HIGHER-DIMENSIONAL ABSOLUTE VERSIONS OF SYMMETRIC, FROBENIUS, AND QUASI-FROBENIUS ALGEBRAS

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ABSTRACT. In this paper, we define and discuss higher-dimensional and absolute versions of symmetric, Frobenius, and quasi-Frobenius algebras. In particular, we compare these with the relative notions defined by Scheja and Storch. We also prove the validity of codimension two-argument for modules over a coherent sheaf of algebras with a 2-canonical module, generalizing a result of the author.

1. INTRODUCTION

(1.1) Let \((R, m)\) be a semilocal Noetherian commutative ring, and \(\Lambda\) a module-finite \(R\)-algebra. In [6], we defined the canonical module \(K_\Lambda\) of \(\Lambda\). The purpose of this paper is two fold, each of which is deeply related to \(K_\Lambda\).

(1.2) In the first part, we define and discuss higher-dimensional and absolute notions of symmetric, Frobenius, and quasi-Frobenius algebras and their non-Cohen–Macaulay versions. In commutative algebra, the non-Cohen–Macaulay version of Gorenstein ring is known as quasi-Gorenstein rings. What we discuss here is a non-commutative version of such rings. Scheja and Storch [7] discussed a relative notion, and our definition is absolute in the sense that it depends only on \(\Lambda\) and is independent of the choice of \(R\). If \(R\) is local, our quasi-Frobenius property agrees with Gorensteinness discussed by Goto and Nishida [1], see Proposition 3.6 and Corollary 3.7.

(1.3) In the second part, we show that the codimension-two argument using the existence of 2-canonical modules in [4] is also still valid in non-commutative settings. For the definition of an \(n\)-canonical module, see (2.8). Codimension-two argument, which states (roughly speaking) that removing a closed subset of codimension two or more does not change the category of coherent sheaves which satisfy Serre’s \((S_2')\) condition, is sometimes used in algebraic geometry, commutative algebra and invariant theory. For example, information on the canonical sheaf and the class group is retained when we remove the singular locus of a normal variety over an algebraically closed field, and then these objects are respectively grasped as the top exterior...

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power of the cotangent bundle and the Picard group of a smooth variety. In [4], almost principal bundles are studied. They are principal bundles after removing closed subsets of codimension two or more.

We prove the following. Let \( X \) be a locally Noetherian scheme, \( U \) an open subset of \( X \) such that \( \text{codim}_X(X \setminus U) \geq 2 \). Let \( i : U \to X \) be the inclusion. Let \( \Lambda \) be a coherent \( \mathcal{O}_X \)-algebra. If \( X \) possesses a 2-canonical module \( \omega \), then the inverse image \( i^* \) induces the equivalence between the category of coherent right \( \Lambda \)-modules which satisfy the \( (S'_2) \) condition and the category of coherent right \( i^*\Lambda \)-modules which satisfy the \( (S'_2) \) condition. The quasi-inverse is given by the direct image \( i_* \). What was proved in [4] was the case that \( \Lambda = \mathcal{O}_X \). If, moreover, \( \omega = \mathcal{O}_X \) (that is to say, \( X \) satisfy the \( (S_2) \) and \( (G_1) \) condition), then the assertion has been well-known, see [3].

\section{2-canonical modules are ubiquitous in algebraic geometry.} If \( \mathcal{I} \) is a dualizing complex of a Noetherian scheme \( X \), then the lowest non-vanishing cohomology group of \( \mathcal{I} \) is semicanonical. A rank-one reflexive sheaf over a normal variety is 2-canonical.

\section{Section 2 is for preliminaries. Section 3 is devoted to the discussion of the first theme mentioned in the paragraph (1.2), while Section 4 is for the second theme mentioned in (1.3).}

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The essential part of this paper has first appeared as [5, sections 9–10]. When it is published as [6], they have been removed after the requirement to shorten the paper (also, the title has been changed slightly). Here we revive them as an independent paper.

