Mathematical Formulation and Numerical Simulation of Bird Flu Infection Process within a Poultry Farm

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Abstract. Bird flu infection processes within a poultry farm are formulated mathematically. A spatial effect is taken into account for the virus concentration with a diffusive term. An infection process is represented in terms of a traveling wave solutions. For a small removal rate, a singular perturbation analysis lead to existence of traveling wave solutions, that correspond to progressive infection in one direction.

Keywords: bird flu, spatial effect, traveling wave solutions, singular perturbation
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INTRODUCTION

Avian influenza or bird flu has been documented since the last decade of the 19th century when E. Perroncito reported on outbreaks of a severe and highly contagious new poultry disease in Northern Italy. Within the last 130 years, more than 30 outbreaks of fowl plague have been reported worldwide [2]. These outbreaks, which are associated with significant economic losses and animal suffering, are most commonly referred to as "high pathogenic avian influenza" or "HPAI" [1]. The infection with highly pathogenic form is transmitted rapidly over a poultry farm and causes domestic birds serious symptoms that eventually lead to death. In practice, spot-check is often conducted to detect infection with H5N1. Even if infection is detected only from one bird, all the birds in the farm are disposed of, which have been causing immense damage to the poultry industry.

Various mathematical models have been proposed in studies of avian influenza. A SI-SIR model was analyzed for infection processes of birds and humans [4]. The transmission dynamics and spatial spread pattern of H5N1-Avian Influenza were studied [5]. The spatial spread of avian flu among flock and human were studied with diffusive terms [3]. The SI-SIR epidemic model describing the transmission of avian influenza among birds and human was proposed [6]. Mathematical models for bird flu infection within a poultry farm were proposed [7, 9, 10, 11].

Bird flu transmission processes involve influenza virus as source of disease, domestic birds as host, and the environment as medium. In a production process of a poultry farm, the entire population of domestic birds is maintained at the manageable capacity for profitable production by supply of new healthy birds for vacancies. Once bird flu intrudes into a poultry farm, some infected birds die at an early stage, and some others live longer. Regardless of being alive or dead, infected birds are the hosts of the virus, unless they are completely removed from the entire population. Mathematical models based on those factors were proposed in study of bird flu infection processes within a poultry farm. The population of susceptible birds and the population of infected birds were formulated [7]. The population of susceptible birds, the population of infected birds, and the virus concentration were modeled [9]. Those models are systems of ordinary differential equations. Spatial effects were incorporated into mathematical modeling [10].

The mathematical study of bird flu infection processes within a poultry farm is continued. In this paper, the mathematical models proposed in [11] are reformulated by considering spatial effect. A singular perturbation analysis lead to existence of traveling wave solutions, that correspond to progressive infection processes in one direction.

MATHEMATICAL MODELS OF BIRD FLU INFECTION PROCESS WITHIN A POULTRY FARM

The bird flu virus transmits through dirt and air, and the virus concentration was incorporated into modeling. Let X, Y, and Z be the population of susceptible birds, the population of infected birds, and the virus concentration in the medium, respectively. The number of susceptible birds to become infected birds per unit time is proportional to the virus concentration in the medium, and it is also proportional to the number of susceptible birds. The decreasing rate of susceptible birds due to infection is σXZ and

$$\frac{dX}{dt} = a\{c-(X+Y)\} - \sigma XZ \tag{1}$$

holds, where σ is a positive constant, a is the rate of supply of new healthy birds for vacancies and c is the capacity of the farm.

The decreasing rate in the population of susceptible birds due to infection is the increasing rate in population of infected birds, and the number of infected birds removed from the entire population is proportional to the number of infected birds itself and

$$\frac{dY}{dt} = \sigma XZ - mY \tag{2}$$

holds, where m is the removal rate. Infected birds are the hosts of influenza virus. The increasing rate of the virus concentration is proportional to the number of infected birds, and the decreasing rate of virus concentration is proportional to the virus concentration itself and

$$\frac{dZ}{dt} = pY - qZ \tag{3}$$

holds, where p and q are positive constants. The equations (1), (2), and (3) lead to the following system of equations

$$\begin{aligned} \frac{dX}{dt} &= a\{c-(X+Y)\} - \omega rXZ, \\ \frac{dY}{dt} &= \omega rXZ - mY, \\ \frac{dZ}{dt} &= p\,(Y - rZ)\,, \end{aligned} \tag{4}$$

where $r = q/p$ and $\omega = \sigma/r$ [9]. Under the assumption that vacancies due to infection are replaced instantly and that the total population always balances with the capacity of the farm, $X+Y=c$ and

$$\frac{dX}{dt} = -\frac{dY}{dt} = -\omega rXZ + m\,(c-X) \tag{5}$$

holds, the system (6) for the populations of the susceptible birds X and the virus concentration in the medium Z is obtained from the system (4).

