Doctor's Thesis

Lyapunov Functionals for Virus-Immune Models with Infinite Delay — One-Strain and Multistrain Models

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Abstract

We have studied the asymptotical behavior of the solutions of the delay differential equation models. To analyze the global stability of the equilibria of those models, Lyapunov functions or Lyapunov functionals are useful and effective. In many papers various techniques to construct Lyapunov functions or functionals and various techniques to prove the nonpositivity of their time derivative have been introduced. For delay differential equations, McCluskey showed the useful integration term. To prove the nonpositivity of the time derivative of the Lyapunov functionals the extension of arithmetic-geometric mean inequality is effective. By these methods the analysis of the global stability of equilibria has progressed. However the general method of the construction of the Lyapunov functions or functionals are unknown.

In this thesis we present a systematic method to construct Lyapunov functionals of several delay differential models of infection disease in vivo. We start with the simple model without delay and finally we consider the complexed multistrain model. The Lyapunov functionals are constructed systematically. In the analysis of the stability of the equilibria, there exist some mathematical difficulties. We have made rigorous solutions for those difficulties.

The thesis is organized as follows. At first we introduce the simple model and describe the notations and definitions. Secondary we analyze a single strain models and prepare the analysis of successive multistrain models. Finally we analyze some multistrain models such as the model with absorption effect and the model with immune variables. The global stability of the equilibria with the competitive exclusion principle and the coexistence of strains are explained rigorously mathematical method.

1 Introduction

There often exist multiple strains of pathogens such as HIV, hepatitis C, Epstein-Barr virus, dengue fever, tuberculosis and malaria. For many diseases, the effect of multiple-strain infections remains still unclear, but theoretical works and experimental results from animal models are clarifying it gradually.

Mathematical modeling and analysis of virus dynamics have been subjects of extensive research. It is needless to say that the stability of equilibria is important. For the analysis of the stability, Lyapunov functions and Lyapunov functionals are useful methods [11]. But the main difficulty we face is the lack of a general method of constructing a Lyapunov function or a Lyapunov functional. For delay differential equations (DDE), McCluskey [12] showed the useful technique using the integration term to construct Lyapunov functionals for DDE. But in many papers after this the calculations for the Lyapunov functionals are often complicated.

In this paper we propose a systematic method with small calculation to construct Lyapunov functionals. At first we start with a simple model and a simple Lyapunov function. Adding several appropriate terms to the known Lyapunov function or functional we can construct a new Lyapunov functional for the complicated model. Also we can construct Lyapunov functionals for the multistrain models by combining the Lyapunov functionals of the single strain models. Those methods in constructing Lyapunov functionals are simple. In the case of multistrain models, the competitive exclusion principle is found for the model with absorption effect without immune variables, and the condition for coexistence of multiple strains is shown for the model with absorption effect and immune variables.

Some recent papers [9], [10] referred to the construction of the Lyapunov functions or functionals. And the infinitely distributed delay was taken into account in several papers [12], [17], [19]. We try to make an accurate description on the mathematical aspect. We describe the definition of the fading memory space, the asymptotic smoothness of the system, the uniform persistence of the virus under $R_0 > 1$, the well-definedness of the Lyapunov functionals and so on.

The contents of this paper is as follows. In section 3, we consider a model with infinitely distributed delay with absorption effect without an immune variable. In section 4, we consider models with an immune variable. For the model with absorption effect without an immune variable, we describe the detail of the mathematical argument. In the model with immune variable, we describe only outline for several repeated mathematical argument.

2 Preliminary model

At first we start with a fundamental one-strain model of infection [15] that contains the uninfected cells x, infected cells y and viruses v such as

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \lambda - \delta x - \beta x v, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \beta x v - a y, \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= a r y - b v. \end{aligned} \tag{2.1}$$

We introduce a generalized simple model, which includes (2.1) as a special case. By using the exponential delay kernel we can reconstruct the model which has three variables from the model with two variables x and v. For example we define the function $y(t)$ as follows :

$$a y(t) = \beta \int_0^\infty g(s) x(t-s) v(t-s) \mathrm{d}s,$$

where $g(s) = \exp\{-as\}/a$. Then the time derivative of $y(t)$ becomes as follows :

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= \beta \int_0^\infty g(s) \left\{ x(t)v(t) - x(t-s)v(t-s) \right\} \mathrm{d}s \\ &= \beta x v - a y, \end{aligned}$$

and the third equation of (2.1) can be rewritten as follows :

$$\frac{\mathrm{d}v}{\mathrm{d}t} = r\beta \int_0^\infty g(s) x(t-s) v(t-s) \mathrm{d}s - bv.$$

Generally the model with the distributional delay should be considered. It includes the case of finite delay or discrete delay as a particular case of it. In this section we introduce some simple models.

Recently, the construction of the Lyapunov functional of the age-structured model is considered concerned with the delay differential equation model.

The function $x(t)$ denotes the population of uninfected cells, $v(t)$ the population of viruses and $y(a,t)$ the age-specific concentration of infected cells with infection age a and time t. The parameter λ denotes the recruitment rate, δ and b the natural death rate of uninfected cells and viruses respectively, β the contact rate, $\eta(a)$ the death rate of infected cells with infection age a, $k(a)$ the virus production rate of infected cells at age a. Then we have

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \lambda - \delta x - \beta x v \\ \frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} &= -\eta(a) y(a,t) \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= \int_0^\infty k(a) y(a,t) \, \mathrm{d}a - bv \\ y(0,t) &= \beta x(t) v(t). \end{aligned} \tag{2.2}$$

The second equation can be solved by the integral along the characteristic curve as follows :

$$\begin{aligned} y(a,t) &= \beta \sigma(a) x(t-a) v(t-a) \quad (0 < a < t) \\ y(a,t) &= y_0(a-t) \exp\left\{ -\int_{a-t}^{a} \eta(s) \mathrm{d}s \right\} \quad (t \le a). \end{aligned}$$

where $y_0(\cdot) = y(\cdot, 0)$. Substitute this equation to the third equation of (2.2), we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda - \delta x - \beta xv$$
$$\frac{\mathrm{d}v}{\mathrm{d}t} = \beta \int_0^t k(a)\sigma(a)v(t-a)x(t-a)\,\mathrm{d}a - bv + F(t),$$

where

$$\sigma(a) = \exp\left\{-\int_0^a \eta(s)\,ds\right\},$$

and

$$F(t) = \int_t^\infty k(a)y_0(a-t)\exp\left\{-\int_{a-t}^a \eta(s)ds\right\}\mathrm{d}a.$$

The asymptotic behavior of this model is the same as the following model [13] by $\lim_{t\to\infty} F(t) = 0$:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda - \delta x - \beta xv$$
$$\frac{\mathrm{d}v}{\mathrm{d}t} = \beta \int_0^\infty k(a)\sigma(a)v(t-a)x(t-a)\,\mathrm{d}a - bv.$$

3 Model with distributed delay with absorption effect without an immune variable

When pathogens infect uninfected cells, the number of pathogens decreases. It is called the effect of absorption. Iggidr *et al.* [7] described the global stability of the model with absorption effect. We add the effect of absorption expressed by ρ, which is less than r, and consider the nonlinear incidence $\mu(x)$ where the function $\mu(x)$ satisfies $\mu(0) = 0$, $\mu(x)$ is strictly increasing and $\mu(x)/x$ is monotone nonincreasing for $x > 0$ as in [10]. Then we have

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \lambda - \delta x - \mu(x)v, \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= \int_0^\infty k(a)\sigma(a)\mu(x(t-a))v(t-a)\mathrm{d}a - \rho\mu(x)v - bv. \end{aligned} \tag{3.1}$$

The parameter λ denotes the recruitment rate of uninfected cells, δ the natural death rate for uninfected cells, β the contact rate between uninfected cells and viruses, b the natural death rate and $k(a)$ the viral production rate of an infected cell with infection-age a and the function $\sigma(a)$ is the probability that an infected cell survives to infection-age a. It is determined by the death rate $\eta(a)$ of infected cells as

$$\sigma(a) = \exp\left(-\int_0^a \eta(s)\mathrm{d}s\right). \tag{3.2}$$

The total number of viral particles produced by an infected cell in its life span is called the burst size r and it is defined by

$$r = \int_0^\infty k(a)\sigma(a)\,\mathrm{d}a, \tag{3.3}$$

where $0 \le k(a) \le K$ for some K and $\sigma(a)$ is integrable. The delay kernel is defined by $g(a) = k(a)\sigma(a)/r$, and the nonnegative function $g(a)$ satisfies $\int_0^\infty g(a)\,\mathrm{d}a = 1$. The model is rewritten to the differential equations with infinite delay as follows

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \lambda - \delta x - \mu(x)v, \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= r\int_0^\infty g(a)\mu(x(t-a))v(t-a)\,\mathrm{d}a - \rho\mu(x)v - bv, \end{aligned} \tag{3.4}$$

with the following initial condition

$$x(\theta) = \phi_0(\theta),\ v(\theta) = \phi_1(\theta),\ \theta \in (-\infty, 0]. \tag{3.5}$$

Due to the infinite delay, an appropriate phase space is required. We use the phase space of fading memory type [1, 5]. For $\Delta > 0$, let

$$C_\Delta = \{\varphi : (-\infty, 0] \to \mathbb{R} \mid \varphi(\theta)\mathrm{e}^{\Delta\theta} \text{ is bounded and uniformly continuous}\}, \tag{3.6}$$

$$Y_\Delta = \{\varphi \in C_\Delta \mid \varphi(\theta) \ge 0 \text{ for all } \theta \le 0\}. \tag{3.7}$$

Define the norm on C_Δ and Y_Δ by

$$\|\varphi\| = \sup_{\theta\le 0} |\varphi(\theta)\mathrm{e}^{\Delta\theta}|. \tag{3.8}$$

We define $\Delta_0 = \liminf_{a\to\infty} \int_0^a \eta(s)\mathrm{d}s/(2a)$. It is positive because we can assume, in a biological point of view, the death rate $\eta(s)$ of infection-age s does not decrease to zero. For any $\Delta \in (0, \Delta_0)$ we can take Δ_1 which satisfies $\Delta < \Delta_1 < \Delta_0$ and there exists a sufficiently large A such that for all $a \geq A$ it holds that

$$\frac{1}{2a}\int_0^a \eta(s)ds \geq \Delta_1. \tag{3.9}$$

Then the following inequality holds :

$$\sigma(a) = \exp\left(-\int_0^a \eta(s)ds\right) \leq \exp(-2\Delta_1 a). \tag{3.10}$$

Thus if the initial functions ϕ_0 and ϕ_1 belong to the phase space C_Δ then the integral of the right hand of the second equation of (3.4) converges. We consider the solution of system (3.4), (x_t, v_t) with the initial condition

$$\begin{gathered} x_0(\theta) = \phi_0(\theta),\ v_0(\theta) = \phi_1(\theta),\ \theta \in (-\infty, 0], \\ \phi_i \in Y_\Delta (i = 0, 1) \text{ and } x(0) > 0, v(0) > 0. \end{gathered} \tag{3.11}$$

3.1 Positivity and boundedness

The positivity and boundedness of the solution are shown as follows.

Proposition 3.1. *Let $x(t)$ and $v(t)$ be the solution of system* (3.4) *with the initial condition* (3.11)*, then $x(t)$ and $v(t)$ are positive for $t > 0$.*

Proof. Suppose that t_1 is the least positive time such that $x(t_1) = 0$, then $x'(t_1) = \lambda > 0$. Therefore there exists small $\epsilon > 0$ such that $x(t) < 0$ for $t = t_1 - \epsilon > 0$. It is a contradiction and it follows that $x(t) > 0$ for all $t > 0$. Similarly suppose that t_1 is the least positive time such that $v(t_1) = 0$, then $v(t) > 0$ for $0 \leq t < t_1$ and

$$\left.\frac{\mathrm{d}v}{\mathrm{d}t}\right|_{t=t_1} = r\int_0^\infty g(a)\mu(x(t_1 - a))v(t_1 - a)\,\mathrm{d}a. \tag{3.12}$$

Then by the positivity of $\mu(x(t_1 - a))v(t_1 - a)$ for $t_1 - a \in (0, t_1]$, the integral of (3.12) is positive for $t = t_1$. Therefore there exists a small $\epsilon > 0$ such that $v(t) < 0$ for $t = t_1 - \epsilon > 0$. It is a contradiction and it follows that $v(t) > 0$ for all $t > 0$. $\square$

It is important that $x(t)$ and $v(t)$ are bounded for sufficient large time t. Moreover it is also important that those are bounded until that time. From the first equation in (3.4) we have

$$x(t) = \left[x(0) + \int_0^t (\lambda - \mu(x)v)e^{\delta\xi}\mathrm{d}\xi\right] e^{-\delta t}.$$

Then

$$x(t) \leq \left[x(0) + \int_0^t \lambda e^{\delta\xi}\mathrm{d}\xi\right] e^{-\delta t} = \frac{\lambda}{\delta} + \left(x(0) - \frac{\lambda}{\delta}\right) e^{-\delta t}.$$

Thus we have the following proposition.

Proposition 3.2. *For each initial condition it holds that*

$$0 < x(t) \leq \max\left\{x(0), \frac{\lambda}{\delta}\right\} \text{ for } t \geq 0, \tag{3.13}$$

and there exists $T > 0$ such that $x(t) \leq \lambda/\delta + 1$ for $t \geq T$.

To see the boundedness of $v(t)$ we define $W_1(t)$ for $t > 0$ as follows

$$W_1(t) = x(t) + \int_0^\infty \mu(x(t-a))v(t-a)e^{-2\Delta_1 a}\mathrm{d}a. \tag{3.14}$$

Lemma 3.3. *Put* $\nu_1 = \min(\delta, 2\Delta_1)$. *For each initial condition it holds that*

$$W_1(t) \le \max\left\{W_1(0), \frac{\lambda}{\nu_1}\right\} \text{ for all } t \ge 0. \tag{3.15}$$

and there exists $T > 0$ *such that* $W_1(t) \le \lambda/\nu_1 + 1$ *for* $t \ge T$.

Proof. At first we confirm that the integral defining $W_1(t)$ converges for fixed each t. By using $\phi_i \in Y_\Delta$, it holds that $\phi_i(\theta) \le \|\phi_i\|e^{-\Delta\theta}$ for $\theta \in (-\infty, 0] \quad (i = 0, 1)$. Therefore the integrand of the second term of $W_1(t)$ is dominated by an integrable function as follows. For any $a \ge t$

$$x(t-a) \le \|\phi_0\|e^{-\Delta(t-a)}, \; v(t-a) \le \|\phi_1\|e^{-\Delta(t-a)}. \tag{3.16}$$

Then by the property of $\mu(x)$ it holds that

$$\mu(x(t-a)) \le \mu(\|\phi_0\|e^{-\Delta(t-a)}).$$

And $\|\phi_0\|e^{-\Delta(t-a)} \ge \|\phi_0\|$ leads

$$\frac{\mu\left(\|\phi_0\|e^{-\Delta(t-a)}\right)}{\|\phi_0\|e^{-\Delta(t-a)}} \le \frac{\mu(\|\phi_0\|)}{\|\phi_0\|}.$$

Therefore

$$\mu(x(t-a)) \le \mu(\|\phi_0\|)e^{-\Delta(t-a)}, \tag{3.17}$$

and

$$\mu(x(t-a))v(t-a)e^{-2\Delta_1 a} \le \mu(\|\phi_0\|)\|\phi_1\|e^{2(\Delta-\Delta_1)a}e^{-2\Delta t}. \tag{3.18}$$

Thus the integral converges because $\Delta - \Delta_1 < 0$.

