\mathbb{R}^2} g(y)d\mu_x(y), \quad (1.1)$$

where $\mu_x$ is a positive measure with compact support contained in the closed ball $B(0, \|x\|)$ of center 0 and radius $\|x\|$, and of norm less than or equal to 1.

And for all function $f$ in $\mathcal{D}(\mathbb{R}^2)$ (the space of $C^\infty$-functions on $\mathbb{R}^2$, with compact support) we have

$$\forall \, y \in \mathbb{R}^2, \quad V_k^W(f)(y) = \int_{\mathbb{R}^2} f(x)dv_y(x), \quad (1.2)$$

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where \( \nu_y \) is a positive measure with support in the set \( \{ x \in \mathbb{R}^2 : \| x \| \geq \| y \| \} \). From the previous results we have deduced that for all functions \( g \) in \( \mathcal{E}(\mathbb{R}^2)^W \) (the subspace of \( \mathcal{E}(\mathbb{R}^2) \) of \( W \)-invariant functions) and \( f \) in \( \mathcal{D}(\mathbb{R}^2)^W \) (the subspace of \( \mathcal{D}(\mathbb{R}^2) \) of \( W \)-invariant functions) we have

\[
\forall x \in \mathbb{R}^2, \quad V_k^W(g)(x) = \int_{\mathbb{R}^2} g(y) d\mu_x^W(y), \tag{1.3}
\]

and

\[
\forall y \in \mathbb{R}^2, \quad V_k(f)(y) = \int_{\mathbb{R}^2} f(x) d\nu_y^W(x), \tag{1.4}
\]

where

\[
\mu_x^W = \frac{1}{|W|} \sum_{w \in W} \mu_{wx} \tag{1.5}
\]

and

\[
\nu_y^W = \frac{1}{|W|} \sum_{w \in W} \nu_{wy} \tag{1.6}
\]

In this paper we prove that for all \( x \in \mathbb{R}^{2 \text{reg}} \) (the regular part of \( \mathbb{R}^2 \)) and \( y \in \mathbb{R}^2 \), the measures \( \mu_x, \mu_x^W \) and \( \nu_y, \nu_y^W \) are absolute continuous with respect to Lebesgue measure on \( \mathbb{R}^2 \). More precisely there exist positive functions \( K(x;y) \) and \( K^W(x;y) \) such that

\[
d\mu_x(y) = K(x;y) dy, \tag{1.7}
\]

\[
d\mu_x^W(y) = K^W(x,y) dy, \tag{1.8}
\]

\[
d\nu_y(x) = K(x,y) A_k(x) dx, \tag{1.9}
\]

\[
d\nu_y^W(x) = K^W(x,y) A_k(x) dx, \tag{1.10}
\]

where \( A_k \) is a weight function on \( \mathbb{R}^2 \) which will be given in the following section (see (2.7)).

The function \( y \to K(x,y) \) and \( y \to K^W(x,y) \) have their support contained in the closed ball \( B(0, \| x \|) \) and satisfy

\[
\int_{\mathbb{R}^2} K(x,y) dy \leq 1, \tag{1.11}
\]

and

\[
\int_{\mathbb{R}^2} K^W(x,y) dy \leq 1. \tag{1.12}
\]

As applications of the previous results, we prove that for all \( \lambda \in \mathbb{C}^2 \) the Opdam-Cherednik kernel \( G_\lambda \) and the Heckman-Opdam hypergeometric function \( F_\lambda \) possess the following integral representations

\[
\forall x \in \mathbb{R}^{2 \text{reg}}, \quad G_\lambda(x) = \int_{\mathbb{R}^2} K(x,y) e^{-i(\lambda,y)} dy, \tag{1.13}
\]
and
\[
\forall \, x \in \mathbb{R}^2_{reg}, \quad F_\lambda(x) = \int_{\mathbb{R}^2} K(x, y)^W e^{-i\langle \lambda, y \rangle} dy. \tag{1.14}
\]

2. The Cherednik operators attached to the root system of type $BC_2$

We consider $\mathbb{R}^2$ with the standard basis $\{e_1, e_2\}$, and inner product $\langle ., . \rangle$ for which this basis is orthonormal. We extend this inner product to a complex bilinear form on $\mathbb{C}^2$.