$$\begin{aligned} \frac{dX}{dt} &= m\,(c-X) - \omega rXZ, \\ \frac{dZ}{dt} &= p\,(c-X-rZ)\,. \end{aligned} \tag{6}$$

There are two stationary points of the system (6), one of which corresponds to the state free of infection and the other corresponds to an endemic state in which part of the populations is infected. The state free of infection is assymptotically stable and the endemic state is unstable for $\omega c - m < 0$, while the state free of infection is unstable and endemic state is assymptotically stable for $\omega c - m > 0$. Those two stationary points coincide for $\omega c - m = 0$. Stability of stationary points were analyzed in the previous study [11]. Figures 1 and 2 show the solutions of the system (6) with initial values $(X_0, Z_0) = (1.0, 0.2)$, $(X_0, Z_0) = (1.0, 0.4)$, $(X_0, Z_0) = (1.0, 0.6)$, $(X_0, Z_0) = (1.0, 0.8)$. Figure 1 shows that solutions converge to the stationary point which corresponds to the state free of infection for $m = 1.5$. Figure 2 shows that solutions converge to the stationary point which corresponds to the endemic state for $m = 0.5$.

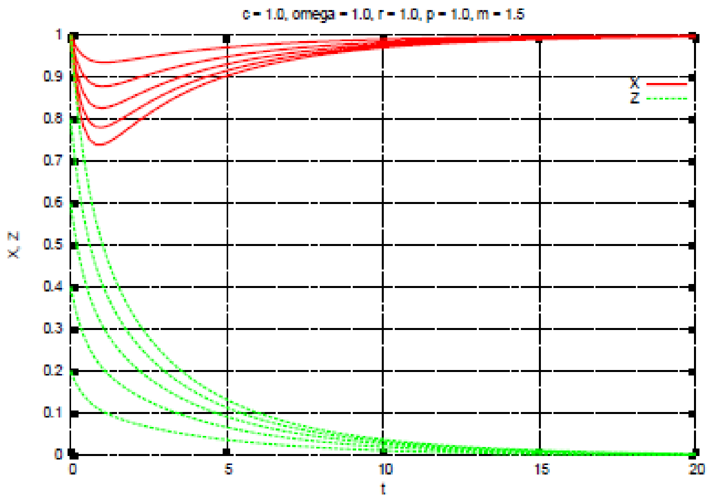

FIGURE 1. Solutions of the ODE system (6) for $\omega c - m < 0$

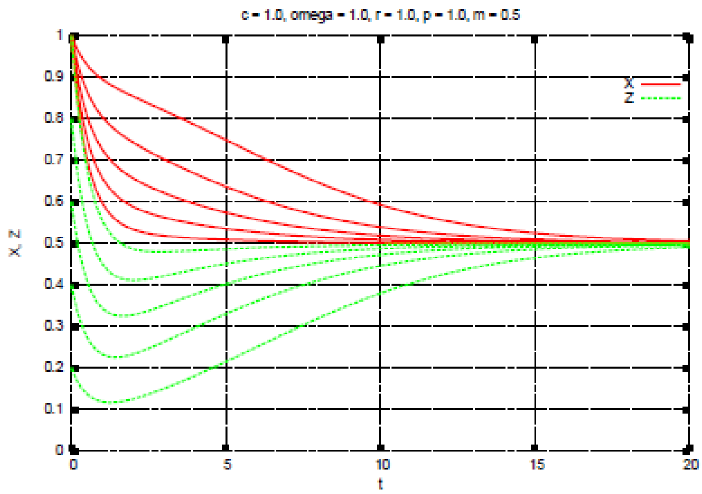

FIGURE 2. Solutions of the ODE system (6) for $\omega c - m > 0$

The viruses diffuse through the medium. In what follows, the effect of virus diffusion is taken into account. In order to take the spatial change into consideration, let x be the one dimensional coordinate variable. It is approriate to assume that the medium is one dimensional because bird cages are lined up in one direction. When a diffusive term is added to the right-hand side of the second equation, the system (6) becomes