Then the time derivative of $W_1(t)$ becomes as follows :

$$
\begin{aligned}
&\frac{\mathrm{d}W_1}{\mathrm{d}t} \\
&= \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_0^t \mu(x(t-a))v(t-a)e^{-2\Delta_1 a}\mathrm{d}a + \int_t^\infty \mu(x(t-a))v(t-a)e^{-2\Delta_1 a}\mathrm{d}a\right\} \\
&= \lambda - \delta x(t) - \mu(x(t))v(t) + \int_0^t \frac{\partial}{\partial t}\left\{\mu(x(t-a))v(t-a)\right\}e^{-2\Delta_1 a}\mathrm{d}a \\
&+ \mu(x(0))v(0)e^{-2\Delta_1 t} + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{-\infty} \mu(x(u))v(u)e^{-2\Delta_1(t-u)}(-\mathrm{d}u) \\
&= \lambda - \delta x(t) - \mu(x(t))v(t) - \int_0^t \frac{\partial}{\partial a}\left\{\mu(x(t-a))v(t-a)\right\}e^{-2\Delta_1 a}\mathrm{d}a \\
&+ \mu(x(0))v(0)e^{-2\Delta_1 t} + \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-2\Delta_1 t}\int_{-\infty}^0 \mu(x(u))v(u)e^{2\Delta_1 u}\mathrm{d}u\right) \\
&= \lambda - \delta x(t) - \mu(x(t))v(t) + \mu(x(0))v(0)e^{-2\Delta_1 t} \\
&- \left\{\left[\mu(x(t-a))v(t-a)e^{-2\Delta_1 a}\right]_0^t - \int_0^t \mu(x(t-a))v(t-a)\left(-2\Delta_1 e^{-2\Delta_1 a}\right)\mathrm{d}a\right\} \\
&- 2\Delta_1 e^{-2\Delta_1 t}\int_{-\infty}^0 \mu(x(u))v(u)e^{2\Delta_1 u}\mathrm{d}u \\
&= \lambda - \delta x(t) - \mu(x(t))v(t) + \mu(x(0))v(0)e^{-2\Delta_1 t} \\
&- \left\{\mu(x(0))v(0)e^{-2\Delta_1 t} - \mu(x(t))v(t)e^0 + 2\Delta_1\int_0^t \mu(x(t-a))v(t-a)e^{-2\Delta_1 a}\mathrm{d}a\right\} \\
&- 2\Delta_1\int_t^\infty \mu(x(t-a))v(t-a)e^{-2\Delta_1 a}\mathrm{d}a \\
&= \lambda - \left(\delta x(t) + 2\Delta_1\int_0^\infty \mu(x(t-a))v(t-a)e^{-2\Delta_1 a}\mathrm{d}a\right).
\end{aligned} \tag{3.19}
$$

Let $\nu_1 = \min(\delta, 2\Delta_1)$ then

$$
\frac{\mathrm{d}W_1}{\mathrm{d}t} \le \lambda - \nu_1\left(x(t) + \int_0^\infty \mu(x(t-a))v(t-a)e^{-2\Delta_1 a}\mathrm{d}a\right) = \lambda - \nu_1 W_1(t). \tag{3.20}
$$

It follows that

$$
W_1(t) \le \frac{\lambda}{\nu_1} + \left(W_1(0) - \frac{\lambda}{\nu_1}\right)e^{-\nu_1 t} \text{ for all } t \ge 0, \tag{3.21}
$$

and the statement of this lemma follows. □

The initial value $W_1(0)$ satisfies the following inequality :

$$
\begin{aligned}
W_1(0) &= x(0) + \int_0^\infty \mu(\phi_0(-a))\phi_1(-a)\mathrm{e}^{-2\Delta_1 a}\mathrm{d}a \\
&\le x(0) + \int_0^\infty \mu(\|\phi_0\|)\mathrm{e}^{\Delta a}\|\phi_1\|\mathrm{e}^{\Delta a}\mathrm{e}^{-2\Delta_1 a}\mathrm{d}a = x(0) + \frac{\mu(\|\phi_0\|)\|\phi_1\|}{2(\Delta_1 - \Delta)}.
\end{aligned} \tag{3.22}
$$

Proposition 3.4. *We can choose positive C and $\widetilde{C}$ that satisfy the following : for each initial condition $v(t) \le \max\{v(0), C/b\}$ for $t \ge 0$, and there exists $T > 0$ such that $v(t) \le \widetilde{C}/b + 1$ for $t \ge T$.*

Proof. By Lemma 3.3, it holds that for $t \geq 0$

$$\int_0^\infty \mu(x(t-a))v(t-a)\mathrm{e}^{-2\Delta_1 a}\mathrm{d}a \leq \max\left\{W_1(0), \frac{\lambda}{\nu_1}\right\}. \tag{3.23}$$

Using this result we will show the boundedness of $v(t)$. Assume that $k(a)$ is bounded and continuous. Let $\|k\|_\infty = \sup_{a\geq 0}\{k(a)\}$ and assume that $\sigma(a)$ is continuous. For $a \geq A$ it holds that $\sigma(a) \leq \mathrm{e}^{-2\Delta_1 a}$ and let $C_1 = \max\left\{\max_{0\leq a\leq A} \sigma(a)\mathrm{e}^{-2\Delta_1 a}, 1\right\}$. Then by using (3.23) it holds that for $t \geq 0$

$$\begin{aligned}\int_0^\infty k(a)\sigma(a)\mu(x(t-a))v(t-a)\,\mathrm{d}a &\leq \int_0^\infty \|k\|_\infty C_1 \mathrm{e}^{-2\Delta_1 a}\mu(x(t-a))v(t-a)\,\mathrm{d}a \\ &= C_1\|k\|_\infty \int_0^\infty \mu(x(t-a))v(t-a)\mathrm{e}^{-2\Delta_1 a}\,\mathrm{d}a \\ &\leq C_1\|k\|_\infty \max\left\{W_1(0), \frac{\lambda}{\nu_1}\right\}. \end{aligned} \tag{3.24}$$

Let $C = C_1\|k\|_\infty \max\{W_1(0), \lambda/\nu_1\}$, then

$$\frac{\mathrm{d}v}{\mathrm{d}t} \leq C - \rho\mu(x)v - bv \leq C - bv. \tag{3.25}$$

Then

$$v(t) \leq \frac{C}{b} + \left(v(0) - \frac{C}{b}\right)\mathrm{e}^{-bt}. \tag{3.26}$$

Therefore $v(t)$ is bounded for $t \geq 0$.

On the other hand there exists $T_1 > 0$ such that $W_1(t) \leq \lambda/\nu_1 + 1$ for $t \geq T_1$ by Lemma 3.3. By the positivity of $x(t)$ it holds that for $t \geq T_1$

$$\int_0^\infty \mu(x(t-a))v(t-a)\mathrm{e}^{-2\Delta_1 a}\mathrm{d}a \leq W_1(t) \leq \lambda/\nu_1 + 1. \tag{3.27}$$

Therefore it holds that for $t \geq T_1$

$$\begin{aligned}\int_0^\infty k(a)\sigma(a)\mu(x(t-a))v(t-a)\,\mathrm{d}a &= C_1\|k\|_\infty \int_0^\infty \mu(x(t-a))v(t-a)\mathrm{e}^{-2\Delta_1 a}\,\mathrm{d}a \\ &\leq C_1\|k\|_\infty \left(\frac{\lambda}{\nu_1} + 1\right). \end{aligned} \tag{3.28}$$

Let $\widetilde{C} = C_1\|k\|_\infty (\lambda/\nu_1 + 1)$, then it holds that for $t \geq T_1$

$$\frac{\mathrm{d}v}{\mathrm{d}t} \leq \widetilde{C} - \rho\mu(x)v - bv \leq \widetilde{C} - bv. \tag{3.29}$$

Then it holds that for $t \geq T_1$

$$v(t) \leq \frac{\widetilde{C}}{b} + \left(v(T_1) - \frac{\widetilde{C}}{b}\right)\mathrm{e}^{-b(t-T_1)}, \tag{3.30}$$

and there exists $T > T_1 > 0$ such that $v(t) \leq \widetilde{C}/b + 1$ for $t \geq T$. This $\widetilde{C}$ does not depend on the initial condition. □

Theorem 3.5. *Let (x, v) be the solution of system (3.4) satisfying the initial condition (3.11). Then there exist M_1 and M_2 such that $\|x_t\| \leq M_1$ and $\|v_t\| \leq M_2$ for $t \geq 0$ where M_1 and M_2 are independent of t .*

Proof. For any $t \geq 0$,

$$\begin{aligned}
\|x_t\| &= \sup_{s \leq 0} x_t(s)\mathrm{e}^{\Delta s} = \sup_{u \leq t} x(u)\mathrm{e}^{\Delta u}\mathrm{e}^{-\Delta t} \\
&= \max\left\{\sup_{u \leq 0} x(u)\mathrm{e}^{\Delta u}\mathrm{e}^{-\Delta t}, \max_{0 \leq u \leq t} x(u)\mathrm{e}^{\Delta u}\mathrm{e}^{-\Delta t}\right\} \\
&\leq \max\left\{\|\phi_0\|\mathrm{e}^{-\Delta t}, \max_{0 \leq u \leq t}\left[\frac{\lambda}{\delta} + \left(x(0) - \frac{\lambda}{\delta}\right)e^{-\delta u}\right]\right\} \\
&\leq \max\left\{\|\phi_0\|,\ x(0),\ \frac{\lambda}{\delta}\right\} = M_1, \qquad (3.31)
\end{aligned}$$

$$\begin{aligned}
\|v_t\| &= \sup_{s \leq 0} v_t(s)\mathrm{e}^{\Delta s} = \sup_{u \leq t} v(u)\mathrm{e}^{\Delta u}\mathrm{e}^{-\Delta t} \\
&= \max\left\{\sup_{u \leq 0} v(u)\mathrm{e}^{\Delta u}\mathrm{e}^{-\Delta t}, \max_{0 \leq u \leq t} v(u)\mathrm{e}^{\Delta u}\mathrm{e}^{-\Delta t}\right\} \\
&\leq \max\left\{\|\phi_1\|\mathrm{e}^{-\Delta t}, \max_{0 \leq u \leq t}\left[\frac{C}{b} + \left(v(0) - \frac{C}{b}\right)e^{-bu}\right]\right\} \\
&\leq \max\left\{\|\phi_1\|,\ v(0),\ \frac{C}{b}\right\} = M_2, \qquad (3.32)
\end{aligned}$$

where M_1 and M_2 are independent of t . □

Therefore the solution (x, v) of (3.4) remains in the phase space $Y_\Delta \times Y_\Delta$ for all time $t \geq 0$, thus $x_t(s) \leq \|x_t\|\mathrm{e}^{-\Delta s}$ and $v_t(s) \leq \|v_t\|\mathrm{e}^{-\Delta s}$ for $s \leq 0$.

Let $X = Y_\Delta \times Y_\Delta$, according to (3.4). Denote by $T(t), t \geq 0$ the family of solution operators corresponding to (3.4) such that $T(t)u(0) = u(t)$ where $u(0), u(t) \in X$. We introduce some notations and terminology: the positive orbit $\gamma^+(u)$ through $u \in X$ is defined as $\gamma^+(u) = \cup_{t \geq 0}\{T(t)u\}$. The ω-limit set $\omega(u)$ of u consists of $y \in X$ such that there is a sequence $t_n \to \infty$ as $n \to \infty$ with $T(t_n)u \to y$ as $n \to \infty$.
The semigroup $T(t)$ is said to be asymptotically smooth, if for any bounded forward invariant subset U of X, there exists a compact set $\mathcal{M}$ such that $d(T(t)U, \mathcal{M}) \to 0$ as $t \to \infty$.

Put $M = \max\left\{\sup_{\phi_0 \in U}\|\phi_0\|, \sup_{\phi_1 \in U}\|\phi_1\|, \lambda/\delta, C/b\right\}$. Since U is bounded, M is finite.

Proposition 3.6. *For any bounded forward invariant subset U of X, define $\mathcal{M}_0$ and $\mathcal{M}$ as follows :*

$$\mathcal{M}_0 := \left\{\varphi \in C_\Delta : \sup_{s \leq 0} \varphi(s)e^{\frac{\Delta}{2}s} \leq M\right\}, \quad \mathcal{M} = \mathcal{M}_0 \times \mathcal{M}_0, \qquad (3.33)$$

then $\lim_{t \to \infty} d(T(t)U, \mathcal{M}) = 0$.

Proof. We know that $\mathcal{M}$ is compact in $C_\Delta \times C_\Delta$ from Lemma 3.2 of [2]. We write $T(t)(\phi_0, \phi_1) = (x_t, v_t)$. We note that $x(t) \leq M$ and $v(t) \leq M$ for all $t \geq 0$. We define the function $\varphi^t(s) \in \mathcal{M}_0$ such that

$$\varphi^t(s) := \begin{cases} x(t+s) & \text{if } -t \leq s \leq 0\ (0 \leq s+t \leq t), \\ x(0)e^{-\frac{\Delta}{2}(s+t)} & \text{if } s \leq -t\ (s+t \leq 0). \end{cases} \qquad (3.34)$$

Separating the interval $(-\infty,0]$ to the two intervals $[-t,0],(-\infty,-t]$, we can conclude that

$$\begin{aligned}
\sup_{-t\le s\le 0}|x_t(s)-\varphi^t(s)|e^{\Delta s} &= 0,\\
\sup_{s\le -t}|x_t(s)-\varphi^t(s)|\ e^{\Delta s} &= \sup_{s+t\le 0}|x(s+t)-x(0)e^{-\frac{\Delta}{2}(s+t)}|e^{\Delta s}\\
&\le \sup_{s+t\le 0}\left(\|\phi_0\|e^{-\Delta(s+t)}+x(0)e^{\frac{-\Delta}{2}(s+t)}\right)e^{\Delta s}\\
&= \sup_{s+t\le 0}\left(\|\phi_0\|e^{-\Delta t}+x(0)e^{\frac{\Delta}{2}(s-t)}\right)\\
&= (\|\phi_0\|+x(0))\ e^{-\Delta t}.
\end{aligned}$$

Therefore we have $d(x_t,\mathcal{M}_0)\le(\|\phi_0\|+x(0))\ e^{-\Delta t}$.

Similarly we define the function $\psi^t(s)\in\mathcal{M}_0$ such that

$$\psi^t(s):=\begin{cases} v(t+s) & \text{if } -t\le s\le 0\ (0\le s+t\le t),\\ v(0)e^{-\frac{\Delta}{2}(s+t)} & \text{if } s\le -t\ (s+t\le 0).\end{cases} \tag{3.35}$$

By the same argument we can also get that

$$\begin{aligned}
\sup_{-t\le s\le 0}|v_t(s)-\psi^t(s)|e^{\Delta s} &= 0,\\
\sup_{s\le -t}|v_t(s)-\psi^t(s)|\ e^{\Delta s} &\le (\|\phi_1\|+v(0))\ e^{-\Delta t}.
\end{aligned}$$

Therefore we have $d(v_t,\mathcal{M}_0)\le(\|\phi_1\|+x(0))\ e^{-\Delta t}$. Thus we get that $\lim_{t\to\infty}d((x_t,v_t),\mathcal{M})=0$ uniformly with respect to $(\phi_0,\phi_1)\in U$. □

Therefore $T(t)$ is asymptotically smooth. The following result of the theory of persistence is taken from [6]:

Theorem 3.7. *Suppose that we have the following:*
(i) X^0 *is open and dense in* X *with* $X^0\cup X_0=X$ *and* $X^0\cap X_0=\emptyset$ *;*
(ii) the solution operators $T(t)$ *satisfy* $T(t):X^0\to X^0, T(t):X_0\to X_0$ *;*
(iii) $T(t)$ *is point dissipative in* X *;*
(iv) $\gamma^+(U)$ *is bounded in* X *if* U *is bounded in* X *;*
(v) $T(t)$ *is asymptotically smooth ;*
(vi) $\mathcal{A}=\cup_{x\in A_b}\omega(x)$ *is isolated and has an acyclic covering* N*, where* A_b *is the global attractor of* $T(t)$ *restricted to* X_0 *and* $N=\cup_{i=1}^k N_i$ *;*
(vii) for each $N_i\in N, W^s(N_i)\cap X^0=\emptyset$ *; where* W^s *refers to the stable set.*
Then $T(t)$ *is a uniform repeller with respect to* X_0*, i.e. there is an* $\eta>0$ *such that for any* $x\in X^0, \liminf_{t\to\infty}d(T(t)x,X_0)\ge\eta$*.*

The basic reproduction number of (3.4) is $R_0=r\mu(\hat{x})/(\rho\mu(\hat{x})+b)$ where $\hat{x}=\lambda/\delta$. Let

$$\begin{aligned}
X^0 &= \{(\phi_0,\phi_1)\in X\big|\phi_0(\theta)\phi_1(\theta)>0 \text{ for some } \theta\le 0\},\\
X_0 &= \{(\phi_0,\phi_1)\in X\big|\phi_0(\theta)\phi_1(\theta)=0 \text{ for all } \theta\le 0\}.
\end{aligned}$$

Theorem 3.8. *Consider system* (3.4)*. Assume that* $R_0>1$ *and* $(x_0,v_0)\in X^0$*, then there exists an* $\eta>0$ *such that*

$$\liminf_{t\to\infty}\|x_t\|\ge\eta,\quad \liminf_{t\to\infty}\|v_t\|\ge\eta.$$

Proof. We check all the conditions of Theorem 3.7. It is easy to see that (i) and (ii) are satisfied. The point dissipativity has been proved in Proposition 3.2 and Proposition 3.4 , so we have (iii). Let $\|\phi_0\|$ and $\|\phi_1\|$ are bounded. Then by (3.31) and (3.32) we obtain (iv). Proposition 3.6 confirms (v).