2.1. The root system of type $BC_2$ and the Cherednik operators on $\mathbb{R}^2$. The root system of type $BC_2$ can be identified with the set $\mathcal{R}$ given by
\[
\mathcal{R} = \{ \pm e_1, \pm e_2, \pm 2e_1, \pm 2e_2 \} \cup \{ \pm e_1 \pm e_2 \}, \tag{2.1}
\]
which can also be written in the form
\[
\mathcal{R} = \{ \pm \alpha_i, i = 1, 2, \ldots, 6 \},
\]
with
\[
\alpha_1 = e_1, \alpha_2 = e_2, \alpha_3 = 2e_1, \alpha_4 = 2e_2, \alpha_5 = (e_1 - e_2), \alpha_6 = (e_1 + e_2). \tag{2.2}
\]
We denote by $\mathcal{R}_+$ the set of positive roots.
\[
\mathcal{R}_+ = \{ \alpha_i, i = 1, 2, \ldots, 6 \}. \tag{2.3}
\]

For $\alpha \in \mathcal{R}$, we consider
\[
r_\alpha(x) = x - \langle \tilde{\alpha}, x \rangle \alpha, \quad \text{with} \quad \tilde{\alpha} = \frac{2 \alpha}{\| \alpha \|^2}, \tag{2.4}
\]
the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^2$ orthogonal to $\alpha$.

The reflections $r_\alpha, \alpha \in \mathcal{R}$, generate a finite group $W$ called the Weyl group associated with $\mathcal{R}$. In this case $W$ is isomorphic to the hyperoctahedral group which is generated by permutations and sign changes of the $e_i, i = 1, 2$.

The multiplicity function $k : \mathcal{R} \to [0, +\infty]$ can be written in the form
\[
k = (k_1, k_2, k_3) \quad \text{where} \quad k_1 \text{ and } k_2 \text{ are the values on the roots } \alpha_1, \alpha_2, \text{ and } \alpha_3, \alpha_4 \text{ respectively, and } k_3 \text{ is the value on the roots } \alpha_5, \alpha_6.
\]

The positive Weyl chamber denoted by $a^+$ is given by
\[
a^+ = \{ x \in \mathbb{R}^2 ; \forall \, \alpha \in \mathcal{R}_+, \langle \alpha, x \rangle > 0 \}. \tag{2.5}
\]

it can also be written in the form
\[
a^+ = \{ (x_1, x_2) \in \mathbb{R}^2 ; x_1 > x_2 > 0 \}. \tag{2.6}
\]

Moreover, let $A_k$ be the weight function
\[
\forall \, x \in \mathbb{R}^2, \quad A_k(x) = \prod_{\alpha \in \mathcal{R}_+} | \sinh(\frac{\alpha}{2}, x) |^{2k(\alpha)}. \tag{2.7}
\]
The Cherednik operators \( T_j, j = 1, 2 \), are defined for functions \( f \) of class \( C^1 \) on \( \mathbb{R}^2 \) by

\[
T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in \mathbb{R}_+} \frac{k(\alpha \langle \alpha, e_j \rangle)}{1 - e^{-\langle \alpha, x \rangle}} \{ f(x) - f(r_\alpha x) \} - \rho_j f(x),
\]

with

\[
\rho_j = \frac{1}{2} \sum_{\alpha \in \mathbb{R}_+} k(\alpha \langle \alpha, e_j \rangle), \quad j = 1, 2.
\]

These operators can also be written in the following form

\[
T_1 f(x) = \frac{\partial}{\partial x_1} f(x) + k_1 \left\{ f(x) - f(r_{\alpha_1} x) \right\} + 2k_2 \left\{ f(x) - f(r_{\alpha_3} x) \right\} + k_3 \left[ \frac{f(x) - f(r_{\alpha_5} x)}{1 - e^{-\langle \alpha_5, x \rangle}} + \frac{f(x) - f(r_{\alpha_6} x)}{1 - e^{-\langle \alpha_6, x \rangle}} \right] - \left( \frac{1}{2} k_1 + k_2 + k_3 \right) f(x),
\]

\[
T_2 f(x) = \frac{\partial}{\partial x_2} f(x) + \left\{ f(x) - f(r_{\alpha_2} x) \right\} + 2k_2 \left\{ f(x) - f(r_{\alpha_4} x) \right\} + k_3 \left[ - \left( \frac{f(x) - f(r_{\alpha_5} x)}{1 - e^{-\langle \alpha_5, x \rangle}} + \frac{f(x) - f(r_{\alpha_6} x)}{1 - e^{-\langle \alpha_6, x \rangle}} \right) - \left( \frac{1}{2} k_1 + k_2 \right) f(x). \right.
\]