\section{Preliminaries}

\section{Throughout this paper, \( R \) denotes a Noetherian commutative ring. For a module-finite \( R \)-algebra \( \Lambda \), a \( \Lambda \)-module means a left \( \Lambda \)-module. \( \Lambda^{\text{op}} \) denotes the opposite algebra of \( \Lambda \), and thus a \( \Lambda^{\text{op}} \)-module is identified with a right \( \Lambda \)-module. A \( \Lambda \)-bimodule means a \( \Lambda \otimes_R \Lambda^{\text{op}} \)-module. The category of finite \( \Lambda \)-modules is denoted by \( \Lambda \text{-mod} \). The category \( \Lambda^{\text{op}} \text{-mod} \) is also denoted by \( \text{mod} \Lambda \).

\section{Let \( (R, \mathfrak{m}) \) be semilocal and \( \Lambda \) be a module-finite \( R \)-algebra. For an \( R \)-module \( M \), the \( \mathfrak{m} \)-adic completion of \( M \) is denoted by \( \hat{M} \). For a finite \( \Lambda \)-module \( M \), by \( \dim M \) or \( \dim_{\Lambda} M \) we mean \( \dim_R M \), which is independent of the choice of \( R \). By \( \text{depth}_M \) or \( \text{depth}_{\Lambda} M \) we mean \( \text{depth}_R(\mathfrak{m}, M) \), which is independent of \( R \). We say that \( M \) is globally Cohen–Macaulay (GCM for
short) if \( \dim M = \text{depth} M \). We say that \( M \) is globally maximal Cohen–Macaulay (GMCM for short) if \( \dim \Lambda = \text{depth} M \). If \( R \) happens to be local, then \( M \) is GCM (resp. GMCM) if and only if \( M \) is Cohen–Macaulay (resp. maximal Cohen–Macaulay) as an \( R \)-module.

\[(2.3)\] For \( M \in \Lambda \text{mod} \), we say that \( M \) satisfies \((S'_n)_{\Lambda,R} \) or \((S'_n)_{R} \) if \( \text{depth}_{R^P} M_P \geq \min(n, \text{ht}_P) \) for every \( P \in \text{Spec} R \) (this notion depends on \( R \)).

\[(2.4)\] Let \( X \) be a locally Noetherian scheme and \( \Lambda \) a coherent \( \mathcal{O}_X \)-algebra. For a coherent \( \Lambda \)-module \( M \), we say that \( M \) satisfies \((S'_n)_{\Lambda,X} \) or \((S'_n)_{X} \) if \( \text{depth}_{\mathcal{O}_{X,x}} M_x \geq \min(n, \text{dim} \mathcal{O}_{X,x}) \) for every \( x \in X \).

\[(2.5)\] Assume that \((R, \mathfrak{m})\) is complete semilocal, and \( \Lambda \neq 0 \) a module-finite \( R \)-algebra. Let \( \mathbb{I} \) be a normalized dualizing complex of \( R \). The lowest non-vanishing cohomology group \( \text{Ext}^{-s}_{R^?}(\Lambda, \mathbb{I}) \) (\( \text{Ext}^{i}_{R}(\Lambda, \mathbb{I}) = 0 \) for \( i < -s \)) is denoted by \( K_{\Lambda} \), and is called the canonical module of \( \Lambda \). If \( \Lambda = 0 \), then we define that \( K_{\Lambda} = 0 \). For basics on the canonical modules, we refer the reader to [6]. Note that \( K_{\Lambda} \) depends only on \( \Lambda \), and is independent of \( R \).

\[(2.6)\] Assume that \((R, \mathfrak{m})\) is semilocal which may not be complete. We say that a finitely generated \( \Lambda \)-bimodule \( K \) is a canonical module of \( \Lambda \) if \( \hat{K} \) is isomorphic to the canonical module \( R^P \otimes_R \Lambda \) for any \( P \in \text{supp} \mathcal{O}_\omega \).

\[(2.7)\] We say that \( \mathcal{O}_\omega \) is an \( R \)-semicanonical right \( \Lambda \)-module if for any \( P \in \text{Spec} R \), \( R_P \otimes_R \mathcal{O}_\omega \) is the right canonical module \( R_P \otimes_R \Lambda \) for any \( P \in \text{supp}_R \mathcal{O}_\omega \).

\[(2.8)\] Let \( C \in \text{mod} \Lambda \). We say that \( C \) is an \( n \)-canonical right \( \Lambda \)-module over \( R \) if \( C \in (S'_n)^R \), and for each \( P \in \text{Spec} R \) with \( \text{ht} P < n \), we have that \( C_P \) is an \( R_P \)-semicanonical right \( \Lambda_P \)-module.