Regarding (vi), clearly $\mathcal{A} = \{E_0\}$ (now $E_0 = (\hat{x}, 0) \in X, \hat{x} = \lambda/\delta$) and isolated. Hence the covering is simply $N = \{E_0\}$, which is acyclic (there is no orbit which connects E_0 to itself in X_0).

It remains to show that $W^s(E_0) \cap X^0 = \emptyset$. Suppose the contrary, there exists a solution $u_t \in X^0$ such that

$$\lim_{t\to\infty} (x_t, v_t) = (\hat{x}, 0).$$

We define the function $V(t)$ such that

$$V(t) = v(t) + r\int_0^\infty \alpha(a)\mu(x(t-a))v(t-a)\mathrm{d}a, \tag{3.36}$$

where $\alpha(a) = \int_a^\infty g(\tau)\mathrm{d}\tau$. By the assumption $v_t \to 0$ we have $V(t) \to 0$ as in the proof of Lemma 3.3. On the other hand differentiating with respect to time gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V(t) &= r\int_0^\infty g(\tau)\mu(x(t-\tau))v(t-\tau)\mathrm{d}\tau - \rho\mu(x)v - bv \\
&\quad + r\int_0^\infty g(\tau)\{\mu(x)v - \mu(x(t-\tau))v(t-\tau)\}\mathrm{d}\tau \\
&= \{r\mu(x) - (\rho\mu(x) + b)\}\, v(t) \\
&= (\rho\mu(x) + b)\left(\frac{r\mu(x)}{\rho\mu(x)+b} - 1\right) v(t).
\end{aligned}$$

Now we take advantage of $R_0 = \dfrac{r\mu(\hat{x})}{\rho\mu(\hat{x}) + b} > 1$: there exists an ϵ and $T > 0$ such that

$$\frac{r\mu(x)}{\rho\mu(x)+b} - 1 > \epsilon,$$

for all $t > T$. Therefore, $V(t)$ goes to infinity or approaches a positive limit as $t \to \infty$. It is a contradiction. Thus we confirmed (vii) and we can apply Theorem 3.7 to obtain that there exists an η such that

$$\liminf_{t\to\infty} \|x_t\| \geq \eta, \quad \liminf_{t\to\infty} \|v_t\| \geq \eta.$$

□

We can apply the following Lemma to x_t and v_t.

Lemma 3.9. *Suppose* $\liminf_{t\to\infty} \|y_t\| \geq \eta$, *then it holds that* $\limsup_{t\to\infty} y(t) \geq \eta'$ *for all* η' *such that* $0 < \eta' < \eta$.

Proof. There exists $T_0 > 0$ such that

$$\|y_t\| = \sup_{\theta\leq 0} y(t+\theta)e^{\Delta\theta} \geq \eta' \text{ for all } t \geq T_0.$$

For arbitrary positive T_1 larger than T_0 let $t > T_1$. Separating the interval $(-\infty, 0]$ to the three intervals $(-\infty, -t], [-t, T_1 - t], [T_1 - t, 0]$, we calculate $\|y_t\|$ with the initial condition $y_0 = \phi \in Y_\Delta$. Let $u = t + \theta$, then

$$\sup_{\theta \le -t} y(t+\theta)e^{\Delta\theta} = \sup_{u \le 0} y(u)e^{\Delta(u-t)} = e^{-\Delta t}\|\phi\|.$$

On the second interval let $K = \sup_{-t \le \theta \le T_1 - t} y(t+\theta) = \sup_{0 \le u \le T_1} y(u)$, then

$$\sup_{0 \le u \le T_1} y(u)e^{\Delta(u-t)} \le Ke^{\Delta(T_1 - t)} = Ke^{\Delta T_1}e^{-\Delta t}.$$

Choosing t large enough makes these two values smaller than $\eta'/2$. Then the value of $\|y_t\|$ is attained at θ_0 in the third interval $T_1 - t < \theta_0 \le 0$. Therefore

$$y(t+\theta_0)e^{\Delta\theta_0} \ge \eta'.$$

Let $u_1 = t + \theta_0 > T_1$ then

$$y(u_1) \ge \eta' e^{-\Delta\theta_0} = \eta' e^{\Delta(-\theta_0)} \ge \eta'.$$

We can choose T_2 larger than u_1 and by the same argument we can obtain $u_2 > T_2$ such that

$$y(u_2) \ge \eta' e^{-\Delta\theta_0} = \eta' e^{\Delta(-\theta_0)} \ge \eta'.$$

Therefore the assertion follows. □

Consequently

$$\limsup_{t\to\infty} x(t) \ge \eta' \text{ and } \limsup_{t\to\infty} v(t) \ge \eta'. \tag{3.37}$$

Proposition 3.10. *Assume $R_0 > 1$. Let (x, v) be a solution of* (3.4) *with* $(x_0, v_0) \in X^0$ *, then there exists a positive η'' such that*

$$\liminf_{t\to\infty} x(t) > \eta'' \textit{ and } \liminf_{t\to\infty} v(t) > \eta''. \tag{3.38}$$

Proof. We use Proposition 3.6 and Theorem 2.2 in [18] with $\rho = x$ or $\rho = v$, where $\rho : X \to (0, \infty)$ is a continuous strictly positive functional on X. □

Now we construct the Lyapunov functional for the model (3.4). We define the functionals

$$W_1^\infty(\phi_t; c) = c\int_0^\infty \alpha(a)H\left(\frac{\phi(t-a)}{c}\right) \mathrm{d}a, \tag{3.39}$$

for our construction of Lyapunov functionals, where $\alpha(a) = \int_a^\infty g(\tau)\mathrm{d}\tau$, $H(u) = u - 1 - \log u$ and c is a positive constant. To guarantee the well-definedness of the integration we require that there exist positive $\tilde{\epsilon}$ and $\tilde{M}$ such that $\tilde{\epsilon} \le \phi(t-a) \le \tilde{M}$ for all $0 \le a < \infty$.

Suppose $R_0 > 1$. Let $(\tilde{x}, \tilde{v})$ be a solution of the equation (3.4) with $(\tilde{x}_0, \tilde{v}_0) \in X^0$. Then combined with Proposition 3.2, Proposition 3.4 and Proposition 3.10, it follows that the ω-limit set Ω of $(\tilde{x}, \tilde{v})$ is non-empty, compact and invariant. It follows that Ω is the union of entire orbits of the equation (3.4). That is, if $(\phi_0, \phi_1) \in Y_\Delta \times Y_\Delta$ is a point in Ω, then there exists an entire solution through (ϕ_0, ϕ_1) such that every point on the solution is in Ω. For the solution (x, v) that lies in Ω, combined with Proposition 3.2, Proposition 3.4 and Proposition 3.10, there exist $\epsilon > 0$ and $M > 0$ such that

$$\epsilon \le x(t) \le M, \quad \epsilon \le v(t) \le M \text{ for all } t \in \mathbb{R}. \tag{3.40}$$

Then the functional (3.39) is well defined for every solution that lies in Ω.

3.2 Stability of the equilibria

Proposition 3.11. *Suppose $R_0 > 1$ and (x, v) is a solution of* (3.4) *that lies in* Ω*, then the time derivative of*

$$U_1(x_t, v_t) = \int_{x^*}^{x} \frac{\mu(\xi) - \mu(x^*)}{\mu(\xi)} \mathrm{d}\xi + \frac{1}{r-\rho}(v - v^* \log v) + \frac{r}{r-\rho} W_1^\infty((\mu(x)v)_t; \mu(x^*)v^*) \tag{3.41}$$

is nonpositive under the condition :

$$1 - \frac{\rho}{r-\rho} \cdot \frac{\mu(x^*)v^*}{\delta x^*} \geq 0. \tag{3.42}$$

Proof. When we assume $R_0 > 1$, there exists an infected equilibrium (x^*, v^*). We can rewrite the first equation of (3.1) as follows

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \delta \cdot (x^* - x) + \mu(x^*)v^* - \mu(x)v, \tag{3.43}$$

and at the equilibrium (x^*, v^*) it holds that $b = (r - \rho)\mu(x^*)$. The time derivative of U_1 along (3.1) becomes as follows

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}U_1(x_t, v_t) &= \left(1 - \frac{\mu(x^*)}{\mu(x)}\right) \{\delta \cdot (x^* - x) + \mu(x^*)v^* - \mu(x)v\} \\
&\quad + \frac{1}{r-\rho}\left(1 - \frac{v^*}{v}\right)\left\{r\mu(x)v - \rho\mu(x)v - bv + r\left(\int_0^\infty g(\tau)\mu(x_t(\tau))v_t(\tau)\mathrm{d}\tau - \mu(x)v\right)\right\} \\
&\quad + \frac{r}{r-\rho}\int_0^\infty g(\tau)\left\{\mu(x)v - \mu(x_t(\tau))v_t(\tau) + \mu(x^*)v^* \log \frac{\mu(x_t(\tau))v_t(\tau)}{\mu(x(t))v(t)}\right\}\mathrm{d}\tau \\
&= \left(1 - \frac{\mu(x^*)}{\mu(x)}\right)\delta x^*\left(1 - \frac{x^*}{x}\right) + \left(1 - \frac{\mu(x^*)}{\mu(x)}\right)\{\mu(x^*)v^* - \mu(x)v\} \\
&\quad + \frac{1}{r-\rho}\left(1 - \frac{v^*}{v}\right)\{(r-\rho)\mu(x)v - (r-\rho)\mu(x^*)v\} \\
&\quad - \frac{r}{r-\rho}\frac{v^*}{v}\left\{\int_0^\infty g(\tau)\mu(x_t(\tau))v_t(\tau)\mathrm{d}\tau - \mu(x)v\right\} \\
&\quad + \frac{r}{r-\rho}\int_0^\infty g(\tau)\mu(x^*)v^* \log \frac{\mu(x_t(\tau))v_t(\tau)}{\mu(x(t))v(t)}\mathrm{d}\tau \qquad (3.44) \\
&= \left(1 - \frac{\mu(x^*)}{\mu(x)}\right)\delta x^*\left(1 - \frac{x^*}{x}\right) + \mu(x^*)v^* - \frac{\mu(x^*)}{\mu(x)}\mu(x^*)v^* - \mu(x)v^* + \mu(x^*)v^* \\
&\quad + \frac{r}{r-\rho}\mu(x^*)v^*\int_0^\infty g(\tau)\left(\frac{\mu(x)}{\mu(x^*)} - \frac{\mu(x_t(\tau))v_t(\tau)}{\mu(x^*)v} + \log \frac{\mu(x_t(\tau))v_t(\tau)}{\mu(x(t))v(t)}\right)\mathrm{d}\tau \\
&= \left(1 - \frac{\mu(x^*)}{\mu(x)}\right)\delta x^*\left(1 - \frac{x^*}{x}\right) + \mu(x^*)v^*\left(2 - \frac{\mu(x^*)}{\mu(x)} - \frac{\mu(x)}{\mu(x^*)}\right) \\
&\quad + \frac{r}{r-\rho}\mu(x^*)v^*\int_0^\infty g(\tau)\left(\frac{\mu(x)}{\mu(x^*)} - \frac{\mu(x_t(\tau))v_t(\tau)}{\mu(x^*)v} + \log \frac{\mu(x_t(\tau))v_t(\tau)}{\mu(x(t))v(t)}\right)\mathrm{d}\tau \\
&= \left(1 - \frac{\mu(x^*)}{\mu(x)}\right)\delta x^*\left(1 - \frac{x^*}{x}\right) - \frac{\rho}{r-\rho}\mu(x^*)v^*\left(1 - \frac{\mu(x^*)}{\mu(x)}\right)\left(1 - \frac{\mu(x)}{\mu(x^*)}\right) \\
&\quad + \frac{r}{r-\rho}\mu(x^*)v^*\int_0^\infty g(\tau)\left(2 - \frac{\mu(x^*)}{\mu(x)} - \frac{\mu(x_t(\tau))v_t(\tau)}{\mu(x^*)v} + \log \frac{\mu(x_t(\tau))v_t(\tau)}{\mu(x(t))v(t)}\right)\mathrm{d}\tau.
\end{aligned}$$

The monotone nonincreasing of $\mu(x)/x$ leads

$$(1-\mu(x^*)/\mu(x))\,(1-x/x^*) \le (1-\mu(x^*)/\mu(x))\,(1-\mu(x)/\mu(x^*))\,, \tag{3.45}$$

and it follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}U_1(x_t,v_t) &\le \left(1-\frac{\rho}{r-\rho}\cdot\frac{\mu(x^*)v^*}{\delta x^*}\right)\delta x^*\left(1-\frac{\mu(x^*)}{\mu(x)}\right)\left(1-\frac{x}{x^*}\right)\\ &+\frac{r}{r-\rho}\mu(x^*)v^*\!\int_0^\infty\! g(\tau)\left(2-\frac{\mu(x^*)}{\mu(x)}-\frac{\mu(x_t(\tau))v_t(\tau)}{\mu(x^*)v}+\log\frac{\mu(x_t(\tau))v_t(\tau)}{\mu(x)v}\right)\mathrm{d}\tau.\end{aligned} \tag{3.46}$$

If

$$1-\frac{\rho}{r-\rho}\cdot\frac{\mu(x^*)v^*}{\delta x^*}\ge 0, \tag{3.47}$$

that is if

$$r\ge\rho\left(1+\frac{v^*\mu(x^*)}{\delta x^*}\right),\text{or } r\ge\rho\cdot\frac{\hat{x}}{x^*}, \tag{3.48}$$

then the derivative is nonpositive and U_1 becomes a Lyapunov functional for the equilibrium (x^*, v^*). □

Theorem 3.12. *If $R_0 > 1$, then all solutions of equation* (3.4) *for which the disease is initially present converge to (x^*, v^*) under the condition* (3.42)*.*

Proof. We can show that the maximal invariant set M in

$$\left\{(x_0,v_0)\in\Omega \;\middle|\; \frac{\mathrm{d}}{\mathrm{d}t}U_1(x_t,v_t)|_{t=0}=0\right\}, \tag{3.49}$$

is the singleton $\{(x^*, v^*)\}$. Let (x, v) be a solution of (3.4) in Ω. Then the ω-limit set and α-limit set are contained in M. They are equal to $\{(x^*, v^*)\}$. Since $U_1(x_t, v_t)$ is nonincreasing along the solution (x, v), (x, v) must be equal to (x^*, v^*) identically. Then the ω-limit set Ω is equal to $\{(x^*, v^*)\}$. It follows that all solutions of equation (3.4) for which disease is initially present converge to (x^*, v^*). □

If there does not exist absorption effect, that is $\rho = 0$, then the condition (3.42) is satisfied.

Assume $R_0 \le 1$ and define the following functional :

$$W_0^\infty((xv)_t)=\int_0^\infty \alpha(a)x(t-a)v(t-a)\mathrm{d}a. \tag{3.50}$$

The integral W_0^∞ can be defined for every bounded solution. Let $(\tilde{x}, \tilde{v})$ be an arbitrary solution. As in the case $R_0 > 1$, the ω-limit set Ω of $(\tilde{x}, \tilde{v})$ is non-empty. Then there exists an entire solution (x, v) through an element $(\phi_0, \phi_1) \in \Omega$.

Proposition 3.13. *Suppose $R_0 \le 1$. Let (x, v) be a solution of* (3.4) *that lies in Ω. Then the time derivative of*

$$U_2(x_t,v_t)=\int_{\hat{x}}^{x}\frac{\mu(\xi)-\mu(\hat{x})}{\mu(\xi)}\mathrm{d}\xi+\frac{1}{r-\rho}v+\frac{r}{r-\rho}W_0^\infty((\mu(x)v)_t), \tag{3.51}$$

is nonpositive.