2.2. The Opdam-Cherednik kernel and the Heckman-Opdam hypergeometric function (see [3, 4, 5, 7]). We denote by \( G_\lambda, \lambda \in \mathbb{C}^2 \), the eigenfunction of the operators \( T_j, j = 1, 2 \). It is the unique analytic function on \( \mathbb{R}^2 \) which satisfies the differential-difference system

\[
\begin{align*}
T_j G_\lambda(x) &= -i \lambda_j G_\lambda(x), \quad j = 1, 2, x \in \mathbb{R}^2, \\
G_\lambda(0) &= 1.
\end{align*}
\]

It is called the Opdam-Cherednik kernel.

We consider the function \( F_\lambda, \lambda \in \mathbb{C}^2 \), defined by

\[
\forall x \in \mathbb{R}^2, \quad F_\lambda(x) = \frac{1}{|W|} \sum_{w \in W} G_\lambda(wx).
\]

It is called the Heckman-Opdam hypergeometric function.

The functions \( G_\lambda \) and \( F_\lambda \) possess the following properties.

i) For all \( x \in \mathbb{R}^2 \) the function \( \lambda \to G_\lambda(x) \) is entire on \( \mathbb{C}^2 \).

ii) We have

\[
\forall x \in \mathbb{R}^2, \forall \lambda \in \mathbb{C}^2, |G_\lambda(x)| \leq G_{1m(\lambda)}(x).
\]

iii) We have

\[
\forall x \in \mathbb{R}^2, \forall \lambda \in \mathbb{R}^2, |G_\lambda(x)| \leq 1.
\]

(See [9]).
vi) The function $G_\lambda, \lambda \in \mathbb{C}^2$, admits the following Laplace type representation
\[
\forall \, x \in \mathbb{R}^2, \, G_\lambda(x) = \int_{\mathbb{R}^2} e^{-i\langle \lambda, y \rangle} d\mu_x(y), \quad (2.16)
\]
where $\mu_x$ is the positive measure given by (1.1).

v) From (2.13), (2.16) we deduce that the function $F_\lambda, \lambda \in \mathbb{C}^2$, possesses the Laplace type representation
\[
\forall \, x \in \mathbb{R}^2, \, F_\lambda(x) = \int_{\mathbb{R}^2} e^{-i\langle \lambda, y \rangle} d\mu^W_x(y), \quad (2.17)
\]
where $\mu^W_x$ is the measure given by (1.3).

3. The transmutation operator and its dual associated with the Cherednik operators attached to the root system of type $BC_2$

**Notations.** We denote by
- $\mathcal{E}(\mathbb{R}^2)$ the space of $C^\infty$-functions on $\mathbb{R}^2$. Its topology is defined by the semi-norms
  \[
  q_{n,K}(\varphi) = \sup_{\|\varphi\|_n \leq n} |D^\mu \varphi(x)|.
  \]
  where $K$ is a compact of $\mathbb{R}^2$, $n \in \mathbb{N}$ and
  \[
  D^\mu = \frac{\partial^{|\mu|}}{\partial \mu_1 x_1 \partial \mu_2 x_2 \partial \mu_3 x_3}, \quad \mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{N}^3, |\mu| = \mu_1 + \mu_2 + \mu_3.
  \]
- $\mathcal{D}(\mathbb{R}^2)$ the space of $C^\infty$-functions on $\mathbb{R}^2$ with compact support. We have
  \[
  \mathcal{D}(\mathbb{R}^2) = \bigcup_{a>0} \mathcal{D}_a(\mathbb{R}^2),
  \]
  where $\mathcal{D}_a(\mathbb{R}^2)$ is the space of $C^\infty$-functions on $\mathbb{R}^2$ with support in the closed ball $B(0, a)$ of center 0 and radius $a$. The topology of $\mathcal{D}_a(\mathbb{R}^2)$ is defined by the semi-norms
  \[
  P_n(\psi) = \sup_{\|\psi\|_n \leq n} |D^\mu \psi(x)|, \quad n \in \mathbb{N}.
  \]
  The space $\mathcal{D}(\mathbb{R}^2)$ is equipped with the inductive limit topology.