$$\begin{aligned} \frac{\partial X}{\partial t} &= m(c-X) - \omega r X Z, \\ \frac{\partial Z}{\partial t} &= p(c-X-rZ) + d\frac{\partial^2 Z}{\partial x^2}, \end{aligned} \tag{7}$$

where d is the diffusion constant. The unknown variables X and Z of the system (6) are now functions of x and t. Solutions $X(x,t)$ and $Z(x,t)$ are defined for $0 \le x \le l$ and $t \ge 0$, and they satisfy the initial conditions

$$X(x,0) = X_0(x) \quad \text{and} \quad Z(x,0) = Z_0(x) \quad (0 \le x \le l),$$

and the boundary conditions

$$\frac{\partial X}{\partial x}(0,t) = \frac{\partial Z}{\partial x}(0,t) = \frac{\partial X}{\partial x}(l,t) = \frac{\partial Z}{\partial x}(l,t) = 0 \qquad (t \geq 0).$$

Note that stationary points of the system of ODE's (6) are solutions of the system of PDE's (7).

TRAVELING WAVE SOLUTIONS OF HOST VIRUS MODEL

Traveling waves are solutions that represent the transportation of information in a single direction [8]. Those solutions often exist in dynamical systems with spatial diffusion, which are represented in a mathematical model. In this study, traveling waves of virus transmission are investigated. Assumed that the solutions of diffusive equation (7) have the shape of a traveling wave as it propagates, let

$$X(x,t) = U(s) \text{ and } Z(x,t) = V(s), \quad s = x - kt, \tag{8}$$

where k is a positive constant. We seek function U and V with the property that they approach constant value at $s \to \pm\infty$. The wave speed k is a priori unknown and must be determined as apart of the solution problem. Subtituting (8) into (7) leads to the system of the second order ordinary differential equations

$$\begin{aligned} -kU' &= m(c-U) - \omega rUV, \\ -kV' &= p(c-U-rV) + dV''. \end{aligned} \tag{9}$$

We write (9) as a simultaneous system of first order equations by defining $V' = W$

$$\begin{aligned} U' &= -\frac{1}{k}\{m(c-U) - \omega rUV\}, \\ V' &= W, \\ W' &= -\delta W - \rho(c-U-rV), \end{aligned} \tag{10}$$

where

$$\delta = \frac{k}{d}, \qquad \rho = \frac{p}{d}.$$

In the (U,V,W) phase space, there are two stationary points for the system (10)

$$(U,V,W) = (c,0,0), \tag{11}$$

$$(U,V,W) = \left(\frac{m}{\omega}, \frac{\omega c - m}{\omega r}, 0\right). \tag{12}$$

The stability of the stationary points (11) and (12) depend on the eigen values of the Jacobian matrix

$$A_1 = \begin{bmatrix} \frac{m}{k} & \frac{\omega rc}{k} & 0 \\ 0 & 0 & 1 \\ \rho & \rho r & -\delta \end{bmatrix}, \tag{13}$$

$$A_2 = \begin{bmatrix} \frac{\omega c}{k} & \frac{mr}{k} & 0 \\ 0 & 1 & 0 \\ \rho & \rho r & -\delta \end{bmatrix}. \tag{14}$$

Numerical results show that eigenvalues of the matrices are all real-valued, and those are one negative eigenvalue and two positive eigenvalues, or two negative eigenvalues and one positive eigenvalue. In any case the stationary points are unstable for any positive value of k.