Proof. The condition $R_0 \leq 1$ is equivalent to $\mu(\hat{x}) - b/(r-\rho) \leq 0$. We can rewrite the model (3.4) as follows

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \delta \cdot (\hat{x} - x) - \mu(x)v, \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= r\mu(x)v - \rho\mu(x)v - bv + r\left(\int_0^\infty g(\tau)\mu(x(t-\tau))v(t-\tau)\mathrm{d}\tau - \mu(x)v\right). \end{aligned} \tag{3.52}$$

The nonpositivity of the time derivative of $U_2(x_t, v_t)$ along (3.52) is shown by the monotonous increase of $\mu(x)$ and $R_0 \leq 1$ as follows :

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}U_2(x_t, v_t) &= \left(1 - \frac{\mu(\hat{x})}{\mu(x)}\right)\{\delta \cdot (\hat{x} - x) - \mu(x)v\} \\ &+ \frac{1}{r-\rho}\left\{r\mu(x)v - \rho\mu(x)v - bv + r\left(\int_0^\infty g(\tau)\mu(x(t-\tau))v(t-\tau)\mathrm{d}\tau - \mu(x)v\right)\right\} \\ &+ \frac{r}{r-\rho}\int_0^\infty g(\eta)\{\mu(x(t))v(t) - \mu(x(t-\eta))v(t-\eta)\}\,\mathrm{d}\eta \\ &= \delta\hat{x}\left(1 - \frac{\mu(\hat{x})}{\mu(x)}\right)\left(1 - \frac{x}{\hat{x}}\right) + (-\mu(x)v + \mu(\hat{x})v) + \frac{1}{r-\rho}\{(r-\rho)\mu(x) - bv\} \\ &= \delta\hat{x}\left(1 - \frac{\mu(\hat{x})}{\mu(x)}\right)\left(1 - \frac{x}{\hat{x}}\right) + \left(\mu(\hat{x}) - \frac{b}{r-\rho}\right)v \leq 0. \end{aligned} \tag{3.53}$$

Thus $U_2(x_t, v_t)$ is a Lyapunov functional for the disease free equilibrium $(\hat{x}, 0)$. □

Theorem 3.14. *If $R_0 \leq 1$, then all solutions converge to the infection-free equilibrium.*

Proof. We can show that the maximal invariant set M in

$$\left\{(x_0, v_0) \in \Omega \;\middle|\; \frac{\mathrm{d}}{\mathrm{d}t}U_2(x_t, v_t)|_{t=0} = 0\right\} \tag{3.54}$$

is the singleton $\{(\lambda/\delta, 0)\}$. As in Theorem 3.12, we can show that all solutions converge to the infection-free equilibrium. □

4 Model with an immune variable

We consider a model with the immune variable z. The model is incorporated with humoral immunity. The system of equations is

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= \lambda - \delta x - \mu(x)v,\\ \frac{\mathrm{d}v}{\mathrm{d}t} &= r\int_0^\infty g(\tau)\mu(x(t-\tau))v(t-\tau)\mathrm{d}\tau - \rho\mu(x)v - bv - pvz,\\ \frac{\mathrm{d}z}{\mathrm{d}t} &= vq(z) - mz,\end{aligned} \tag{4.1}$$

where $vq(z)$ represents the activation of immunity by infected cells and m represents the death rate of immune variable.

Kajiwara *et al.* [10] studied a method of construction of Lyapunov functions for an ODE model containing those terms. Several types of $q(z)$ are considered. Nowak and Bangham [15] and Murase *et al.* [14] used $q(z) = qz$. Inoue *et al.* [8] used $q(z) = q$. Gomez–Acevedo and Li [4] used $q(z) = qz/(z + K)$, where q and K are constants. In this paper we examine two types of $q(z)$. The first type is $q(z) = q$, and the second type is $q(z) = qz$, where q is constant. Using an argument in Section 3 we can choose $Y_\Delta \times Y_\Delta \times \mathbb{R}_{\geq 0}$ as a phase space for system (4.1), and the integral in (4.1) converges. The basic reproduction number of (4.1) is $R_0 = r\mu(\hat{x})/(\rho\mu(\hat{x}) + b)$ where $\hat{x} = \lambda/\delta$.

The disease free equilibrium of the model (4.1) is $E_0 = (\lambda/\delta, 0, 0)$. The infected equilibrium is (x^*, v^*, z^*) that satisfies following equations :

$$\lambda - \delta x^* - \mu(x^*)v^* = 0, \tag{4.2}$$

$$\{(r-\rho)\mu(x^*) - b - pz^*\}\, v^* = 0, \tag{4.3}$$

$$v^* q(z^*) - mz^* = 0, \tag{4.4}$$

where $v^* > 0$.

4.1 Positivity and boundedness

By a similar argument in Proposition 3.1 we have the positivity of x, v and z. We will show the boundedness of them. By (4.1) we obtain

$$\frac{\mathrm{d}x}{\mathrm{d}t} \leq \lambda - \delta x, \tag{4.5}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} \leq r\int_0^\infty g(\tau)\mu(x_t)v_t d\tau - \rho\mu(x)v - bv. \tag{4.6}$$

Then by a similar argument in Proposition 3.2 and Proposition 3.4 we can see the boundedness of $x(t)$ and $v(t)$.

In the case $q(z) = q$, let V be the upper bound of v. By the third equation of (4.1) we obtain

$$\frac{\mathrm{d}z}{\mathrm{d}t} \leq qV - mz. \tag{4.7}$$

Then we have the followings :

$$z \leq \frac{qV}{m} + \left(z(0) - \frac{qV}{m}\right)e^{-mt}, \tag{4.8}$$

therefore it holds that

$$z \le \max\left\{z(0), \frac{qV}{m}\right\} \text{ for all } t \ge 0, \tag{4.9}$$

and there exists a T such that $z \le qV/m + 1$ for all $t \ge T$.

In the case $q(z) = qz$, we define the following function :

$$W_2(t) = x(t) + v(t) + \frac{p}{q}z(t) + (\rho+1)\int_0^\infty \mu(x(t-a))v(t-a)e^{-2\Delta_1 a}\mathrm{d}a.$$

Then the time derivative of $W_2(t)$ becomes as follows :

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}W_2 &= \lambda - \delta x - \mu(x)v + \int_0^\infty k(a)\sigma(a)\mu(x(t-a))v(t-a)\mathrm{d}a - \rho\mu(x)v - bv - pvz \\
&+ \frac{p}{q}(qvz - mz) + (\rho+1)\left\{\mu(x)v - 2\Delta_1\int_0^\infty \mu(x(t-a))v(t-a)e^{-2\Delta_1 a}\mathrm{d}a\right\} \\
&\le \lambda - \delta x - \frac{b}{2}v - m\frac{p}{q}z - 2\Delta_1(\rho+1)\int_0^\infty \mu(x(t-a))v(t-a)e^{-2\Delta_1 a}\mathrm{d}a \\
&+ C_1\|k\|_\infty \int_0^\infty \mu(x(t-a))v(t-a)e^{-2\Delta_1 a}\mathrm{d}a \\
&\le C_1\|k\|_\infty\left(\frac{\lambda}{\delta}+1\right) + \lambda \\
&- \left(\delta x + \frac{b}{2}v + m\cdot\frac{p}{q}z + 2\Delta_1(\rho+1)\int_0^\infty \mu(x(t-a))v(t-a)e^{-2\Delta_1 a}\mathrm{d}a\right).
\end{aligned} \tag{4.10}$$

Let $C_2 = C_1\|k\|_\infty(\lambda/\delta + 1) + \lambda$ and $\nu_2 = \min(\delta, b/2, m, 2\Delta_1)$ then

$$\frac{\mathrm{d}}{\mathrm{d}t}W_2 \le C_2 - \nu_2 W_2,$$

and it holds that

$$W_2 \le \max\left\{W_2(0), \frac{C_2}{\nu_2}\right\} \text{ for all } t \ge 0. \tag{4.11}$$

Therefore there exists $T_2(> T)$ such that $W_2 \le C_2/\nu_2 + 1$ for all $t \ge T_2$. The positivity of x and v leads the boundedness of z.

4.2 Stability of the infected equilibrium : $q(z) = qz$

If $R_0 > 1$ then the infected equilibrium exists. At first we consider the case $q(z) = qz$. If $z(0) = 0$ then $z(t) \equiv 0$ for all $t \ge 0$ by the third equation of (4.1). Then the model is reduced to the model (3.4) without an immune variable. At most two infected equilibria are obtained from (4.2),(4.3) and (4.4). The one equilibrium $E^\dagger$ is obtained from $z^\dagger = 0$ and $v^\dagger > 0$. Then $x^\dagger$ is determined by $\mu(x^\dagger) = b/(r-\rho)$. Since $R_0 > 1$ then it holds that $\mu(\lambda/\delta) > b/(r-\rho)$ and $\lambda/\delta > x^\dagger$. The equilibrium is $E^\dagger = (x^\dagger, (\lambda - \delta x^\dagger)/\mu(x^\dagger), 0)$. The other equilibrium $E^\ddagger$ is obtained from $z^\ddagger > 0$. Then $v^\ddagger = m/q$. Thus $x^\ddagger$ is determined by $m/q = (\lambda - \delta x^\ddagger)/\mu(x^\ddagger)$. The immune variable $z^\ddagger$ is determined by $z^\ddagger = \{(r-\rho)\mu(x^\ddagger) - b\}/p$. If $(r-\rho)\mu(x^\ddagger) - b \le 0$ then there does not exist an equilibrium with positive immune variable and the infected equilibrium is only $E^\dagger$. Else if it holds that $(r-\rho)\mu(x^\ddagger) - b > 0$ then there exists the equilibrium

$E^{\ddagger} = (x^{\ddagger}, m/q, \{(r-\rho)\mu(x^{\ddagger}) - b\}/p)$ with positive immune variable where $x^{\ddagger} > x^{\dagger}$. The condition $(r-\rho)\mu(x^{\ddagger}) - b > 0$ is equivalent to the following inequality :

$$\frac{m}{q} < \frac{\lambda - \delta x^{\dagger}}{\mu(x^{\dagger})}. \tag{4.12}$$

Figure 1 shows the relationship of these two infected equilibria and the disease free equilibrium.

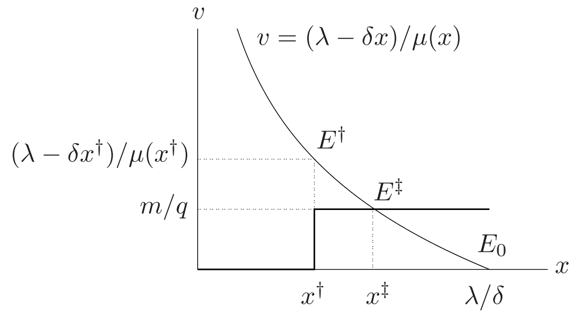

Figure 1: $m/q < (\lambda - \delta x^{\dagger})/\mu(x^{\dagger})$

Suppose (4.12) , we define $X^0 = \{(\phi_0, \phi_1, \alpha) \in X \big| \phi_0(\theta)\phi_1(\theta) > 0 \text{ for some } \theta \le 0\}$ and $X_0 = \{(\phi_0, \phi_1, \alpha) \in X \big| \phi_0(\theta)\phi_1(\theta) = 0 \text{ for all } \theta \le 0\}$. Then the attractor $\mathcal{A}$ restricted to X_0 is $\{E_0\}$. A similar argument as Theorem 3.8, Lemma 3.9 and Proposition 3.10 leads the following Proposition :

Proposition 4.1. *Assume $R_0 > 1$. Let (x, v, z) be a solution of (4.1) with $(x_0, v_0, z(0)) \in X^0$, then there exists a positive η'' such that*

$$\liminf_{t\to\infty} x(t) > \eta'' \text{ and } \liminf_{t\to\infty} v(t) > \eta''. \tag{4.13}$$

Suppose $R_0 > 1$. Let $(\tilde{x}, \tilde{v}, \tilde{z})$ be a solution of the equation (4.1) with $(\tilde{x}_0, \tilde{v}_0, \tilde{z}(0)) \in X^0$. Then the ω-limit set Ω of $(\tilde{x}, \tilde{v}, \tilde{z})$ is non-empty, compact and invariant. The set Ω is the union of entire orbits of the equation (4.1). That is , if $(\phi_0, \phi_1, \alpha) \in Y_\Delta \times Y_\Delta \times \mathbb{R}_+$ is an omega limit point of $(\tilde{x}, \tilde{v}, \tilde{z})$, then there exists a solution through (ϕ_0, ϕ_1, α) such that every point on the solution is in Ω. As in the model without immune variable, For the solution that lies in Ω, there exist $\epsilon > 0$ and $M > 0$ such that

$$\epsilon \le x(t) \le M, \quad \epsilon \le v(t) \le M \text{ for all } t \in \mathbb{R}. \tag{4.14}$$

Then the functional defined by (3.39) is well defined for every solution that lies in Ω.

Proposition 4.2. *Suppose that $R_0 > 1$ and that there exists the infected equilibrium $E^{\ddagger}$ with positive immune variable. Let (x, v, z) be a solution of (4.1) that lies in Ω, then the time derivative of*

$$U_3(x_t, v_t, z) = U_1(x_t, v_t) + \frac{p}{r-\rho} \int_{z^{\ddagger}}^{z} \frac{\tau - z^{\ddagger}}{q\tau} \mathrm{d}\tau \tag{4.15}$$

is nonpositive under the condition :

$$r > \rho \left(1 + \frac{v^{\ddagger}\mu(x^{\ddagger})}{\delta x^{\ddagger}}\right), \text{ or } x^{\ddagger} > \frac{\rho}{r} \cdot \hat{x}. \tag{4.16}$$

Proof. If there exists the interior equilibrium $(x^\ddagger, v^\ddagger, z^\ddagger)$, we define the functional U_3 as follows :

$$\begin{aligned} U_3(x_t, v_t, z) = \int_{x^\ddagger}^{x} \frac{\mu(\xi) - \mu(x^\ddagger)}{\mu(\xi)} \mathrm{d}\xi + \frac{1}{r-\rho}(v - v^\ddagger \log v) + \frac{p}{r-\rho}\int_{z^\ddagger}^{z} \frac{\tau - z^\ddagger}{q\tau}\mathrm{d}\tau \\ + \frac{r}{r-\rho} W_1^\infty((\mu(x)v)_t; \mu(x^\ddagger)v^\ddagger). \end{aligned} \tag{4.17}$$

We consider the following modified system with absorption effect :

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \lambda - \delta x - \mu(x)v, \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= r\int_0^\infty g(\tau)\mu(x(t-\tau))v(t-\tau)\mathrm{d}\tau - \rho\mu(x)v - (b + pz^\ddagger)v. \end{aligned} \tag{4.18}$$

This model has the same equilibrium as (4.1) and a Lyapunov functional similar to U_1. We rewrite the system (4.1) as follows :

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \lambda - \delta x - \mu(x)v, \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= r\int_0^\infty g(\tau)\mu(x(t-\tau))v(t-\tau)\mathrm{d}\tau - \rho\mu(x)v - (b + pz^\ddagger)v - (pz - pz^\ddagger)v, \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= vqz - mz. \end{aligned} \tag{4.19}$$

The time derivative of U_3 along (4.19) is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}U_3(x_t, v_t, z) &= \frac{\mathrm{d}}{\mathrm{d}t}U_1 + \frac{1}{r-\rho}\left(1 - \frac{v^\ddagger}{v}\right)\left\{-(pz - pz^\ddagger)v\right\} + \frac{p}{r-\rho}\frac{\mathrm{d}}{\mathrm{d}t}\int_{z^\ddagger}^{z}\frac{\tau - z^\ddagger}{q(\tau)}\mathrm{d}\tau \\ &\leq \frac{r}{r-\rho}\delta x^\ddagger\left\{1 - \frac{\rho}{r}\left(1 + \frac{\mu(x^\ddagger)v^\ddagger}{\delta x^\ddagger}\right)\right\}\left(1 - \frac{\mu(x^\ddagger)}{\mu(x)}\right)\left(1 - \frac{x}{x^\ddagger}\right) \\ &+ \frac{r}{r-\rho}\mu(x^\ddagger)v^\ddagger\int_0^\infty g(\tau)\left(2 - \frac{\mu(x^\ddagger)}{\mu(x)} - \frac{\mu(x_t(\tau))v_t(\tau)}{\mu(x^\ddagger)v} + \log\frac{\mu(x_t(\tau))v_t(\tau)}{\mu(x)v}\right)\mathrm{d}\tau \\ &+ \frac{p}{r-\rho}\left(1 - \frac{v^\ddagger}{v}\right)(-v)\left(z - z^\ddagger\right) + \frac{p}{r-\rho}\frac{z - z^\ddagger}{qz}\left(vqz - mz\right). \end{aligned} \tag{4.20}$$

The second term is nonpositive by the extension of arithmetic-geometric mean inequality [9]. The last two terms become

$$\begin{aligned} &\frac{p}{r-\rho}\left(-v + v^\ddagger\right)\left(z - z^\ddagger\right) + \frac{p}{r-\rho}\frac{z - z^\ddagger}{qz}\left(vqz - mz\right) \\ =&\frac{p}{r-\rho}\left(v^\ddagger - v\right)\left(z - z^\ddagger\right) + \frac{p}{r-\rho}(z - z^\ddagger)\left(v - \frac{m}{q}\right) \\ =&\frac{p}{r-\rho}\left(z - z^\ddagger\right)\left(v^\ddagger - \frac{m}{q}\right) \end{aligned}$$

This is zero because $v^\ddagger = m/q$. Therefore when $r > \rho\left(1 + \frac{v^\ddagger\mu(x^\ddagger)}{\delta x^\ddagger}\right) = \rho \cdot \frac{\hat{x}}{x^\ddagger}$ the derivative of U_3 holds nonpositive and U_3 is a Lyapunov functional for the equilibrium $E^\ddagger = (x^\ddagger, m/q, z^\ddagger)$. □

Theorem 4.3. *If $R_0 > 1$ and there exists the equilibrium $E^\ddagger = (x^\ddagger, q/m, z^\ddagger)$, then all solutions of* (4.1) *for which the disease is initially present converge to the infected equilibrium $(x^\ddagger, m/q, z^\ddagger)$ under the condition* (4.16)*.*

Proof. We can show that the maximal invariant set M in

$$\left\{(x_0, v_0, z(0)) \in \Omega \;\middle|\; \frac{\mathrm{d}}{\mathrm{d}t} U_1(x_t, v_t, z)|_{t=0} = 0\right\}, \tag{4.21}$$

is the singleton $\{(x^\ddagger, v^\ddagger, z^\ddagger)\}$. Let (x, v, z) be a solution of (4.1) in Ω. Then the ω-limit set and α-limit set are contained in M. They are equal to $\{(x^\ddagger, v^\ddagger, z^\ddagger)\}$. Since $U_3(x_t, v_t, z)$ is nonincreasing along the solution (x, v, z), (x, v, z) must be equal to $(x^\ddagger, v^\ddagger, z^\ddagger)$ identically. Then the ω-limit set Ω is equal to $\{(x^\ddagger, v^\ddagger, z^\ddagger)\}$. It follows that all solutions of (4.1) for which disease is initially present converge to $(x^\ddagger, v^\ddagger, z^\ddagger)$. □

When there does not exist absorption effect, that is $\rho = 0$, then the condition (4.16) is obviously satisfied.