By using the measure $\mu_x$ given by (1.1) we define by applying the same method as in [8], the transmutation operator $V_k$ called also the trigonometric Dunkl intertwining operator relating to the root system of type $BC_2$ by
\[
\forall \, x \in \mathbb{R}^2, \quad V_k(g)(x) = \int_{\mathbb{R}^2} g(y)d\mu_x(y), \quad g \in \mathcal{E}(\mathbb{R}^2). \quad (3.1)
\]
The operator $V_k$ is the unique linear topological isomorphism from $E(\mathbb{R}^2)$ onto itself satisfying the transmutation relations
\begin{equation}
\forall x \in \mathbb{R}^2, \quad T_j V_k(g)(x) = V_k\left( \frac{\partial}{\partial y_j} g \right)(x), \quad j = 1, 2.
\end{equation}
and the condition
\begin{equation}
V_k(g)(0) = g(0). \tag{3.3}
\end{equation}

The dual $^t V_k$ of the operator $V_k$ is defined by the following duality relation
\begin{equation}
\int_{\mathbb{R}^2} ^t V_k(f)(y)g(y)dy = \int_{\mathbb{R}^2} V_k(g)(x)f(x)A_k(x)dx, \tag{3.4}
\end{equation}
with $f$ in $D(\mathbb{R}^2)$ and $g$ in $E(\mathbb{R}^2)$.

The operator $^t V_k$ is a linear topological isomorphism from $D(\mathbb{R}^2)$ onto itself satisfying the transmutation relations
\begin{equation}
\forall y \in \mathbb{R}^2, \quad ^t V_k((T_j + S_j)f)(y) = \frac{\partial}{\partial y_j} ^t V_k(f)(y), \quad j = 1, 2, \tag{3.5}
\end{equation}
where $S_j$ is the operator on $D(\mathbb{R}^2)$ given by
\begin{equation}
\forall x \in \mathbb{R}^2, \quad S_j(h)(x) = \sum_{\alpha \in \mathbb{R}_+} k(\alpha) \langle \alpha, e_j \rangle h(r_{\alpha}x).
\end{equation}

The operator $^t V_k$ is an integral transform, more precisely we have
\begin{equation}
\forall y \in \mathbb{R}^2, \quad ^t V_k(f)(y) = \int_{\mathbb{R}^2} f(x)d\nu_y(x), \quad f \in D(\mathbb{R}^2), \tag{3.6}
\end{equation}
where $\nu_y$ is the measure given by (1.2).

**Remark 3.1.** By using the measure $\mu_x$ given by (1.1) we have defined and studied in [8] the transmutation operator $V_k^W$ on $E(\mathbb{R}^2)^W$ relating to the root system of type $BC_2$ and we have studied also its dual $^t V_k^W$ on $D(\mathbb{R}^2)^W$. We have given some properties of these operators and we have proved that they are positive integral transforms.

**Notation.** We denote by $B(c, a)$ the open ball of $\mathbb{R}^2$ of center $c \in \mathbb{R}^2$ and radius $a > 0$, and by $\bar{B}(c, a)$ its closure.

**Proposition 3.2.** Let $y_0 \in \mathbb{R}^2$ and $a > 0$. We consider the sequence $\{g_n\}_{n \in \mathbb{N}\setminus\{0\}}$ of functions in $D(\mathbb{R}^2)$, positive, increasing such that :
\begin{equation}
\forall n \in \mathbb{N}\setminus\{0\}, \text{supp} \ g_n \subset \bar{B}(y_0, a), \forall y \in B(y_0, a - \frac{1}{n}), \ g_n(y) = 1,
\end{equation}
and
\begin{equation}
\forall y \in \mathbb{R}^2, \quad \lim_{n \to +\infty} g_n(y) = 1_{B(y_0, a)}(y),
\end{equation}
where \(1_{B(y_0,a)}\) is the characteristic function of the ball \(B(y_0,a)\). We have

\[
\forall x \in \mathbb{R}^2, \lim_{n \to +\infty} V_k(g_n)(x) = \lim_{n \to +\infty} \int_{\mathbb{R}^2} g_n(y) d\mu_x(y) = \int_{\mathbb{R}^2} 1_{B(y_0,a)}(y) d\mu_x(y).
\]