Considering a new time scale $t = k\tau$ in (10) gives

$$\begin{aligned} \dot{U} &= -\{m(c-U) - \omega rUV\}, \\ \dot{V} &= kW, \\ \dot{W} &= k\{-\delta W - \rho(c-U-rV)\}, \end{aligned} \tag{15}$$

where . denotes the derivative with respect to τ while $'$ denotes the derivatives with respect to s. In the limit $k \longrightarrow 0$, the the system (15) becomes

$$\begin{aligned} \dot{U} &= -\{m(c-U) - \omega rUV\}, \\ \dot{V} &= 0, \\ \dot{W} &= 0. \end{aligned} \tag{16}$$

This is the center manifold of system (10) in the limit, that is

$$\begin{aligned} U &= \frac{mc}{m+\omega rV}, \\ V &= Constant, \\ W &= Constant. \end{aligned} \tag{17}$$

On the center minifold, the system (10) reduces to

$$\begin{aligned} V' &= W, \\ W' &= -\delta W - \rho\left(c - \frac{mc}{m+\omega rV} - rV\right). \end{aligned} \tag{18}$$

The system (18) has two stationary points

$$(V,W) = (0,0), \tag{19}$$

$$(V,W) = \left(\frac{\omega c - m}{\omega r}, 0\right). \tag{20}$$

The Jacobian matrices J_1 and J_2 associated with stationary points (19) and (20) are

$$J_1 = \begin{bmatrix} 0 & 1 \\ -\frac{\rho\omega cr}{m} + \rho r & -\delta \end{bmatrix}, \tag{21}$$

$$J_2 = \begin{bmatrix} 0 & 1 \\ -\frac{\rho mr}{\omega c} + \rho r & -\delta \end{bmatrix}, \tag{22}$$

respectively. The eigenvalues of J_1 are

$$\lambda_{\pm} = -\frac{k}{2d} \pm \frac{\sqrt{(-k/d)^2 - 4(pr/dm)(\omega c - m)}}{2}. \tag{23}$$

The eigenvalues of J_2 are

$$\mu_{\pm} = -\frac{k}{2d} \pm \frac{\sqrt{(-k/d)^2 - 4(pr/dc\omega)(m - \omega c)}}{2}. \tag{24}$$

For $\omega c - m < 0$, λ_- is negative and λ_+ is positive, but $\mu_\pm$ are both negative. Under this condition, the stationary point (19) is unstable and the stationary point (20) is asymptotically stable. Conversely, for $\omega c - m > 0$, $\lambda_\pm$ are both negative, but μ_- is negative and μ_+ is positive. Under this condition, the stationary point (19) is asymptotically stable and the stationary point (20) is unstable.

Figures 3 and 4 show solutions of the system (18) with initial values $(V_0, W_0) = (0, 0.5)$. Figure 3 shows that solutions converge to the stationary point (19) for $m = 0.5$, which confirms that the stationary point (19) is asymptotically stable for $\omega c - m > 0$. Figure 4 shows that solutions diverge to the stationary point (20) for $m = 1.5$. The stasionary point (20) is practically insignificant for $\omega c - m < 0$. So, traveling waves of the system (7) exist for $\omega c - m > 0$. Figure 5 and 6 show the vector fields of the ODE system (18) and progressive waves of bird flu infection respectively.