Suppose that there does not exist the infected equilibrium $E^\ddagger$. We define $X^0 = \{(\phi_0, \phi_1, \alpha) \in X \,|\, \phi_0(\theta)\phi_1(\theta) > 0 \text{ for some } \theta \le 0\}$ and $X_0 = \{(\phi_0, \phi_1, \alpha) \in X \,|\, \phi_0(\theta)\phi_1(\theta) = 0 \text{ for all } \theta \le 0\}$. Then the attractor $\mathcal{A}$ restricted to X_0 is $\{E_0\}$. Let Ω be the ω-limit set of the solution of (4.1). Then by the same argument for Proposition 3.10, the functional (3.39) is well defined for every solution that lies in Ω. Figure 2 shows the absence of $E^\ddagger$.

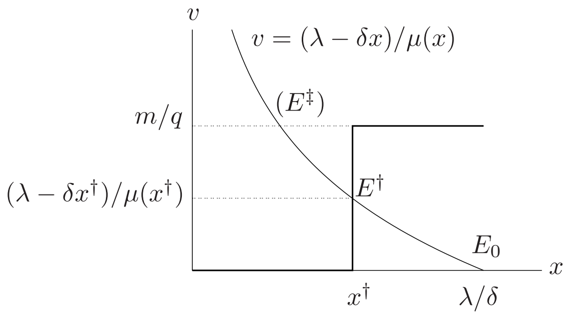

Figure 2: $m/q > (\lambda - \delta x^\dagger)/\mu(x^\dagger)$

Proposition 4.4. *Suppose that $R_0 > 1$, the infected equilibrium is only $E^\dagger = (x^\dagger, v^\dagger, 0)$. Let (x, v, z) be a solution that lies in Ω, then the time derivative of*

$$U_4(x_t, v_t, z) = U_1(x_t, v_t) + \frac{p}{r-\rho}\int_0^z \frac{1}{q}\mathrm{d}\tau, \tag{4.22}$$

is nonpositive under the condition :

$$r > \rho\left(1 + \frac{v^\dagger \mu(x^\dagger)}{\delta x^\dagger}\right), \text{ or } x^\dagger > \frac{\rho}{r}\cdot \hat{x}. \tag{4.23}$$

Proof. The functional U_4 is defined as follows :

$$\begin{aligned} U_4(x_t, v_t, z) = \int_{x^\dagger}^{x} \frac{\mu(\xi) - \mu(x^\dagger)}{\mu(\xi)}\mathrm{d}\xi + \frac{1}{r-\rho}(v - v^\dagger \log v) + \frac{p}{r-\rho}\int_0^z \frac{1}{q}\mathrm{d}\tau \\ + \frac{r}{r-\rho} W_1^\infty((\mu(x)v)_t; \mu(x^\dagger)v^\dagger). \end{aligned} \tag{4.24}$$

The time derivative of U_4 along (4.19) is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}U_4(x_t, v_t, z) &= \frac{\mathrm{d}}{\mathrm{d}t}U_1 + \frac{1}{r-\rho}\left(1 - \frac{v^\dagger}{v}\right)(-pzv) + \frac{p}{r-\rho}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^z \frac{1}{q}\mathrm{d}\tau \\
&\leq \frac{r}{r-\rho}\delta x^\dagger \left\{1 - \frac{\rho}{r}\left(1 + \frac{\mu(x^\dagger)v^\dagger}{\delta x^\dagger}\right)\right\}\left(1 - \frac{\mu(x^\dagger)}{\mu(x)}\right)\left(1 - \frac{x}{x^\dagger}\right) \\
&+ \frac{r}{r-\rho}\mu(x^\dagger)v^\dagger \int_0^\infty g(\tau)\left(2 - \frac{\mu(x^\dagger)}{\mu(x)} - \frac{\mu(x_t(\tau))v_t(\tau)}{\mu(x^\dagger)v} + \log\frac{\mu(x_t(\tau))v_t(\tau)}{\mu(x)v}\right)\mathrm{d}\tau \\
&+ \frac{p}{r-\rho}\left(-v + v^\dagger\right)z + \frac{p}{r-\rho}\frac{1}{q}\left(vqz - mz\right). \qquad (4.25)
\end{aligned}$$

The second term is nonpositive by the extension of arithmetic-geometric mean inequality. The last two terms become

$$\begin{aligned}
&\frac{p}{r-\rho}\left(-v + v^\dagger\right)z + \frac{p}{r-\rho}\frac{1}{q}\left(vq - m\right)z \\
=&\frac{p}{r-\rho}\left(v^\dagger - \frac{m}{q}\right)z. \qquad (4.26)
\end{aligned}$$

When there does not exist $E^\ddagger$, it holds that $(r-\rho)\mu(x^\ddagger) - b \leq 0$. This implies $x^\ddagger \leq x^\dagger$ and $v^\ddagger \geq v^\dagger$, that is $m/q \geq v^\dagger$. Therefore (4.26) is nonpositive. Consequently when $r > \rho\left(1 + \frac{v^\dagger\mu(x^\dagger)}{\delta x^\dagger}\right) = \rho \cdot \frac{\hat{x}}{x^\dagger}$ the derivative of U_4 holds nonpositive and U_4 is a Lyapunov functional at $E^\dagger = (x^\dagger, v^\dagger, 0)$. □

Theorem 4.5. *If $R_0 > 1$ and the infected equilibrium is only $E^\dagger$, then all solutions of equation* (4.1) *for which the disease is initially present converge to the infected equilibrium* $(x^\dagger, v^\dagger, 0)$ *under the condition* (4.23).

Proof. We can show that the maximal invariant set M in

$$\left\{(x_0, v_0, z(0)) \in \Omega \;\middle|\; \frac{\mathrm{d}}{\mathrm{d}t}U_4(x_t, v_t, z)|_{t=0} = 0\right\} \qquad (4.27)$$

is the singleton $\{(x^\dagger, v^\dagger, 0)\}$. Let (x, v, z) be a solution of (4.1) in Ω. Then the ω-limit set and α-limit set are contained in M. They are equal to $\{(x^\dagger, v^\dagger, 0)\}$. Since $U_4(x_t, v_t, z)$ is nonincreasing along the solution (x, v, z), (x, v, z) must be equal to $(x^\dagger, v^\dagger, 0)$ identically. Then the ω-limit set Ω is equal to $\{(x^\dagger, v^\dagger, 0)\}$. It follows that all solutions of (4.1) for which disease is initially present converge to $(x^\dagger, v^\dagger, 0)$. □

4.3 Stability of the infected equilibrium : $q(z) = q$

Now we consider the case $q(z) = q$. The infected equilibrium derived from (4.2),(4.3) and (4.4) is only $E^* = (x^*, v^*, z^*)$ where $v^* = mz^*/q, z^* = \{(r-\rho)\mu(x^*) - b\}/p)$. We define $X^0 = \{(\phi_0, \phi_1, \alpha) \in X \big| \phi_0(\theta)\phi_1(\theta) > 0 \text{ for some } \theta \leq 0\}$ and $X_0 = \{(\phi_0, \phi_1, \alpha) \in X \big| \phi_0(\theta)\phi_1(\theta) = 0 \text{ for all } \theta \leq 0\}$. Then the attractor $\mathcal{A}$ restricted to X_0 is $\{E_0\}$. Let Ω be the ω-limit set of the solution of (4.1). Then by the same argument for Proposition 3.10, the functional (3.39) is well defined for every solution that lies in Ω.

Proposition 4.6. *Suppose $R_0 > 1$. Let (x, v, z) be a solution of (4.1) that lies in Ω, then the time derivative of*

$$U_5(x_t, v_t, z) = U_1(x_t, v_t) + \frac{p}{r-\rho}\int_{z^*}^{z} \frac{\tau - z^*}{q} \mathrm{d}\tau \tag{4.28}$$

is nonpositive under the condition :

$$r > \rho\left(1 + \frac{v^*\mu(x^*)}{\delta x^*}\right), \text{ or } x^* > \frac{\rho}{r}\cdot \hat{x}. \tag{4.29}$$

Proof. If $R_0 > 1$ then there exists the interior equilibrium (x^*, v^*, z^*). The functional U_5 is

$$\begin{aligned} U_5(x_t, v_t, z) = \int_{x^*}^{x} \frac{\mu(\xi) - \mu(x^*)}{\mu(\xi)}\mathrm{d}\xi + \frac{1}{r-\rho}(v - v^* \log v) + \frac{p}{r-\rho}\int_{z^*}^{z} \frac{\tau - z^*}{q}\mathrm{d}\tau \\ + \frac{r}{r-\rho} W_1^\infty((\mu(x)v)_t; \mu(x^*)v^*). \end{aligned} \tag{4.30}$$

We consider the following modified system with absorption effect:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \lambda - \delta x - \mu(x)v, \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= r\int_0^\infty g(\tau)\mu(x(t-\tau))v(t-\tau)\mathrm{d}\tau - \rho\mu(x)v - (b + pz^*)v. \end{aligned} \tag{4.31}$$

This model has the same equilibrium as (4.1) and a Lyapunov functional similar to U_1. We rewrite the system (4.1) as follows :

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \lambda - \delta x - \mu(x)v, \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= r\int_0^\infty g(\tau)\mu(x(t-\tau))v(t-\tau)\mathrm{d}\tau - \rho\mu(x)v - (b + pz^*)v - (pz - pz^*)v, \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= vq - mz. \end{aligned} \tag{4.32}$$

If we note $m = v^*q/z^*$, the time derivative of U_5 along (4.32) is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}U_5(x_t, v_t, z) &= \frac{\mathrm{d}}{\mathrm{d}t}U_1 + \frac{1}{r-\rho}\left(1 - \frac{v^*}{v}\right)\{-(pz - pz^*)v\} + \frac{p}{r-\rho}\frac{\mathrm{d}}{\mathrm{d}t}\int_{z^*}^{z}\frac{\tau - z^*}{q}\mathrm{d}\tau \\ &\le \frac{r}{r-\rho}\delta x^*\left\{1 - \frac{\rho}{r}\left(1 + \frac{\mu(x^*)v^*}{\delta x^*}\right)\right\}\left(1 - \frac{\mu(x^*)}{\mu(x)}\right)\left(1 - \frac{x}{x^*}\right) \\ &+ \frac{r}{r-\rho}\mu(x^*)v^*\int_0^\infty g(\tau)\left(2 - \frac{\mu(x^*)}{\mu(x)} - \frac{\mu(x_t(\tau))v_t(\tau)}{\mu(x^*)v} + \log\frac{\mu(x_t(\tau))v_t(\tau)}{\mu(x)v}\right)\mathrm{d}\tau \\ &+ \frac{p}{r-\rho}\left(1 - \frac{v^*}{v}\right)(-v)(z - z^*) + \frac{p}{r-\rho}\frac{z - z^*}{q}(vq - mz). \end{aligned} \tag{4.33}$$

The second term is nonpositive by the extension of arithmetic-geometric mean inequality. Since $q(z) = q$ then the last two terms become

$$\begin{aligned} &\frac{p}{r-\rho}(-v + v^*)(z - z^*) + \frac{p}{r-\rho}\frac{z - z^*}{q}\left(vq - \frac{qv^*}{z^*}z\right) \\ =&\frac{pv^*}{r-\rho}\left(1 - \frac{v}{v^*}\right)(z - z^*) + \frac{pv^*}{r-\rho}(z - z^*)\left(\frac{v}{v^*} - \frac{z}{z^*}\right) \\ =&\frac{pv^*}{r-\rho}z\left(1 - \frac{z^*}{z}\right)\left(1 - \frac{z}{z^*}\right) \end{aligned}$$

This is nonpositive by arithmetic-geometric mean inequality.

Therefore when $r > \rho\left(1 + \frac{v^*\mu(x^*)}{\delta x^*}\right) = \rho \cdot \frac{\hat{x}}{x^*}$ the derivative of U_5 holds nonpositive and U_5 is a Lyapunov functional for (x^*, v^*, z^*). □

Theorem 4.7. *If $R_0 > 1$, then all solutions of* (4.1) *for which the disease is initially present converge to the infected equilibrium (x^*, v^*, z^*) under the condition* (4.29).

Proof. We can show that the maximal invariant set M in

$$\left\{(x_0, v_0, z(0)) \in \Omega \;\middle|\; \frac{\mathrm{d}}{\mathrm{d}t}U_5(x_t, v_t, z)|_{t=0} = 0\right\}, \tag{4.34}$$

is the singleton $\{(x^*, v^*, z^*)\}$. Let (x, v, z) be a solution of (4.1) in Ω. Then the ω-limit set and α-limit set are contained in M. They are equal to $\{(x^*, v^*, z^*)\}$. Since $U_5(x_t, v_t, z)$ is nonincreasing along the solution (x, v, z), (x, v, z) must be equal to (x^*, v^*, z^*) identically. Then the ω-limit set Ω is equal to $\{(x^*, v^*, z^*)\}$. It follows that all solutions of (4.1) for which disease is initially present converge to (x^*, v^*, z^*). □

4.4 Stability of the infection-free equilibrium

We consider the case $R_0 \leq 1$. Let (x, v, z) be an arbitrary solution. As in the case $R_0 > 1$, the ω-limit set Ω of (x, v, z) is non-empty and the Lyapunov functional is well-defined for every solution.

Proposition 4.8. *Suppose $R_0 \leq 1$. Let (x, v, z) be a solution of* (4.1) *that lies in Ω. Then the time derivative of*

$$U_6(x_t, v_t, z) = U_2(x_t, v_t) + \frac{p}{r-\rho}\int_0^z \frac{\tau}{q(\tau)}\mathrm{d}\tau, \tag{4.35}$$

is nonpositive.