3. **Symmetric and Frobenius algebras**

\[(3.1)\] Let \((R, \mathfrak{m})\) be a Noetherian semilocal ring, and \( \Lambda \) a module-finite \( R \)-algebra. Let \( K_{\Lambda} \) denote the canonical module of \( \Lambda \), see [6].

We say that \( \Lambda \) is quasi-symmetric if \( \hat{\Lambda} \) is the canonical module of \( \Lambda \). That is, \( \Lambda \cong K_{\Lambda} \) as \( \Lambda \)-bimodules. It is called symmetric if it is quasi-symmetric and GCM. Note that \( \Lambda \) is quasi-symmetric (resp. symmetric) if and only if \( \hat{\Lambda} \) is so. Note also that quasi-symmetric and symmetric are absolute notion,
and is independent of the choice of $R$ in the sense that the definition does not change when we replace $R$ by the center of $\Lambda$, because $K_\Lambda$ is independent of the choice of $R$.

(3.2) For (non-semilocal) Noetherian ring $R$, we say that $\Lambda$ is locally quasi-symmetric (resp. locally symmetric) over $R$ if for any $P \in \text{Spec} R$, $\Lambda_P$ is a quasi-symmetric (resp. symmetric) $R_P$-algebra. This is equivalent to say that for any maximal ideal $m$ of $R$, $\Lambda_m$ is quasi-symmetric (resp. symmetric), see [6, (7.6)].

In the case that $(R,m)$ is semilocal, $\Lambda$ is locally quasi-symmetric (resp. locally symmetric) over $R$ if it is quasi-symmetric (resp. symmetric), but the converse is not true in general.

Lemma 3.3. Let $(R,m)$ be a Noetherian semilocal ring, and $\Lambda$ a module-finite $R$-algebra. Then the following are equivalent.

1. $\Lambda_\Lambda$ is the right canonical module of $\Lambda$.
2. $\Lambda_\Lambda$ is the left canonical module of $\Lambda$.

Proof. We may assume that $R$ is complete. Then replacing $R$ by a Noether normalization of $R/\text{ann}_R \Lambda$, we may assume that $R$ is regular and $\Lambda$ is a faithful $R$-module.

We prove $1 \Rightarrow 2$. By [6, Lemma 5.10], $K_\Lambda$ satisfies $(S'_2)^R$. By assumption, $\Lambda_\Lambda$ satisfies $(S'_2)^R$. As $R$ is regular and $\dim R = \dim \Lambda$, $K_\Lambda = \Lambda^* = \text{Hom}_R(\Lambda, R)$. So we get an $R$-linear map

\[ \varphi : \Lambda \otimes_R \Lambda \to R \]

such that $\varphi(ab \otimes c) = \varphi(a \otimes bc)$ and that the induced map $h : \Lambda \to \Lambda^*$ given by $h(a)(c) = \varphi(a \otimes c)$ is an isomorphism (in mod $\Lambda$). Now $\varphi$ induces a homomorphism $h' : \Lambda \to \Lambda^*$ in $\Lambda$ mod given by $h'(c)(a) = \varphi(a \otimes c)$. To verify that this is an isomorphism, as $\Lambda$ and $\Lambda^*$ are reflexive $R$-modules, we may localize at a prime $P$ of $R$ of height at most one, and then take a completion, and hence we may further assume that $\dim R \leq 1$. Then $\Lambda$ is a finite free $R$-module, and the matrices of $h$ and $h'$ are transpose each other. As the matrix of $h$ is invertible, so is that of $h'$, and $h'$ is an isomorphism.