The function \(x \to \mu_x(B(y_0,a)) = \int_{\mathbb{R}^2} 1_{B(y_0,a)}(y) d\mu_x(y)\), which can also be denoted by \(V_k(1_{B(y_0,a)})(x)\) is defined almost every where on \(\mathbb{R}^2\) (see [1, p. 17]), measurable and for all \(f \in D(\mathbb{R}^2)\), we have

\[
\int_{\mathbb{R}^2} \mu_x(B(y_0,a)) f(x) A_k(x) dx = \int_{B(y_0,a)} t V_k(f)(y) dy. \tag{3.7}
\]

Proof. For all \(n \in \mathbb{N}\setminus\{0\}\), the function \(V_k(g_n)\) belongs to \(\mathcal{E}(\mathbb{R}^2)\). Then we obtain the results of this proposition from the continuity of the operator \(V_k\) from \(\mathcal{E}(\mathbb{R}^2)\) into itself, the monotonic convergence theorem and the relation (3.4). \(\square\)

**Remark 3.3.** There exists a \(\sigma\)-algebra \(m\) in \(\mathbb{R}^2\) which contains all Borel sets in \(\mathbb{R}^2\). Then for all \(E \in m\), the function \(x \to \mu_x(E)\) is defined almost every where on \(\mathbb{R}^2\), measurable and we have the following relation

\[
\int_{\mathbb{R}^2} \mu_x(E) f(x) A_k(x) dx = \int_{E} t V_k(f)(y) dy, \quad f \in D(\mathbb{R}^2). \tag{3.8}
\]

**Proposition 3.4.** Let \(x_0 \in \mathbb{R}^2\) and \(a > 0\). We consider the sequence \(\{f_n\}_{n \in \mathbb{N}\setminus\{0\}}\) of functions in \(D(\mathbb{R}^2)\), positive, increasing such that:

\[
\forall n \in \mathbb{N}\setminus\{0\}, \text{supp} f_n \subset \bar{B}(x_0,a), \forall x \in B(x_0,a - \frac{1}{n}), f_n(x) = 1,
\]

and

\[
\forall x \in \mathbb{R}^2, \lim_{n \to +\infty} f_n(x) = 1_{B(x_0,a)}(x),
\]

where \(1_{B(x_0,a)}\) is the characteristic function of the ball \(B(x_0,a)\). We have

\[
\forall y \in \mathbb{R}^2, \lim_{n \to +\infty} V_k(f_n)(y) = \lim_{n \to +\infty} \int_{\mathbb{R}^2} f_n(x) d\nu_y(x) = \int_{\mathbb{R}^2} 1_{B(x_0,a)}(x) d\nu_y(x).
\]

The function \(y \to \nu_y(B(x_0,a)) = \int_{\mathbb{R}^2} 1_{B(x_0,a)}(x) d\nu_y(x)\), which can also be denoted by \(t V_k(1_{B(x_0,a)})(y)\) is defined almost every where on \(\mathbb{R}^2\) (see [1, p.
measurable and for all \(g\) in \(\mathcal{E}(\mathbb{R}^2)\), we have
\[
\int_{\mathbb{R}^2} \nu_y(B(x_0, a))g(y)dy = \int_{B(x_0, a)} V_k(g)(x)A_k(x)dx. \tag{3.9}
\]

**Proof.** For all \(n \in \mathbb{N}\setminus\{0\}\), the function \(V_k(f_n)\) belongs to \(\mathcal{D}(\mathbb{R}^2)\). Then the continuity of the operator \(V_k\) from \(\mathcal{D}(\mathbb{R}^2)\) into itself, the monotonic convergence theorem and the relation (3.4) imply the results of this proposition. \(\Box\)

### 3.1. Absolute continuity of the measure \(\nu_y\).

The purpose of this subsection is to prove that for all \(y \in \mathbb{R}^2\), the measure \(\nu_y\) is absolute continuous with respect to the Lebesgue measure on \(\mathbb{R}^2\). **Notation.** We denote by \(\lambda\) the Lebesgue measure on \(\mathbb{R}^2\).