Proof. If $R_0 \leq 1$ then there exists no interior equilibrium and the only equilibrium is $(\hat{x}, 0, 0)$. The functional U_6 is

$$U_6(x_t, v_t, z) = \int_{\hat{x}}^{x} \frac{\mu(\xi) - \mu(\hat{x})}{\mu(\xi)}\mathrm{d}\xi + \frac{1}{r-\rho}v + \frac{p}{r-\rho}\int_0^z \frac{\tau}{q(\tau)}\mathrm{d}\tau + \frac{r}{r-\rho}W_0^{\infty}((\mu(x)v)_t). \tag{4.36}$$

Then the time derivative of U_6 along (4.1) is as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}U_6(x_t, v_t, z) = \delta \cdot \left(1 - \frac{\mu(\hat{x})}{\mu(x)}\right)(x - \hat{x}) + \left(\mu(\hat{x}) - \frac{b}{r-\rho}\right)v - \frac{pm}{r-\rho}\frac{z^2}{q(z)}. \tag{4.37}$$

Since $R_0 \leq 1$ is equivalent to $\mu(\hat{x}) \leq b/(r-\rho)$, then the derivative becomes nonpositive and U_6 is a Lyapunov functional for the equilibrium $(\hat{x}, 0, 0)$. □

Theorem 4.9. *If $R_0 \leq 1$, then all solutions converge to the infection-free equilibrium.*

Proof. We can show that the maximal invariant set M in

$$\left\{(x_0, v_0, z(0)) \in \Omega \;\middle|\; \frac{\mathrm{d}}{\mathrm{d}t}U_6(x_t, v_t, z)|_{t=0} = 0\right\}, \tag{4.38}$$

is the singleton $\{(\lambda/\delta, 0, 0)\}$. As in Theorem 3.14 , we can show that all solutions converge to the infection-free equilibrium. □

5 Multistrain model with absorption effect without immune variables

In many infection processes, there often exist multiple strains in pathogens. We consider whether the competitive exclusion principle holds or the coexistence of multiple strains holds in multistrain models. In this section we will show that the competitive exclusion principle holds in the model with absorption effect without immune variables.

We consider the multistrain model with absorption extended from the single strain model (3.1). If the variable v_i denotes the virus population of i-th strain, we have

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= \lambda - \delta x - \sum_{i=1}^{n} \beta_i \mu(x) v_i, \\ \frac{\mathrm{d}v_i}{\mathrm{d}t} &= r_i \beta_i \int_0^\infty g_i(\tau)\mu(x(t-\tau))v_i(t-\tau)\mathrm{d}\tau - \rho_i \beta_i \mu(x) v_i - b_i v_i, \\ & \qquad\qquad\qquad\qquad\qquad\qquad (i = 1.2.\ldots, n),\end{aligned} \tag{5.1}$$

with the initial condition

$$x(\theta) = \phi_0(\theta), v_i(\theta) = \phi_i(\theta), \text{for } \theta \le 0, \qquad (i = 1, 2, \ldots, n), \tag{5.2}$$

where

$$\phi_0(\theta), \phi_i(\theta) \in Y_\Delta. \tag{5.3}$$

When the i-th strain satisfies the condition $\phi_0(\theta)\phi_i(\theta) > 0$ for some $\theta \le 0$ we say that the strain is *present*.

All parameters concerned with i-th strain are expressed with i. The delay kernel $g_i(\tau)$ is defined by

$$g_i(\tau) = \frac{k_i(\tau)\sigma_i(\tau)}{r_i}, \quad r_i = \int_0^\infty k_i(\tau)\sigma_i(\tau)\mathrm{d}\tau.$$

The phase space of (5.1) is $Y_\Delta \times Y_\Delta^n$. We assume $\rho_i < r_i$ and define $\tilde{R}_0^i = (r_i - \rho_i)\beta_i\mu(\hat{x})/b_i$ for every i-th strain, where $\hat{x} = \lambda/\delta$. The parameter $\tilde{R}_0^i$ quantifies the strength of infection. We may assume

$$\tilde{R}_0^1 > \tilde{R}_0^2 > \cdots > \tilde{R}_0^n, \tag{5.4}$$

without loss of generality. The exceptional case $\tilde{R}_0^i = \tilde{R}_0^{i+1}$ for some i is not considered in this paper. For $j \ne i$ it holds that

$$\tilde{R}_0^j \ne \tilde{R}_0^i \Leftrightarrow \frac{(r_j - \rho_j)\beta_j\mu(\hat{x})}{b_j} \ne \frac{(r_i - \rho_i)\beta_i\mu(\hat{x})}{b_i} \Leftrightarrow \frac{(r_j - \rho_j)\beta_j}{b_j} \ne \frac{(r_i - \rho_i)\beta_i}{b_i}. \tag{5.5}$$

We define an order relation by

$$i \trianglelefteq j \Leftrightarrow \tilde{R}_0^i \le \tilde{R}_0^j, \quad i \triangleleft j \Leftrightarrow \tilde{R}_0^i < \tilde{R}_0^j, \quad \text{and} \quad 0 \triangleleft i \text{ for all } i \in \mathbb{N}_n \cup \{0\}.$$

In our paper we assume (5.4), thus for the subset $J \subset \mathbb{N}_n$ the $\max^{\triangleleft} J$ is uniquely defined by

$$\max^{\triangleleft} J = i \Leftrightarrow i \in J \text{ and } j \trianglelefteq i \text{ for all } j \in J.$$

If the initial condition of the i-th strain is $\phi_0(\theta)\phi_i(\theta) = 0$ for all $\theta \le 0$ then $v_i(t) \equiv 0$ for all $t \ge 0$. Else if the initial condition of the i-th strain is $\phi_0(\theta)\phi_i(\theta) > 0$ for some $\theta \le 0$ then the positivity and boundedness of $x(t)$ and $v_i(t)$ are shown in the same way as in Section 3.

At an equilibrium $(x^*, \boldsymbol{v}^*)$ with $\boldsymbol{v}^* = (v_1^*, \ldots, v_n^*)$, the following equations hold :

$$\lambda - \delta x^* - \sum_{i=1}^{n} \beta_i \mu(x^*) v_i^* = 0, \tag{5.6}$$

$$\{(r_i - \rho_i)\beta_i \mu(x^*) - b_i\} v_i^* = 0, \qquad (i = 1, 2, \ldots, n). \tag{5.7}$$

By the equation (5.6) it holds that $x^* \leq \lambda/\delta$ because $v_i^* \geq 0$. If it holds that $v_i^* > 0$ for some i then $x^* < \lambda/\delta$ holds.

We define $\hat{x}_i$ by the equation (5.7) as follows :

$$\frac{(r_i - \rho_i)\beta_i}{b_i} = \frac{1}{\mu(\hat{x}_i)}. \tag{5.8}$$

By the equation (5.5) it holds that if $i \neq j$ then $\hat{x}_i \neq \hat{x}_j$ and if $j \triangleleft i$ then $\hat{x}_i < \hat{x}_j$. By the equation (5.7) it holds that $x^* = \hat{x}_i$ or $v_i^* = 0$.

If $\tilde{R}_0^i < 1$ then $x^* \leq \lambda/\delta < \hat{x}_i$, that is $x^* \neq \hat{x}_i$ and it leads $v_i^* = 0$.

If $\tilde{R}_0^i = 1$ then $x^* \leq \lambda/\delta = \hat{x}_i$. If $x^* \neq \hat{x}_i$ then $v_i^* = 0$. Else if $x^* = \hat{x}_i = \lambda/\delta$ then by (5.6) it holds that $v_j^* = 0$ for all $j \in \mathbb{N}_n$. In any case it holds that $v_i^* = 0$.

If $\tilde{R}_0^i > 1$ then $\hat{x}_i < \lambda/\delta$. If $v_i^* > 0$ then $x^* = \hat{x}_i$. Therefore it holds that $x^* \neq \hat{x}_j$ and $v_j^* = 0$ for all $j \neq i$. Then the equilibrium is $E_i(\hat{x}_i, \hat{\boldsymbol{v}})$, where $(\hat{\boldsymbol{v}})_i = \hat{v}_i = \dfrac{\lambda - \delta\hat{x}_i}{\mu(\hat{x}_i)} > 0$ and $(\hat{\boldsymbol{v}})_j = 0$ for all $j \neq i$. Else if it holds that $x^* \neq \hat{x}_i$ for any i then the equilibrium is $E_0 = (\lambda/\delta, 0, \cdots, 0)$.

5.1 The competitive exclusion principle

For the variable $x(t)$ Proposition 3.2 also holds in this model (5.1). For each initial condition it holds that

$$0 < x(t) \leq \max\left\{x(0), \frac{\lambda}{\delta}\right\} \text{ for } t \geq 0, \tag{5.9}$$

and there exists $T > 0$ such that $x(t) \leq \lambda/\delta + 1$ for $t \geq T$.

Proposition 5.1. *We can choose positive C_i's and $\widetilde{C_i}$'s that satisfy the following : for each initial condition, $v_i(t) \leq \max\{v_i(0), C_i/b_i\}$ for $t \geq 0$, and there exists $T > 0$ such that $v_i(t) \leq \widetilde{C_i}/b_i + 1$ for $t \geq T$.*

Proof. We define the functional $W_i(t)$ by the same way as Lemma 3.3 as follows :

$$W_i(t) = x(t) + \beta_i \int_0^{\infty} \mu(x(t-a)) v_i(t-a) e^{-2\Delta_1 a} \mathrm{d}a, \tag{5.10}$$

where $\Delta < \Delta_1 < \Delta_0$. Then the time derivative of $W_i(t)$ becomes as follows :

$$\begin{aligned}
&\frac{\mathrm{d}W_i}{\mathrm{d}t} \\
&= \frac{\mathrm{d}x}{\mathrm{d}t} + \beta_i \frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_0^t \mu(x(t-a))v_i(t-a)e^{-2\Delta_1 a}\mathrm{d}a + \int_t^\infty \mu(x(t-a))v_i(t-a)e^{-2\Delta_1 a}\mathrm{d}a\right\} \\
&= \lambda - \delta x(t) - \sum_{j=1}^{n}\beta_j \mu(x(t))v_j(t) + \beta_i \int_0^t \frac{\partial}{\partial t}\left\{\mu(x(t-a))v_i(t-a)\right\} e^{-2\Delta_1 a}\mathrm{d}a \\
&\quad + \beta_i \mu(x(0))v_i(0)e^{-2\Delta_1 t} + \beta_i \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{-\infty} \mu(x(u))v_i(u)e^{-2\Delta_1 (t-u)}(-\mathrm{d}u) \\
&\le \lambda - \delta x(t) - \beta_i \mu(x(t))v_i(t) - \beta_i \int_0^t \frac{\partial}{\partial a}\left\{\mu(x(t-a))v_i(t-a)\right\} e^{-2\Delta_1 a}\mathrm{d}a \\
&\quad + \beta_i \mu(x(0))v_i(0)e^{-2\Delta_1 t} + \beta_i \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-2\Delta_1 t}\int_{-\infty}^0 \mu(x(u))v_i(u)e^{2\Delta_1 u}\mathrm{d}u\right) \\
&= \lambda - \delta x(t) - \beta_i \mu(x(t))v_i(t) + \beta_i \mu(x(0))v_i(0)e^{-2\Delta_1 t} \\
&\quad - \beta_i \left\{\left[\mu(x(t-a))v_i(t-a)e^{-2\Delta_1 a}\right]_0^t - \int_0^t \mu(x(t-a))v_i(t-a)\left(-2\Delta_1 e^{-2\Delta_1 a}\right)\mathrm{d}a\right\} \\
&\quad - 2\Delta_1 \beta_i e^{-2\Delta_1 t}\int_{-\infty}^0 \mu(x(u))v_i(u)e^{2\Delta_1 u}\mathrm{d}u \\
&= \lambda - \delta x(t) - \beta_i \mu(x(t))v_i(t) + \beta_i \mu(x(0))v_i(0)e^{-2\Delta_1 t} \\
&\quad - \beta_i \left\{\mu(x(0))v_i(0)e^{-2\Delta_1 t} - \mu(x(t))v_i(t)e^0 + \Delta_1 \int_0^t \mu(x(t-a))v_i(t-a)e^{-2\Delta_1 a}\mathrm{d}a\right\} \\
&\quad - 2\Delta_1 \beta_i \int_t^\infty \mu(x(t-a))v_i(t-a)e^{-2\Delta_1 a}\mathrm{d}a \\
&= \lambda - \left(\delta x(t) + 2\Delta_1 \beta_i \int_0^\infty \mu(x(t-a))v_i(t-a)e^{-2\Delta_1 a}\mathrm{d}a\right).
\end{aligned} \tag{5.11}$$

Let $\nu_1 = \min(\delta, 2\Delta_1)$ then

$$\frac{\mathrm{d}W_i}{\mathrm{d}t} \le \lambda - \nu_1\left(x(t) + \beta_i \int_0^\infty \mu(x(t-a))v_i(t-a)e^{-2\Delta_1 a}\mathrm{d}a\right) = \lambda - \nu_1 W_i(t). \tag{5.12}$$

It follows that

$$W_i(t) \le \frac{\lambda}{\nu_1} + \left(W_i(0) - \frac{\lambda}{\nu_1}\right)e^{-\nu_1 t} \text{ for all } t \ge 0, \tag{5.13}$$

and

$$W_i(t) \le \max\left\{W_i(0), \frac{\lambda}{\nu_1}\right\} \text{ for all } t \ge 0. \tag{5.14}$$

Therefore there exists $T_1 > 0$ such that $W_i(t) \le \lambda/\nu_1 + 1$ for $t \ge T_1$.

Using this result we can show the boundedness of $v_i(t)$. It holds that

$$\int_0^\infty \mu(x(t-a))v_i(t-a)\mathrm{e}^{-2\Delta_1 a}\mathrm{d}a \le \frac{1}{\beta_i}\max\left\{W_i(0), \frac{\lambda}{\nu_1}\right\}. \tag{5.15}$$

Assume that $k_i(a)$ is bounded and continuous. Let $\|k_i\|_\infty = \sup_{a\geq 0}\{k_i(a)\}$ and assume that $\sigma_i(a)$ is continuous. For $a \geq A$ it holds that $\sigma_i(a) \leq \mathrm{e}^{-2\Delta_1 a}$ and let $C_0^i = \max\left\{\max_{0\leq a\leq A}\sigma_i(a)\mathrm{e}^{-2\Delta_1 a}, 1\right\}$. Then by using (5.15) it holds that for $t \geq 0$

$$\begin{aligned}\int_0^\infty k_i(a)\sigma_i(a)\mu(x(t-a))v_i(t-a)\,\mathrm{d}a &\leq \int_0^\infty \|k_i\|_\infty C_0^i \mathrm{e}^{-2\Delta_1 a}\mu(x(t-a))v_i(t-a)\,\mathrm{d}a \\ &= C_0^i\|k_i\|_\infty \int_0^\infty \mu(x(t-a))v_i(t-a)\mathrm{e}^{-2\Delta_1 a}\,\mathrm{d}a \\ &\leq \frac{C_0^i\|k_i\|_\infty}{\beta_i}\max\left\{W_i(0), \frac{\lambda}{\nu_1}\right\}. \end{aligned} \tag{5.16}$$

Let $C_i = C_0^i\|k_i\|_\infty \max\{W_i(0), \lambda/\nu_1\}$, then

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} \leq C_i - \rho_i\mu(x)v_i - b_iv_i \leq C_i - b_iv_i. \tag{5.17}$$

Then

$$v_i(t) \leq \frac{C_i}{b_i} + \left(v_i(0) - \frac{C_i}{b_i}\right)\mathrm{e}^{-b_it}. \tag{5.18}$$

Therefore $v_i(t)$ is bounded for $t \geq 0$.

Let $\widetilde{C_i} = C_0^i\|k_i\|_\infty(\lambda/\nu_1 + 1)$. Then by the same argument of Proposition 3.4 we can confirm that there exist $T > T_1 > 0$ such that $v_i(t) \leq \widetilde{C_i}/b_i + 1$ for $t \geq T$. □

We will state that the competitive exclusion principle holds in the multistrain model with the absorption effect without immune variables under the condition with some parameters. That is, the only i-th strain survives which is *present* and has the largest parameter $\tilde{R}_0^i$ larger than unity.

Let us define the set of strains that can potentially survive as $\mathcal{S}$ as follows :

$$\mathcal{S} = \{i \in \mathbb{N}_n | \tilde{R}_0^i > 1\}, \quad \mathbb{N}_n := \{1, 2, \cdots, n\}. \tag{5.19}$$

For $J \subset \mathcal{S}$ let us consider the phase space $X^{\mathcal{S}\backslash J}$ as follows:

$$X^{\mathcal{S}\backslash J} = \{(\psi_0, \psi_1, \ldots, \psi_n) \in X \big| \; \psi_0(\theta)\psi_i(\theta) = 0 \text{ for all } \theta \leq 0 \; \forall i \in (\mathcal{S}\backslash J)\}. \tag{5.20}$$

Let us consider the semiflow $\{U_{S\backslash J}(t)\}_{t\geq 0} : X^{S\backslash J} \to X^{S\backslash J}$ such that for $u_0 \in X$, $U_{S\backslash J}(t)u_0$ is a solution of (5.1).