$2 \Rightarrow 1$ follows from $1 \Rightarrow 2$, considering the opposite ring. □

Definition 1. Let $(R,m)$ be semilocal. We say that $\Lambda$ is a pseudo-Frobenius $R$-algebra if the equivalent conditions of Lemma 3.3 are satisfied. If $\Lambda$ is GCM in addition, then it is called a Frobenius $R$-algebra. Note that these definitions are independent of the choice of $R$. Moreover, $\Lambda$ is pseudo-Frobenius (resp. Frobenius) if and only if $\hat{\Lambda}$ is so. For a general $R$, we say that $\Lambda$ is locally pseudo-Frobenius (resp. locally Frobenius) over $R$ if $\Lambda_P$ is pseudo-Frobenius (resp. Frobenius) for $P \in \text{Spec} R$. 
Lemma 3.4. Let \((R, \mathfrak{m})\) be semilocal. Then the following are equivalent.

1. \((K_\mathfrak{A})_\mathfrak{A}\) is projective in \(\text{mod } \mathfrak{A}\).
2. \(\mathfrak{A}(K_\mathfrak{A})\) is projective in \(\mathfrak{A}\text{ mod}\),

where \(\mathfrak{A}\) denotes the \(\mathfrak{m}\)-adic completion.

Proof. We may assume that \((R, \mathfrak{m}, k)\) is complete regular local and \(\Lambda\) is a faithful \(R\)-module. Let \(\bar{}\) denote the \(\mathfrak{m}\)-adic completion. Then \(\bar{\Lambda}\) is a finite dimensional \(k\)-algebra. So \(\text{mod } \bar{\Lambda}\) and \(\bar{\Lambda}\text{ mod}\) have the same number of simple modules, say \(n\). An indecomposable projective module in \(\text{mod } \Lambda\) is nothing but the projective cover of a simple module in \(\text{mod } \bar{\Lambda}\). So \(\text{mod } \Lambda\) and \(\bar{\Lambda}\text{ mod}\) have \(n\) indecomposable projectives. Now \(\text{Hom}_R(?, R)\) is an equivalence between \(\text{add}(K_\Lambda)\Lambda\) and \(\text{add}_\Lambda \Lambda\). It is also an equivalence between \(\text{add}_\Lambda (K_\Lambda)\) and \(\text{add}_\Lambda \Lambda\). So both \(\text{add}(K_\Lambda)\Lambda\) and \(\text{add}_\Lambda (K_\Lambda)\) also have \(n\) indecomposables. So 1 is equivalent to \(\text{add}(K_\Lambda)\Lambda = \text{add}_\Lambda \Lambda\). 2 is equivalent to \(\text{add}_\Lambda (K_\Lambda) = \text{add}_\Lambda \Lambda\). So 1\(\iff\)2 is proved simply applying the duality \(\text{Hom}_R(?, R)\). \(\Box\)

(3.5) Let \((R, \mathfrak{m})\) be semilocal. If the equivalent conditions in Lemma 3.4 are satisfied, then we say that \(\Lambda\) is pseudo-quasi-Frobenius. If it is GCM in addition, then we say that it is quasi-Frobenius. These definitions are independent of the choice of \(R\). Note that \(\Lambda\) is pseudo-quasi-Frobenius (resp. quasi-Frobenius) if and only if \(\bar{\Lambda}\) is so.

Proposition 3.6. Let \((R, \mathfrak{m})\) be semilocal. Then the following are equivalent.

1. \(\Lambda\) is quasi-Frobenius.
2. \(\Lambda\) is GCM, and \(\text{dim } \Lambda = \text{idim}_\Lambda \Lambda\), where \(\text{idim}\) denotes the injective dimension.
3. \(\Lambda\) is GCM, and \(\text{dim } \Lambda = \text{idim } \Lambda\Lambda\).