**Proposition 3.5.** For \(x \in \mathbb{R}^2_{\text{reg}}\), there exists a unique positive function \(K(x, .)\) integrable with respect to the Lebesgue measure \(\lambda\), and a positive measure \(\mu^y_x\) on \(\mathbb{R}^2\) such that for every Borel set \(E\), we have
\[
\mu_x(E) = \int_E K(x, y)dy + \mu^y_x(E). \tag{3.10}
\]

**Proof.** We deduce (3.10) from (1.1) and Theorem 6.9 of [6, p.129-130] and Theorem 8.6 and its Corollary of [6, p. 166]. \(\Box\)

**Remark 3.6.**

i) The supports of the function \(y \to K(x, y)\) and the measure \(\mu^y_x\) are contained in the ball \(\bar{B}(0, \|x\|)\).

ii) The measures \(\mu^y_x\) and the Lebesgue measure \(\lambda\) are mutually singular.

iii) From Theorem 8.6, p. 166 and Definition 8.3, p.164, of [6], we have
\[
K(x, y) = \lim_{a \to 0} \frac{\mu_x(B(y, a))}{\lambda(B(y, a))}. \tag{3.11}
\]

**Proposition 3.7.** We consider \(x \in \mathbb{R}^2_{\text{reg}}\) and a positive function \(f\) in \(\mathcal{D}(\mathbb{R}^2)\) with support contained in the ball \(\bar{B}(0, R), R > 0\).

i) For all Borel set \(E\), we have
\[
\int_E N^f(y)dy = \int_{B(0, R)} \mu^y_x(E)f(x)A_k(x)dx, \tag{3.12}
\]
where
\[
N^f(y) = V_k(f)(y) - \int_{B(0, R)} K(x, y)f(x)A_k(x)dx. \tag{3.13}
\]

ii) We have
\[
\forall y \in \mathbb{R}^2, N^f(y) \geq 0. \tag{3.14}
\]
Proof.  

i) By using the relations (3.8), (3.10), we obtain

\[
\int_E V_k(f)(y)dy = \int_{B(0,R)} \mu_x(E) f(x) A_k(x)dx
\]

\[= \int_{B(0,R)} \left[ \int_E K(x,y)dy + \mu^*_x(E) \right] f(x) A_k(x)dx.
\]

We deduce (3.12) by applying Fubini-Tonelli’s theorem to the second member.

ii) From the relation (3.12), the positivity of the measure \(\mu^*_x\) implies that for all Borel set \(E\), we have

\[
\int_E \mathcal{N}^f(y)dy \geq 0.
\]

Thus

\[\forall y \in \mathbb{R}^2, \mathcal{N}^f(y) \geq 0.\]

\[\square\]

**Proposition 3.8.** The measure \(\Lambda^f\) on \(\mathbb{R}^2\), given for all Borel set \(E\) by

\[
\Lambda^f(E) = \int_E \mathcal{N}^f(y)dy,
\]

(3.15)

is positive and bounded.

**Proof.** - The relation (3.14) gives the positivity of the measure \(\Lambda^f\).

- From the relations (3.15), (3.12), for all Borel set \(E\) we have

\[
\Lambda^f(E) \leq \int_{B(0,R)} \|\mu^*_x\| f(x) A_k(x)dx.
\]

(3.16)

On the other hand by using (3.10), we obtain for \(x \in \mathbb{R}^2_{reg}\),

\[
\mu^*_x(E) \leq \mu_x(E),
\]

thus

\[
\|\mu^*_x\| \leq \|\mu_x\| \leq 1.
\]

By using this result, the relation (3.16) implies that for all Borel set \(E\), we have

\[
\Lambda^f(E) \leq M_f,
\]

where

\[
M_f = \int_{B(0,R)} f(x) A_k(x)dx.
\]

Then the measure \(\Lambda^f\) is bounded.  \(\square\)

**Proposition 3.9.** Let \(x \in \mathbb{R}^2_{reg}\) and \(f\) the function given in Proposition 3.7.


i) For all Borel set $E$ we have

$$\Lambda^J(E) = 0.$$  \hfill (3.17)

ii) For $y \in \mathbb{R}^2$, we have

$$^tV_k(f)(y) = \int_{B(0,R)} K(x,y)f(x)\mathcal{A}_k(x)dx.$$ \hfill (3.18)