Let $s = \max^{\triangleleft} J$ for $J \neq \emptyset$. We also consider the sets N and ∂N as follows :

$$\begin{aligned} N &= \{(x, \boldsymbol{v}) \in X^{\mathcal{S}\backslash J} \; \big| \; \phi_0(\theta)\phi_s(\theta) > 0 \text{ for some } \theta \leq 0\}, \\ \partial N &= \{(x, \boldsymbol{v}) \in X^{\mathcal{S}\backslash J} \; \big| \; \phi_0(\theta)\phi_s(\theta) = 0 \text{ for all } \theta \leq 0\}. \end{aligned} \tag{5.21}$$

For the initial condition $(\phi_0, \phi_1, \ldots, \phi_n) \in X$ let us define $\mathcal{J} \subset \mathcal{S}$ by

$$\mathcal{J} = \{i \in \mathcal{S} \mid \phi_0(\theta)\phi_i(\theta) > 0 \text{ for some } \theta \leq 0\}. \tag{5.22}$$

When the i-th strain is *not present*, then $v_i(t) \equiv 0$ for all $t \geq 0$. Let $\tilde{u} = (\tilde{x}, \tilde{\boldsymbol{v}})$ be a solution of (5.1). We can confirm that the semigroup associated with this model is asymptotically smooth as in Proposition 3.6. The ω-limit set Ω of $\tilde{u}$ is non-empty. Then there exists an entire solution u through an element $(\phi_0, \boldsymbol{\phi}) \in \Omega$, where $\boldsymbol{\phi} = (\phi_1, \cdots, \phi_n)$.

Theorem 5.2. *Suppose $\mathcal{J} = \emptyset$. Let $u = (x, \boldsymbol{v})$ be a solution of* (5.1) *that lies in* Ω. *Then the time derivative of*

$$\begin{aligned} U_1^M(u) = &\int_{x^*}^{x} \frac{\mu(\eta) - \mu(x^*)}{\mu(\eta)} \mathrm{d}\eta + \sum_{i=1}^{n} \frac{1}{r_i - \rho_i} v_i \\ &+ \sum_{i=1}^{n} \frac{r_i}{r_i - \rho_i} \beta_i \int_0^\infty \alpha_i(\eta) \mu(x(t-\eta)) v_i(t-\eta) \mathrm{d}\eta \end{aligned} \tag{5.23}$$

is nonpositive, where $x^* = \lambda/\delta$ and $\alpha_i(a) = \int_a^\infty g_i(\tau)\mathrm{d}\tau$.

Proof. The time derivative of $U_1^M(u)$ along (5.1) becomes

$$\begin{aligned} \frac{\mathrm{d}U_1^M}{\mathrm{d}t} &= \delta x^* \left(1 - \frac{\mu(x^*)}{\mu(x)}\right)\left(1 - \frac{x}{x^*}\right) + \sum_{i=1}^{n} \left(\beta_i \mu(x^*) - \frac{b_i}{r_i - \rho_i}\right) v_i \\ &+ \sum_{j=1}^{n} \frac{1}{r_j - \rho_j} r_j \beta_j \left(\int_0^\infty g_i(\tau)\mu(x(t-\tau))v_j(t-\tau)\mathrm{d}\tau - \mu(x(t))v_j(t)\right) \\ &+ \sum_{j=1}^{n} \frac{r_j}{r_j - \rho_j} \beta_j \int_0^\infty g_i(\tau)\left(\mu(x(t))v_j(t) - \mu(x(t-\tau))v_j(t-\tau)\right)\mathrm{d}\tau \\ &= \delta x^* \left(1 - \frac{\mu(x^*)}{\mu(x)}\right)\left(1 - \frac{x}{x^*}\right) + \sum_{i=1}^{n} \left(\beta_i \mu(x^*) - \frac{b_i}{r_i - \rho_i}\right) v_i. \end{aligned}$$

The nonpositivity of the time derivative of $U_1^M(u)$ along (5.1) is shown by $v_i \equiv 0$ for $i \in \mathcal{S}$, $\tilde{R}_0^i \le 1$ for $i \notin \mathcal{S}$ and the monotonous increase of $\mu(x)$. □

Theorem 5.3. *If $\mathcal{J} = \emptyset$, then all solutions converge to the infection free equilibrium E_0.*

Proof. We can show that the maximal invariant set M in

$$\left\{(x_0, \boldsymbol{v}_0) \in \Omega \;\middle|\; \frac{\mathrm{d}}{\mathrm{d}t} U_1^M(x_t, \boldsymbol{v}_t)|_{t=0} = 0\right\}, \text{where } (\boldsymbol{v}_t)_i = (v_i)_t, \tag{5.24}$$

is the singleton $\{(\lambda/\delta, 0, \cdots, 0)\}$. Let u be a solution of (5.1) in Ω. Then the ω-limit set and α-limit set are contained in M. They are equal to $\{E_0\}$. Since $U_1^M(u)$ is nonincreasing along the solution u, u must be equal to E_0 identically. Then the ω-limit set Ω is equal to $\{E_0\}$. It follows that all solutions of equation (5.1) for which no strain in $\mathcal{S}$ is *present* converges to E_0. □

Theorem 5.4. *If $\mathcal{J} \neq \emptyset$ then every solution on $X^{\mathcal{S}\setminus\mathcal{J}}$ converges to the equilibrium E_s under the condition :*

$$1 - \frac{\rho_s}{r_s - \rho_s} \cdot \frac{\beta_s \mu(\hat{x}_s)\hat{v}_s}{\delta \hat{x}_s} \ge 0, \tag{5.25}$$

where $s = \max^{\lhd} \mathcal{J}$.

We will show this Theorem 5.4 by mathematical induction on $\#\mathcal{J}$. For mathematical induction of Theorem 5.4 we assume that

$$1 - \frac{\rho_i}{r_i - \rho_i} \cdot \frac{\beta_i \mu(\hat{x}_i)\hat{v}_i}{\delta \hat{x}_i} \ge 0 \text{ for every } i \in \mathcal{S}. \tag{5.26}$$

This inequality (5.26) holds under the condition that the absorption effect ρ_i is sufficiently small. If ρ_i is zero then the inequality holds unconditionally .

5.1.1 Case $\#\mathcal{J} = 1$

When $\#\mathcal{J} = 1$, it can be shown in the same way as the case of single strain. Let $\mathcal{J} = \{s\}$ and define $(N, \partial N)$ as (5.21). Every strain except s-th is *not present* or will be extinct by the condition $\tilde{R}_0 \leq 1$.

When the s-th strain is *present* then we will show that the solution will not converge to the equilibrium E_0. Let define the function $V_s(t)$ as follows :

$$V_s(t) = v_s(t) + r_s\beta_s \int_0^\infty \alpha_s(a)\mu(x(t-a))v_s(t-a)\mathrm{d}a, \tag{5.27}$$

where $\alpha_s(a) = \int_a^\infty g_s(\tau)\mathrm{d}\tau$. Suppose that the solution converges to the disease free equilibrium E_0, then $v_s(t) \to 0, x(t) \to \lambda/\delta$ as $t \to \infty$. Therefore it holds that $V_s(t) \to 0$ as $t \to \infty$. On the other hand it holds that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V_s(t) &= \{(r_s - \rho_s)\beta_s\mu(x(t)) - b_s\}\, v_s(t) \\ &= \frac{b_s}{\mu(\hat{x}_s)}\left(\mu(x(t)) - \mu(\hat{x}_s)\right) v_s(t).\end{aligned} \tag{5.28}$$

The inequality $\hat{x}_s < \lambda/\delta$ leads $\mu(\hat{x}_s) < \mu(\lambda/\delta)$. Thus for some $\epsilon > 0$ there exists a $T > 0$ such that $\mu(x(t)) - \mu(\hat{x}_s) \geq \epsilon$ for all $t \geq T$. Therefore it holds that $\frac{\mathrm{d}}{\mathrm{d}t}V_s(t) > 0$ for all $t \geq T$ and it is a contradiction.

Now we can apply Theorem 4.2 in [6] and the infection is persistent. We define the following functional for our construction of Lyapunov functionals,

$$W_1^\infty(\phi; c) = c\int_0^\infty \alpha_s(a)H\left(\frac{\phi(t-a)}{c}\right)\mathrm{d}a, \tag{5.29}$$

where $H(u) = u - 1 - \log u$ and c is a positive constant for $\phi \in Y_\Delta$. To guarantee the well-definedness of the integration we require that there exist positive $\tilde{\epsilon}$ and $\tilde{M}$ such that $\tilde{\epsilon} \leq \phi(t-a) \leq \tilde{M}$ for all $0 \leq a < \infty$. A similar argument as Theorem 3.8 , Lemma 3.9 and Proposition 3.10 leads the following Proposition :

Proposition 5.5. *Assume $\tilde{R}_0^i > 1$. Let $(x, \boldsymbol{v})$ be an entire solution of* (5.1) *with $(x_0, \boldsymbol{v}_0) \in N$, then there exists a positive η'' such that*

$$\liminf_{t\to\infty} x(t) > \eta'' \text{ and } \liminf_{t\to\infty} v_i(t) > \eta'', \quad (\text{ for } i = 1, 2, \ldots, n). \tag{5.30}$$

Let $\tilde{u} = (\tilde{x}, \tilde{\boldsymbol{v}})$ be a solution of equation (5.1) with $(\tilde{x}_0, \tilde{\boldsymbol{v}}_0) \in N$. Then the ω-limit set Ω of $(\tilde{x}, \tilde{\boldsymbol{v}})$ is non-empty, compact and invariant. It follows that Ω is the union of the entire orbits of the equation (5.1). That is, if $(\phi_0, \boldsymbol{\phi}) \in Y_\Delta \times Y_\Delta^n$ is a point in Ω, then there exists an entire solution through $(\phi_0, \boldsymbol{\phi})$ such that every point on the solution is in Ω. For the solution $u = (x, \boldsymbol{v})$ that lies in Ω, combined with Proposition 5.1 and Proposition 3.10, there exist $\epsilon > 0$ and $M > 0$ such that

$$\epsilon \leq x(t) \leq M, \quad \epsilon \leq v_s(t) \leq M \text{ for all } t \in \mathbb{R}. \tag{5.31}$$

Then the functional (5.29) is well-defined for every entire solution that lies in Ω.

Proposition 5.6. *Let $u = (x, \boldsymbol{v})$ be an entire solution of* (5.1) *that lies in* Ω*, then the time derivative of*

$$
\begin{aligned}
U_2^M(u) = &\int_{\hat{x}_s}^{x} \frac{\mu(\eta) - \mu(\hat{x}_s)}{\mu(\eta)} \mathrm{d}\eta + \frac{1}{r_s - \rho_s}(v_s - \hat{v}_s \log v_s) + \sum_{i \neq s} \frac{1}{r_i - \rho_i} v_i \\
&+ \frac{r_s}{r_s - \rho_s} \beta_s W_1^\infty((\mu(x)v_s)_t; \mu(\hat{x}_s)\hat{v}_s) \\
&+ \sum_{i \neq s} \frac{r_i}{r_i - \rho_i} \beta_i \int_0^\infty \alpha_i(\eta)\mu(x(t-\eta))v_i(t-\eta)\mathrm{d}\eta
\end{aligned}
\tag{5.32}
$$

is nonpositive under the condition (5.25)*, where* $s = \max^{\triangleleft} \mathcal{J}$*.*

Proof. We denote the time derivative of $U_2^M(u)$ by using the following S_i^{A} for $i = 0, 1, \ldots, n$ as

$$
\begin{aligned}
S_0^{\mathrm{A}} &= \delta \hat{x}_s \left(1 - \frac{\mu(\hat{x}_s)}{\mu(x)}\right)\left(1 - \frac{x}{\hat{x}_s}\right), \\
S_s^{\mathrm{A}} &= \beta_s \mu(\hat{x}_s)\hat{v}_s \left(2 - \frac{\mu(\hat{x}_s)}{\mu(x)} - \frac{\mu(x)}{\mu(\hat{x}_s)}\right), \\
S_i^{\mathrm{A}} &= \left(\beta_i \mu(\hat{x}_s) - \frac{b_i}{r_i - \rho_i}\right) v_i, \quad (i \neq s).
\end{aligned}
\tag{5.33}
$$

And we define P_s^{Ad} as

$$
\begin{aligned}
P_s^{\mathrm{Ad}} = \frac{r_s \beta_s \mu(\hat{x})\hat{v}_s}{r_s - \rho_s} \int_0^\infty g_s(\tau) \Bigg(2 - \frac{\mu(\hat{x})}{\mu(x)} - \frac{\mu(x(t-\tau))v_s(t-\tau)}{\mu(\hat{x})v_s} \\
+ \log \frac{\mu(x(t-\tau))v_s(t-\tau)}{\mu(x)v_s}\Bigg) \mathrm{d}\tau.
\end{aligned}
\tag{5.34}
$$

Then the time derivative of $U_2^M(u)$ along (5.1) becomes

$$
\begin{aligned}
&\frac{\mathrm{d}U_2^M}{\mathrm{d}t} = S_0^{\mathrm{A}} + S_s^{\mathrm{A}} + \sum_{i \neq s} S_i^{\mathrm{A}} \\
&+ \frac{1}{r_s - \rho_s}\left(1 - \frac{\hat{v}_s}{v_s}\right) r_s \beta_s \left(\int_0^\infty g_s(\tau)\mu(x(t-\tau))v_s(t-\tau)\mathrm{d}\tau - \mu(x(t))v_s(t)\right) \\
&+ \frac{r_s}{r_s - \rho_s}\beta_s \int_0^\infty g_s(\tau)\left(\mu(x)v_s - \mu(x(t-\tau))v_s(t-\tau) + \mu(\hat{x}_s)\hat{v}_s \log \frac{\mu(x(t-\tau))v_s(t-\tau)}{\mu(x)v_s}\right)\mathrm{d}\tau \\
&+ \sum_{i \neq s} \frac{r_i}{r_i - \rho_i}\beta_i \left(\int_0^\infty g_i(\tau)\mu(x(t-\tau))v_i(t-\tau)\mathrm{d}\tau - \mu(x(t))v_i(t)\right) \\
&+ \sum_{i \neq s} \frac{r_i}{r_i - \rho_i}\beta_i \int_0^\infty g_i(\tau)\left\{\mu(x(t))v_i(t) - \mu(x(t-\tau))v_i(t-\tau)\right\}\mathrm{d}\tau
\end{aligned}
$$

$$
\begin{aligned}
&= S_0^{\mathrm{A}} + \beta_s \mu(\hat{x}_s)\hat{v}_s \left(2 - \frac{\mu(\hat{x}_s)}{\mu(x)} - \frac{\mu(x)}{\mu(\hat{x}_s)}\right) \\
&+ \frac{r_s \beta_s \mu(\hat{x}_s)\hat{v}_s}{r_s - \rho_s} \int_0^\infty g_s(\tau) \left(\frac{\mu(x)}{\mu(\hat{x}_s)} - \frac{\mu(x(t-\tau))v_s(t-\tau)}{\mu(\hat{x}_s)v_s} + \log \frac{\mu(x(t-\tau))v_s(t-\tau)}{\mu(x)v_s} \right) \mathrm{d}\tau \\
&+ \sum_{i \neq s} S_i^{\mathrm{A}} \\
&= S_0^{\mathrm{A}} - \frac{\rho_s}{r_s - \rho_s} \beta_s \mu(\hat{x}_s)\hat{v}_s \left(2 - \frac{\mu(\hat{x}_s)}{\mu(x)} - \frac{\mu(x)}{\mu(\hat{x}_s)}\right) \\
&+ \frac{r_s \beta_s \mu(\hat{x}_s)\hat{v}_s}{r_s - \rho_s} \int_0^\infty g_s(\tau) \left(2 - \frac{\mu(\hat{x}_s)}{\mu(x)} - \frac{\mu(x(t-\tau))v_s(t-\tau)}{\mu(\hat{x}_s)v_s} + \log \frac{\mu(x(t-\tau))v_s(t-\tau)}{\mu(x)v_s} \right) \mathrm{d}\tau \\
&+ P_s^{\mathrm{A}d} + \sum_{i \neq s} S_i^{\mathrm{A}} \\
&= S_0^{\mathrm{A}} - \frac{\rho_s}{r_s - \rho_s} \beta_s \mu(\hat{x}_s)\hat{v}_s \left(1 - \frac{\mu(\hat{x}_s)}{\mu(x)}\right) \left(1 - \frac{\mu(x)}{\mu(\hat{x}_s)}\right) + P_s^{\mathrm{A}d} + \sum_{i \neq s} S_i^{\mathrm{A}}.
\end{aligned} \tag{5.35}
$$