Proof. 1\(\Rightarrow\)2. By definition, \(\Lambda\) is GCM. To prove that \(\text{dim } \Lambda = \text{idim}_\Lambda \Lambda\), we may assume that \(R\) is local. Then by [1, (3.5)], we may assume that \(R\) is complete. Replacing \(R\) by the Noetherian normalization of \(R/\text{ann}_R \Lambda\), we may assume that \(R\) is a complete regular local ring of dimension \(d\), and \(\Lambda\) its maximal Cohen–Macaulay (that is, finite free) module. As \(\text{add}_\Lambda \Lambda = \text{add}_\Lambda (K_\Lambda)\) by the proof of Lemma 3.4, it suffices to prove \(\text{idim}_\Lambda (K_\Lambda) = d\). Let \(\Pi_R\) be the minimal injective resolution of the \(R\)-module \(R\). Then \(\mathcal{J} = \text{Hom}_R(\Lambda, \Pi_R)\) is an injective resolution of \(K_\Lambda = \text{Hom}_R(\Lambda, R)\) as a left \(\Lambda\)-module. As the length of \(\mathcal{J}\) is \(d\) and

\[
\text{Ext}^d_\Lambda(\Lambda/\mathfrak{m}\Lambda, K_\Lambda) \cong \text{Ext}^d_R(\Lambda/\mathfrak{m}\Lambda, R) \neq 0,
\]

we have that \(\text{idim}_\Lambda (K_\Lambda) = d\).
2⇒1. We may assume that $R$ is complete regular local and $\Lambda$ is maximal Cohen–Macaulay. By [1, (3.6)], we may further assume that $R$ is a field. Then $\Lambda$ is injective. So $(K_\Lambda)_{\Lambda} = \text{Hom}_R(\Lambda, R)$ is projective, and $\Lambda$ is quasi-Frobenius, see [8, (IV.3.7)].

$1\Leftrightarrow 3$ is proved similarly. □

**Corollary 3.7.** Let $R$ be arbitrary. Then the following are equivalent.

1. For any $P \in \text{Spec } R$, $\Lambda_P$ is quasi-Frobenius.
2. For any maximal ideal $m$ of $R$, $\Lambda_m$ is quasi-Frobenius.
3. $\Lambda$ is a Gorenstein $R$-algebra in the sense that $\Lambda$ is a Cohen–Macaulay $R$-module, and $\text{idim}_{\Lambda_P} \Lambda_P = \dim \Lambda_P$ for any $P \in \text{Spec } R$.

**Proof.** $1\Rightarrow 2$ is trivial.

$2\Rightarrow 3$. By Proposition 3.6, we have $\text{idim}_{\Lambda_m} \Lambda_m = \dim \Lambda_m$ for each $m$. Then by [1, (4.7)], $\Lambda$ is a Gorenstein $R$-algebra.

$3\Rightarrow 1$ follows from Proposition 3.6. □

(3.8) Let $R$ be arbitrary. We say that $\Lambda$ is a quasi-Gorenstein $R$-algebra if $\Lambda_P$ is pseudo-quasi-Frobenius for each $P \in \text{Spec } R$.

**Definition 2** (Scheja–Storch [7]). Let $R$ be general. We say that $\Lambda$ is symmetric (resp. Frobenius) relative to $R$ if $\Lambda$ is $R$-projective, and $\Lambda^* := \text{Hom}_R(\Lambda, R)$ is isomorphic to $\Lambda$ as a $\Lambda$-bimodule (resp. as a right $\Lambda$-module).

It is called quasi-Frobenius relative to $R$ if the right $\Lambda$-module $\Lambda^*$ is projective.

**Lemma 3.9.** Let $(R, m)$ be local.

1. If $\dim \Lambda = \dim R$, $R$ is quasi-Gorenstein, and $\Lambda^* \cong \Lambda$ as $\Lambda$-bimodules (resp. $\Lambda^* \cong \Lambda$ as right $\Lambda$-modules, $\Lambda^*$ is projective as a right $\Lambda$-module), then $\Lambda$ is quasi-symmetric (resp. pseudo-Frobenius, pseudo-quasi-Frobenius).

2. Assume that $R$ is Gorenstein. If $\Lambda$ is symmetric (resp. Frobenius, quasi-Frobenius) relative to $R$, then $\Lambda$ is symmetric (resp. Frobenius, quasi-Frobenius).