**Proof.**  

i) From the relations (3.15), (3.12), for all Borel set $E$ the measure $\Lambda^J$ possesses also the following form

$$\Lambda^J(E) = \int_{B(0,R)} \mu^*_x(E)f(x)\mathcal{A}_k(x)dx.$$ \hfill (3.19)

On the other hand from Proposition 3.8 the measure $\Lambda^J$ is absolute continuous with respect to the Lebesgue measure $\lambda$ and from Remark 3.6 ii) the measure $\mu^*_x, x \in B(0,R)$, and the Lebesgue measure $\lambda$ are mutually singular. Then from Proposition 6.8 (f), p.129, of [6], the measure $\Lambda^J$ and $\mu^*_x, x \in B(0,R)$, are mutually singular. By using the definition of measures mutually singular (see p. 128 of [6]), we deduce (3.17) from (3.19).

ii) By using the i) and (3.15), (3.13), we obtain (3.18).

□

**Theorem 3.10.** For all $f$ in $\mathcal{D}(\mathbb{R}^2)$ we have

$$\forall y \in \mathbb{R}^2, \; ^tV_k(f)(y) = \int_{\mathbb{R}^2} K(x,y)f(x)\mathcal{A}_k(x)dx.$$ \hfill (3.20)

**Proof.** We obtain (3.20) by writing $f = f^+ - f^-$ and by using Proposition 3.9 ii).  \hfill □

**Remark 3.11.** Theorem 3.10 shows that for all $y \in \mathbb{R}^2$ the measure $\nu_y$ given by the relation (1.2), is absolute continuous with respect to the measure $\mathcal{A}_k(x)dx$. More precisely we have

$$dv_y(x) = K(x,y)\mathcal{A}_k(x)dx.$$ \hfill (3.21)

3.2. **Absolute continuity of the measure $\mu_x$.** The purpose of this subsection is to prove that for all $x \in \mathbb{R}^2_{\text{reg}}$ the measure $\mu_x$ is absolute continuous with respect to the Lebesgue measure on $\mathbb{R}^2$.

**Theorem 3.12.** For all $g$ in $\mathcal{E}(\mathbb{R}^2)$ and $x_0 \in \mathbb{R}^2_{\text{reg}}$, we have

$$V_k(g)(x_0) = \int_{\mathbb{R}^2} K(x_0,y)g(y)dy.$$ \hfill (3.22)
Proof. By writing $g = g^+ - g^-$ it suffices to prove the theorem for $g$ positive. From the relation (3.9) we have

$$\frac{1}{\lambda(B(x_0,a))} \int_{B(x_0,a)} V_k(g)(x) \mathcal{A}_k(x) \, dx = \int_{\mathbb{R}^2} g(y) \frac{\nu_y(B(x_0,a))}{\lambda(B(x_0,a))} \, dy. \quad (3.23)$$

By using the relation (3.20), and by applying Fubini-Tonelli’s theorem to the second member of (3.23), we obtain

$$\frac{1}{\lambda(B(x_0,a))} \int_{B(x_0,a)} V_k(g)(x) \mathcal{A}_k(x) \, dx = \frac{1}{\lambda(B(x_0,a))} \int_{B(x_0,a)} \left[ \int_{\mathbb{R}^2} K(x,y) g(y) \, dy \right] \mathcal{A}_k(x) \, dx.$$

By applying the relation (2) of [6, p.168], to the two members of this relation we get

$$\mathcal{A}_k(x_0) V_k(g)(x_0) = \mathcal{A}_k(x_0) \int_{\mathbb{R}^2} K(x_0,y) g(y) \, dy,$$

as

$$\mathcal{A}_k(x_0) \neq 0 \iff x_0 \in \mathbb{R}^2_{\text{reg}},$$

thus for $x_0 \in \mathbb{R}^2_{\text{reg}}$, we have

$$V_k(g)(x_0) = \int_{\mathbb{R}^2} K(x_0,y) g(y) \, dy.$$

\[\square\]