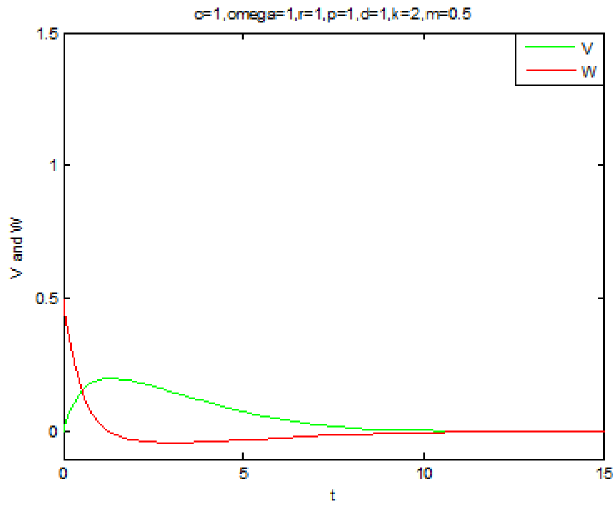

FIGURE 3. Solutions of the ODE system (18) for $\omega c - m > 0$

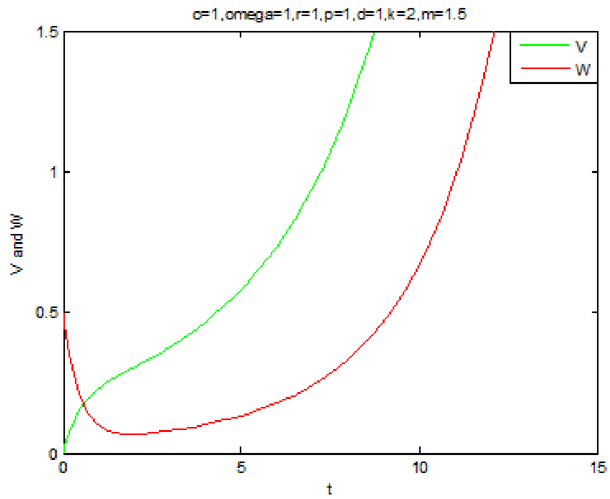

FIGURE 4. Solutions of the ODE system (18) for $\omega c - m < 0$

DISCUSSION

It has been shown in Section 3 that the stationary points of the system (10) correspond to the state free of infection. After perturbation, the stationary points of the system (10) leads to convergence to the stationary point (19) when $m = 0.5$ and it is practically insignificant to the stationary point (20) when $m = 1.5$. For $\omega c - m > 0$, there is a unique heteroclinic orbit of system (18) connecting the stationary points (19) and (20). This shows that the system (10) has a heteroclinic orbit for $\omega c - m > 0$ for all small k, which leads to a conclusion that the system (17) admits a traveling wave solution for all small positive k. This result leads to the conclusion that intrusion of bird flu leads to infection of the entire population for a small removal rate of. The removal of infected birds is essential for maintenance of the state free of infection. In a typical bird house of a poultry farm, birds are kept in cages lined up in one direction. The host-virus model (7) is appropriate. Our analysis has shown that an infection process of bird flu within a poultry farm is interpreted as a traveling wave solution. This should be taken into consideration of strategies for prevention of bird flu outbreak within a poultry farm.

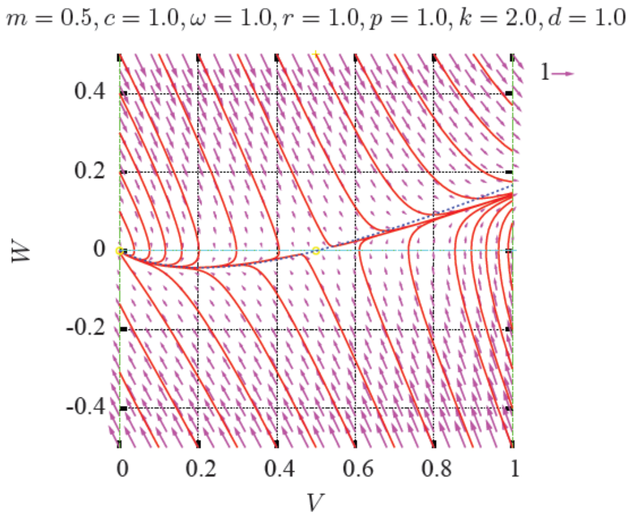

FIGURE 5. Vector fields of the ODE system (18)

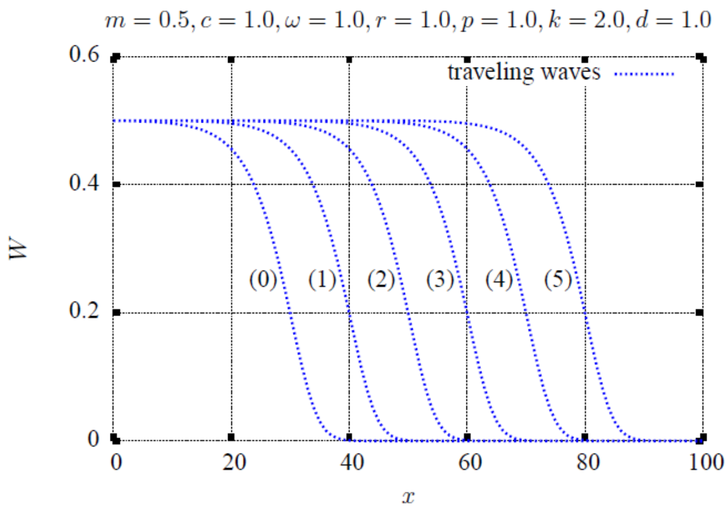

FIGURE 6. Progressive waves of bird flu infection (0):t=0, (1):t=5, (2):t=10, (3):t=15, (4):t=20, (5):t=25

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