3. If $\Lambda$ is nonzero and $R$-projective, then $\Lambda$ is quasi-symmetric (resp. pseudo-Frobenius, pseudo-quasi-Frobenius) if and only if $R$ is quasi-Gorenstein and $\Lambda$ is symmetric (resp. Frobenius, quasi-Frobenius) relative to $R$.

4. If $\Lambda$ is nonzero and $R$-projective, then $\Lambda$ is symmetric (resp. Frobenius, quasi-Frobenius) if and only if $R$ is Gorenstein and $\Lambda$ is symmetric (resp. Frobenius, quasi-Frobenius) relative to $R$.

**Proof.** We can take the completion, and we may assume that $R$ is complete local.
1. Let \( d = \dim \Lambda = \dim R \), and let \( \mathbb{I} \) be the normalized dualizing complex (see [6, (5.2)]) of \( R \). Then
\[
K_\Lambda = \text{Ext}_R^{-d}(\Lambda, \mathbb{I}) \cong \text{Hom}_R(\Lambda, H^{-d}(\mathbb{I})) \cong \text{Hom}(\Lambda, K_R) \cong \text{Hom}(\Lambda, R) = \Lambda^*
\]
as \( \Lambda \)-bimodules, and the result follows.

2. We may assume that \( \Lambda \) is nonzero. As \( R \) is Cohen–Macaulay and \( \Lambda \) is a finite projective \( R \)-module, \( \Lambda \) is a maximal Cohen–Macaulay \( R \)-module. By 1, the result follows.

3. The ‘if’ part follows from 1. We prove the ‘only if’ part. As \( \Lambda \) is \( R \)-projective and nonzero, \( \dim \Lambda = \dim R \). As \( \Lambda \) is \( R \)-finite free, \( K_\Lambda \cong \text{Hom}_R(\Lambda, K_R) \cong \Lambda^* \otimes_R K_R \). As \( K_\Lambda \) is \( R \)-free and \( \Lambda^* \otimes_R K_R \) is nonzero and isomorphic to a direct sum of copies of \( K_R \), we have that \( K_R \) is \( R \)-projective, and hence \( R \) is quasi-Gorenstein, and \( K_R \cong R \). Hence \( K_\Lambda \cong \Lambda^* \), and the result follows. 4 follows from 3 easily. \( \square \)

(3.10) Let \((R, m)\) be semilocal. Let a finite group \( G \) act on \( \Lambda \) by \( R \)-algebra automorphisms. Let \( \Omega = \Lambda^* \otimes_R \Lambda^* \) as \( \Lambda \)-bimodules, and the result follows.

(3.11) We simply call an \( RG \)-module a \( G \)-module. We say that \( M \) is a \((G, \Lambda)\)-module if \( M \) is a \( G \)-module, \( \Lambda \)-module, the \( R \)-module structures coming from that of the \( G \)-module structure and the \( \Lambda \)-module structure agree, and \( g(am) = (ga)(gm) \) for \( g \in G \), \( a \in \Lambda \), and \( m \in M \). A \((G, \Lambda)\)-module and an \( \Omega \)-module are one and the same thing.

(3.12) By the action \(((a \otimes a')g)a_1 = a(ga_1)a' \), we have that \( \Lambda \) is a \((\Lambda \otimes \Lambda^\text{op}) \ast G \)-module in a natural way. So it is an \( \Omega \)-module by the action \((ag)a_1 = a(ga_1)\). It is also a right \( \Omega \)-module by the action \( a_1(ag) = g^{-1}(a_1a) \). If the action of \( G \) on \( \Lambda \) is trivial, then these actions make an \( \Omega \)-bimodule.

(3.13) Given an \( \Omega \)-module \( M \) and an \( RG \)-module \( V \), \( M \otimes_R V \) is an \( \Omega \)-module by \((ag)(m \otimes v) = (ag)m \otimes gv \). \( \text{Hom}_R(M, V) \) is a right \( \Omega \)-module by \((\varphi(ag))(m) = g^{-1}\varphi(a(gm)) \). It is easy to see that the standard isomorphism
\[
\text{Hom}_R(M \otimes_R V, W) \rightarrow \text{Hom}_R(M, \text{Hom}_R(V, W))
\]
is an isomorphism of right \( \Omega \)-modules for a left \( \Omega \)-module \( M \) and \( G \)-modules \( V \) and \( W \).