Remark 3.13. From Theorem 3.12 and the relation (1.1) we deduce that for all $x \in \mathbb{R}^2_{\text{reg}}$ the measure $\mu_x$ is absolute continuous with respect to the Lebesgue measure on $\mathbb{R}^2$. More precisely we have

$$d\mu_x(y) = K(x,y) \, dy. \quad (3.24)$$

Corollary 3.14. i) For all $\lambda \in \mathbb{C}^2$ and $x \in \mathbb{R}^2_{\text{reg}}$, we have

$$G_\lambda(x) = \int_{\mathbb{R}^2} K(x,y) e^{-i(\lambda \cdot y)} \, dy. \quad (3.25)$$

ii) For all $x \in \mathbb{R}^2_{\text{reg}}$, we have

$$\int_{\mathbb{R}^2} K(x,y) \, dy \leq 1. \quad (3.26)$$

iii) For all $x \in \mathbb{R}^2_{\text{reg}}$, we have

$$\text{supp} K(x, \cdot) \subset \overline{B(0, \|x\|)}. \quad (3.27)$$

Proof. We deduce the results of this Corollary from (1.1), (2.17), and Theorem 3.12. \[\square\]
Theorem 3.15. We have
\[ \forall x \in \mathbb{R}^2_{\text{reg}}, \lim_{\|\lambda\| \to +\infty} G_\lambda(x) = 0. \quad (3.28) \]

Proof. From the relation (3.26) the function \( K(x,\cdot) \) is integrable on \( \mathbb{R}^2 \) with respect to the Lebesgue measure on \( \mathbb{R}^2 \). Then we deduce (3.28) from the relation (3.25) and Riemann-Lebesgue Lemma for the usual Fourier transform on \( \mathbb{R}^2 \). \( \square \)

3.3. Absolute continuity of the measures \( \nu_y^W \) and \( \mu_x^W \).

Theorem 3.16. For all \( f \) in \( D(\mathbb{R}^2)^W \), we have
\[ \forall y \in \mathbb{R}^2, \quad V_k^W(f)(y) = \int_{\mathbb{R}^2} K^W(x,y)f(x)A_k(x)dx, \quad (3.29) \]
where \( K^W(x,y) \) is the function given by
\[ K^W(x,y) = \frac{1}{|W|^2} \sum_{w,w' \in W} K(wx,w'y). \quad (3.30) \]

Proof. The relations (1.4), (1.6) and Theorem 3.10 imply the relations (3.29), (3.30). \( \square \)

Theorem 3.17. For all \( g \) in \( E(\mathbb{R}^2)^W \), we have
\[ \forall x \in \mathbb{R}^2_{\text{reg}}, \quad V_k^W(g)(x) = \int_{\mathbb{R}^2} K^W(x,y)g(y)dy, \quad (3.31) \]
where \( K^W(x,y) \) is the function given by the relation (3.30).

Proof. We deduce (3.31) from the relations (1.3), (1.5) and Theorem 3.12. \( \square \)

Remark 3.18. Theorems 3.16, 3.17 show that the measures \( \nu_y^W, y \in \mathbb{R}^2 \) and \( \mu_x^W, x \in \mathbb{R}^2_{\text{reg}} \), given respectively by the relations (1.4), (1.3), are absolute continuous with respect to the Lebesgue measure on \( \mathbb{R}^2 \). More precisely we have
\[ d\nu_y^W(x) = K^W(x,y)A_k(x)dx, \quad (3.32) \]
and
\[ d\mu_x^W(y) = K^W(x,y)dy. \quad (3.33) \]

Corollary 3.19. i) For all \( \lambda \in \mathbb{C}^2 \) and \( x \in \mathbb{R}^2_{\text{reg}} \), we have
\[ F_\lambda(x) = \int_{\mathbb{R}^2} K^W(x,y)e^{-i\langle \lambda, y \rangle}dy. \quad (3.34) \]
ii) For all $x \in \mathbb{R}^2_{\text{reg}}$, we have
\[
\int_{\mathbb{R}^2} K^W(x, y) dy \leq 1.
\] (3.35)

iii) For all $x \in \mathbb{R}^2_{\text{reg}}$, we have
\[
\text{supp} K^W(x, \cdot) \subset B(0, \|x\|).
\] (3.36)

Proof. The relations (1.3), (2.18) and Theorem 3.17, imply the results of this Corollary.

\[ \square \]

Theorem 3.20. We have
\[
\forall x \in \mathbb{R}^2_{\text{reg}}, \lim_{\|\lambda\| \to +\infty} F_{\lambda}(x) = 0.
\] (3.37)

Proof. We deduce (3.37) from the relations (3.34), (3.35) and Riemann-Lebesgue Lemma for the usual Fourier transform on $\mathbb{R}^2$.

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References