(3.14) Now consider the case \( \Lambda = R \). Then the pairing \( \phi : RG \otimes_R RG \rightarrow R \) given by \( \phi(g \otimes g') = \delta_{gg',e} \) (Kronecker’s delta) is non-degenerate, and induces
an $RG$-bimodule isomorphism $\Omega = RG \to (RG)^* = \Omega^*$. As $\Omega = RG$ is a finite free $R$-module, we have that $\Omega = RG$ is symmetric relative to $R$.

**Lemma 3.15.** If $\Lambda$ is quasi-symmetric (resp. symmetric) and the action of $G$ on $\Lambda$ is trivial, then $\Omega$ is quasi-symmetric (resp. symmetric).

**Proof.** Taking the completion, we may assume that $R$ is complete. Then replacing $R$ by a Noether normalization of $R/\text{ann}_R \Lambda$, we may assume that $R$ is a regular local ring, and $\Lambda$ is a faithful $R$-module. As the action of $G$ on $\Lambda$ is trivial, $\Omega = \Lambda \otimes_R RG$ is quasi-symmetric (resp. symmetric), as can be seen easily. $\square$

(3.16) In particular, if $\Lambda$ is commutative quasi-Gorenstein (resp. Gorenstein) and the action of $G$ on $\Lambda$ is trivial, then $\Omega = \Lambda G$ is quasi-symmetric (resp. symmetric).

(3.17) In general, $\Omega \Lambda \cong \Lambda \otimes_R RG$ as $\Omega$-modules.

**Lemma 3.18.** Let $M$ and $N$ be right $\Omega$-modules, and let $\varphi : M \to N$ be a homomorphism of right $\Lambda$-modules. Then $\psi : M \otimes_R RG \to N \otimes_R RG$ given by
\[
\psi(m \otimes g) = g(\varphi(g^{-1}m)) \otimes g
\]
is an $\Omega$-homomorphism. In particular,
1. If $\varphi$ is a $\Lambda$-isomorphism, then $\psi$ is an $\Omega$-isomorphism.
2. If $\varphi$ is a split monomorphism in $\text{mod } \Lambda$, then $\psi$ is a split monomorphism in $\text{mod } \Omega$.

**Proof.** Straightforward. $\square$

**Proposition 3.19.** Let $G$ be a finite group acting on $\Lambda$. Set $\Omega := \Lambda \ast G$.

1. If the action of $G$ on $\Lambda$ is trivial and $\Lambda$ is quasi-symmetric (resp. symmetric), then so is $\Omega$.
2. If $\Lambda$ is pseudo-Frobenius (resp. Frobenius), then so is $\Omega$.
3. If $\Lambda$ is pseudo-quasi-Frobenius (resp. quasi-Frobenius), then so is $\Omega$.

**Proof.** 1 is Lemma 3.15. To prove 2 and 3, we may assume that $(R, \mathfrak{m})$ is complete regular local and $\Lambda$ is a faithful module.

2. 

\[(K_\Omega)_\Omega \cong \text{Hom}_R(\Lambda \otimes_R RG, R) \cong \text{Hom}_R(\Lambda, R) \otimes (RG)^* \cong K_\Lambda \otimes RG\]
as right $\Omega$-modules. It is isomorphic to $\Lambda_\Omega \otimes RG \cong \Omega_\Omega$ by Lemma 3.18, 1, since $K_\Lambda \cong \Lambda$ in $\text{mod } \Lambda$. Hence $\Omega$ is pseudo-Frobenius. If, in addition, $\Lambda$ is Cohen–Macaulay, then $\Omega$ is also Cohen–Macaulay, and hence $\Omega$ is Frobenius.

3 is proved similarly, using Lemma 3.18, 2. $\square$

Note that the assertions for Frobenius and quasi-Frobenius properties also follow easily from Lemma 3.9 and [7, (3.2)].
4. Codimension-two argument

(4.1) This section is the second part of this paper. In this section, we show that the codimension-two argument using the existence of 2-canonical modules in [4] is still valid in non-commutative settings, as announced in (1.3).

(4.2) Let $X$ be a locally Noetherian scheme, $U$ its open subscheme, and $\Lambda$ a coherent $O_X$-algebra. Let $i : U \hookrightarrow X$ be the inclusion.

(4.3) Let $M \in \text{mod } \Lambda$. That is, $M$ is a coherent right $\Lambda$-module. Then by restriction, $i^* M \in \text{mod } i^* \Lambda$.

(4.4) For a quasi-coherent $i^* \Lambda$-module $N$, we have an action

$$i_* N \otimes_{O_X} \Lambda \xrightarrow{1 \otimes u} i_* N \otimes_{O_X} i_* i^* \Lambda \rightarrow i_* (N \otimes_{O_U} i^* \Lambda) \xrightarrow{u} i_* N,$$

where $u$ is the unit map for the adjoint pair $(i^*, i_*)$. So we get a functor $i^* : \text{Mod } i^* \Lambda \rightarrow \text{Mod } \Lambda$, where $\text{Mod } i^* \Lambda$ (resp. $\text{Mod } \Lambda$) denote the category of quasi-coherent $i^* \Lambda$-modules (resp. $\Lambda$-modules).

**Lemma 4.5.** Let the notation be as above. Assume that $U$ is large in $X$ (that is, $\text{codim}_X (X \setminus U) \geq 2$). If $M \in (S'_2)^{\Lambda^{\text{op}},X}$, then the canonical map $u : M \rightarrow i_* i^* M$ is an isomorphism.

**Proof.** Follows immediately from [4, (7.31)].

**Proposition 4.6.** Let the notation be as above, and let $U$ be large in $X$. Assume that there is a 2-canonical right $\Lambda$-module. Then we have the following.

1. If $N \in (S'_2)^{i^* \Lambda^{\text{op}},U}$, then $i_* N \in (S'_2)^{i^* \Lambda^{\text{op}},X}$.
2. $i^* : (S'_2)^{i^* \Lambda^{\text{op}},X} \rightarrow (S'_2)^{i^* \Lambda^{\text{op}},U}$ and $i_* : (S'_2)^{i^* \Lambda^{\text{op}},U} \rightarrow (S'_2)^{i^* \Lambda^{\text{op}},X}$ are quasi-inverse each other.

**Proof.** The question is local, and we may assume that $X$ is affine.

1. There is a coherent subsheaf $Q$ of $i_* N$ such that $i^* Q = i^* i_* N = N$ by [2, Exercise II.5.15]. Let $V$ be the $\Lambda$-submodule of $i_* N$ generated by $Q$.

That is, the image of the composite

$$Q \otimes_{O_X} \Lambda \rightarrow i_* N \otimes_{O_X} \Lambda \rightarrow i_* N.$$

Note that $V$ is coherent, and $i^* Q \subset i^* V \subset i^* i_* N = i^* Q = N$.

Let $C$ be a 2-canonical right $\Lambda$-module. Let $?^\dagger := \text{Hom}_{\Lambda^{\text{op}}} (? , C)$, $\Gamma = \text{End}_{\Lambda} C$, and $?^\ddagger := \text{Hom}_{\Gamma} (?) , C$. Let $M$ be the double dual $V^{\dagger \ddagger}$. Then $M \in (S'_2)^{i^* \Lambda^{\text{op}},X}$, and hence

$$M \cong i_* i^* M \cong i_* i^* (V^{\dagger \ddagger}) \cong i_* (i^* V)^{\dagger \ddagger} \cong i_* (N^{\dagger \ddagger}) \cong i_* N.$$

So $i_* N \cong M$ lies in $(S'_2)^{i^* \Lambda^{\text{op}},X}$. 
2 follows from 1 and Lemma 4.5 immediately. □

References


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