The monotone nonincrease of $\mu(x)/x$ leads

$$
(1 - \mu(\hat{x}_s)/\mu(x))\,(1 - x/\hat{x}_s) \leq (1 - \mu(\hat{x}_s)/\mu(x))\,(1 - \mu(x)/\mu(\hat{x}_s))\,, \tag{5.36}
$$

and it follows that

$$
\frac{\mathrm{d}U_2^M}{\mathrm{d}t} \leq \left(1 - \frac{\rho_s}{r_s - \rho_s} \cdot \frac{\beta_s \mu(\hat{x}_s)\hat{v}_s}{\delta \hat{x}_s}\right) S_0^{\mathrm{A}} + P_s^{\mathrm{Ad}} + \sum_{i \neq s} S_i^{\mathrm{A}}. \tag{5.37}
$$

If it holds that

$$
1 - \frac{\rho_s}{r_s - \rho_s} \cdot \frac{\beta_s \mu(\hat{x}_s)\hat{v}_s}{\delta \hat{x}_s} \geq 0 \left(\Leftrightarrow \hat{x}_s \geq \frac{\rho_s}{r_s} \cdot \frac{\lambda}{\delta} \right), \tag{5.38}
$$

then $\mathrm{d}U_2/\mathrm{d}t$ is nonpositive. □

Proposition 5.7. *All solutions of equation* (5.1) *for which the s-th strain is present converge to E_s under the condition* (5.38)*, where $s = \max^{\triangleleft} \mathcal{J}$.*

Proof. We can show that the maximal invariant set M in

$$
\left\{ (x_0, \boldsymbol{v}_0) \in \Omega \;\middle|\; \frac{\mathrm{d}}{\mathrm{d}t} U_2^M(x_t, \boldsymbol{v}_t)|_{t=0} = 0 \right\}, \tag{5.39}
$$

is the singleton $\{E_s\}$. Let $u = (x_t, \boldsymbol{v}_t)$ be a solution of (5.1) in Ω. Then the ω-limit set and α-limit set are contained in M. They are equal to $\{E_s\}$. Since $U_2^M(x_t, \boldsymbol{v}_t)$ is nonincreasing along the solution u, u must be equal to E_s identically. Then the ω-limit set Ω is equal to $\{E_s\}$. It follows that all solutions of equation (5.1) for which disease is present converge to E_s. □

5.1.2 Case $\#\mathcal{S} \geq 2$ and $2 \leq \#\mathcal{J} \leq \#\mathcal{S}$

In this section we assume that $\#\mathcal{S} \geq 2$ and $\#\mathcal{J} \geq 2$ for a subset $\mathcal{J} \subset \mathcal{S}$. Our induction hypothesis is concerned with the validity of Theorem 5.4 for each subset $\mathcal{J}' \subset \mathcal{S}$ such that $1 \leq \#\mathcal{J}' < \#\mathcal{J}$.

Let $s = \max^{\triangleleft} \mathcal{J}$. We define N and ∂N by (5.21). Let $\mathcal{J}' \subset \mathcal{J} \backslash \{s\}$ then by $\#\mathcal{J}' < \#\mathcal{J}$ we can use induction hypothesis when $\#\mathcal{J}' \geq 1$. That is if $\mathcal{J}' \neq \emptyset$ then there exists an $s' = \max^{\lhd} \mathcal{J}'$ such that the equilibrium $E_{s'}$ is attractive, where $\mathcal{J}' = \{i \in \mathcal{J}\backslash\{s\} \mid i\text{-th strain is } \textit{initially present}\}$. Else if $\mathcal{J}' = \emptyset$ then by Theorem 5.3 the solution converges to the equilibrium E_0. Therefore the attractor of the boundary semiflow is $E_0 \cup \bigcup_{i\in\mathcal{J}\backslash\{s\}} E_i$.

Now we will show that the solution does not converge to the equilibrium in ∂N from N. We define the function $V_s(t)$ by (5.27)

$$V_s(t) = v_s(t) + r_s\beta_s \int_0^\infty \alpha_s(a)\mu(x(t-a))v_s(t-a)\mathrm{d}a.$$

Then the time derivative of this function is

$$\frac{\mathrm{d}}{\mathrm{d}t}V_s(t) = \frac{b_s}{\mu(\hat{x}_s)}\left(\mu(x(t)) - \mu(\hat{x}_s)\right)v_s(t). \tag{5.40}$$

If $v_s(t)$ is present, then it holds that $v_s > 0$ for all $t \geq 0$. If it holds that $u \to E_k(\tilde{R}_0^k < \tilde{R}_0^s$ i.e. $\hat{x}_k > \hat{x}_s)$ as $t \to \infty$ then it holds that $v_s \to 0$, $x \to \hat{x}_k > 0$ and $V_s(t) \to 0$.

On the other hand for $\epsilon > 0$ there exists $T > 0$ such that $\mu(x(t)) - \mu(\hat{x}_s) \geq \epsilon$ for all $t \geq T$. Thus it holds that $\frac{\mathrm{d}}{\mathrm{d}t}V_s(t) > 0$ for all $t \geq T$. It is a contradiction.

Proposition 5.8. *For all $j, k \in \mathcal{J}' \cup \{0\}$, let $u(t)$ be a nontrivial (nonconstant) entire solution of* (5.1) *such that*

$$\lim_{t\to-\infty} u(t) = E_j \text{ and } \lim_{t\to\infty} u(t) = E_k,$$

then $j \lhd k$, where $\mathcal{J}' = \{i \in \mathcal{J}\backslash\{s\} \mid i\text{-th strain is initially present}\}$.

Proof. At first we will show that $j \unlhd k$. We define the function $V_j(t)$ by (5.27)

$$V_j(t) = v_j(t) + r_j\beta_j \int_0^\infty \alpha_j(a)\mu(x(t-a))v_j(t-a)\mathrm{d}a.$$

The time derivative is

$$\frac{\mathrm{d}}{\mathrm{d}t}V_j(t) = \frac{b_j}{\mu(\hat{x}_j)}\left(\mu(x(t)) - \mu(\hat{x}_j)\right)v_j(t). \tag{5.41}$$

Then if $j \rhd k$ and $v_k(t)$ is present, then it holds that $1/\mu(\hat{x}_j) > 1/\mu(\hat{x}_k)$ and $x \to \hat{x}_k$. If $u \to E_k$ as $t \to \infty$, then it holds that $v_j \to 0$ and $x \to \hat{x}_k$. Therefore for an $\epsilon > 0$ there exist a $T > 0$ such that $1/\mu(\hat{x}_j) - 1/\mu(x(t)) \geq \epsilon$ for all $t \geq T$. Thus it holds that $\frac{\mathrm{d}}{\mathrm{d}t}V_j(t) > 0$ for all $t \geq T$ and it is a contradiction. Therefore if $u \to E_k$ then $j \unlhd k$.

Secondary we will show that it holds that $j \neq k$. We define the following functional :

$$\begin{aligned} U_3^M(u) = &\int_{\hat{x}_k}^x \frac{\mu(\eta) - \mu(\hat{x}_k)}{\mu(\eta)}\mathrm{d}\eta + \frac{1}{r_k - \rho_k}(v_k - \hat{v}_k \log v_k) + \sum_{i\neq k} \frac{1}{r_i - \rho_i} v_i \\ &+ \frac{r_k}{r_k - \rho_k}\beta_k W_1^\infty((\mu(x)v_k)_t; \mu(\hat{x}_k)\hat{v}_k) \\ &+ \sum_{i\neq k} \frac{r_i}{r_i - \rho_i}\beta_i \int_0^\infty \alpha_i(\eta)\mu(x(t-\eta))v_i(t-\eta)\mathrm{d}\eta. \end{aligned} \tag{5.42}$$

The functional $W_1^\infty((\mu(x)v_k)_t; \mu(\hat{x}_k)\hat{v}_k)$ is well-defined as follows .

We assume that $j = k$, then it holds that $\lim_{t\to-\infty} u(t) = E_k$ and $\lim_{t\to\infty} u(t) = E_k$. Then there exists a $T_1 > 0$ such that $v_k(t) \leq \hat{v}_k + \epsilon$ for all $t \leq -T_1$ and $v_k(t) \leq \hat{v}_k + \epsilon$ hold for $\epsilon > 0$ and for all $t \geq T_1$. The function v_k is continuous on the interval $[-T_1, T_1]$ and it is upper bounded. Also for an $\epsilon > 0$ there exists a $T_2 > 0$ such that $v_k(t) \geq \hat{v}_k - \epsilon$ for all $t \leq -T_2$ and $v_k(t) \geq \hat{v}_k - \epsilon$ for all $t \geq T_2$ hold. On the interval $[-T_2, T_2]$ the function v_k is continuous and lower bounded. Therefore the infinite integration is well-defined and the functional is well-defined.

Then the time derivative of $U_3^M(u)$ becomes as follows:

$$\frac{\mathrm{d}U_3^M}{\mathrm{d}t} \leq \left(1 - \frac{\rho_k}{r_k - \rho_k} \cdot \frac{\beta_k \mu(\hat{x}_k)\hat{v}_k}{\delta \hat{x}_k}\right) S_0^{\mathrm{A}} + P_k^{\mathrm{Ad}} + \sum_{i \neq k} S_i^{\mathrm{A}}. \tag{5.43}$$

This is nonpositive under the condition

$$1 - \frac{\rho_k}{r_k - \rho_k} \cdot \frac{\beta_k \mu(\hat{x}_k)\hat{v}_k}{\delta \hat{x}_k} \geq 0 \left(\Leftrightarrow \hat{x}_k \geq \frac{\rho_k}{r_k} \cdot \frac{\lambda}{\delta}\right). \tag{5.44}$$

It holds that $\mathrm{d}U_3^M/\mathrm{d}t = 0$ if and only if $v_i = 0$ for $i \neq k$, $v_k = \hat{v}_k$ and $x = \hat{x}_k$. The value of U_3^M for E_k as $t \to -\infty$ is the same as the value for E_k as $t \to \infty$. The nonpositivity of the derivative of U_3^M leads that U_3^M should be constant. It is a contradiction. Thus it holds that $j \neq k$. □

Therefore there does not exist a chain of boundary equilibria or homoclinic solution. Now we can apply Theorem 4.2 in [6] and the flow $U_{\mathcal{S}\setminus J}$ on $(N, \partial N)$ is persistent. Thus for the ω-limit set of the solution, the integration W_1^∞ is well-defined and the functional U_3^M is also well-defined. Then the following Proposition holds.

Proposition 5.9. *Let Ω be the ω-limit set of a solution of* (5.1) *and let u be an entire solution that lies in Ω. Then the time derivative of*

$$\begin{aligned} U_2^M(u) = &\int_{\hat{x}_s}^{x} \frac{\mu(\eta) - \mu(\hat{x}_s)}{\mu(\eta)} \mathrm{d}\eta + \frac{1}{r_s - \rho_s}(v_s - \hat{v}_s \log v_s) + \sum_{i \neq k} \frac{1}{r_i - \rho_i} v_i \\ &+ \frac{r_s}{r_s - \rho_s} \beta_s W_1^\infty((\mu(x)v_s)_t; \mu(\hat{x}_s)\hat{v}_s) \\ &+ \sum_{i \neq k} \frac{r_i}{r_i - \rho_i} \beta_i \int_0^\infty \alpha_i(\eta)\mu(x(t-\eta))v_i(t-\eta)\mathrm{d}\eta \end{aligned} \tag{5.45}$$

is nonpositive under the condition (5.44) *for $s = \max^{\triangleleft} J$.*

Proposition 5.10. *All solutions of equation* (5.1) *for which the s-th strain is present converge to E_s under the condition* (5.38)*, where $s = \max^{\triangleleft} J$.*

Proof. We can show that the maximal invariant set M in

$$\left\{(x_0, \boldsymbol{v}_0) \in \Omega \;\middle|\; \frac{\mathrm{d}}{\mathrm{d}t} U_2^M(x_t, \boldsymbol{v}_t)|_{t=0} = 0\right\}, \tag{5.46}$$

is the singleton $\{E_s\}$. Let $u = (x, \boldsymbol{v})$ be a solution of (5.1) in Ω. Then the ω-limit set and α-limit set are contained in M. They are equal to $\{E_s\}$. Since $U_2^M(x_t, \boldsymbol{v}_t)$ is nonincreasing along the solution u, u must be equal to E_s identically. Then the ω-limit set Ω is equal to $\{E_s\}$. □

Thus Theorem 5.4 has been proved.

6 Multistrain model with absorption effect and immune variables

Inoue *et al.* [8] analyed the dynamics of ODE models of humoral immunity. For all models, they proved the global stability of the disease steady state. Our purpose is to analyze a system with delay. In this section we consider the multistrain model with absorption effect and immune variables extended from the single strain model (4.1) such that

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \lambda - \delta x - \sum_{i=1}^{n} \beta_i \mu(x) v_i, \\ \frac{\mathrm{d}v_i}{\mathrm{d}t} &= r_i \beta_i \int_0^\infty g_i(\tau)\mu(x(t-\tau))v_i(t-\tau)\mathrm{d}\tau - \rho_i\beta_i\mu(x)v_i - b_i v_i - p_i v_i z_i \\ \frac{\mathrm{d}z_i}{\mathrm{d}t} &= q_i v_i - m_i z_i \qquad (i = 1, 2, \ldots, n). \end{aligned} \tag{6.1}$$

We consider the case the immune activity increases at a rate proportional to the number of virus.

The phase space of (6.1) is $X = Y_\Delta \times Y_\Delta^n \times \mathbb{R}_+^n$. When we define $\tilde{R}_0^i = (r_i - \rho_i)\beta_i\mu(\lambda/\delta)/b_i$, we may assume

$$\tilde{R}_0^1 > \tilde{R}_0^2 > \cdots > \tilde{R}_0^n, \tag{6.2}$$

without loss of generality. The exceptional case $\tilde{R}_0^i = \tilde{R}_0^{i+1}$ for some i is not considered in this paper. For $i \lhd j$ it holds that

$$\tilde{R}_0^i < \tilde{R}_0^j \Leftrightarrow \frac{(r_i - \rho_i)\beta_i\mu(\lambda/\delta)}{b_i} < \frac{(r_j - \rho_j)\beta_j\mu(\lambda/\delta)}{b_j} \Leftrightarrow \frac{1}{\mu(\hat{x}_i)} < \frac{1}{\mu(\hat{x}_j)} \Leftrightarrow \hat{x}_i > \hat{x}_j. \tag{6.3}$$

We define the set of strains which can potentially survive as $\mathcal{S}$ defined by

$$\mathcal{S} = \{i \in \mathbb{N}_n | \tilde{R}_0^i > 1\}. \tag{6.4}$$

Let J be a subset of $\mathcal{S}$. We introduce a phase space $X^{\mathcal{S}\setminus J} = \{(\psi_0, \psi_0, \ldots, \psi_0, \alpha_1, \ldots, \alpha_n) \in X \mid \psi_0(\theta)\psi_i(\theta) = 0 \text{ for all } \theta \le 0\ \forall i \in (\mathcal{S}\setminus J)\}$ and consider the semiflow $\{U_{S\setminus J}(t)\}_{t\ge 0} : X^{S\setminus J} \to X^{S\setminus J}$ such that for $u_0 \in X$, $U_{S\setminus J}(t)u_0$ is a solution of (6.1).

The positivity and boundedness of $x(t)$ and $v_i(t)$ are shown in the same way as Proposition 5.1, where $\phi_0(\theta)\phi_i(\theta) > 0$ for some $\theta \le 0$. And it holds that $x(t) \le \max\{x(0), \lambda/\delta\}, \limsup_{t\to\infty} x(t) \le \lambda/\delta$.

When the initial condition of immune variable is $z_i(0) > 0$, the positivity of z_i is shown as follows. If $t_0 > 0$ is the least positive time such that $z_i(t_0) = 0$ then the third equation of (6.1) leads

$$\frac{\mathrm{d}z_i}{\mathrm{d}t}\Big|_{t=t_0} = q_i v_i(t_0) - m_i z_i(t_0) = q_i v_i(t_0) > 0. \tag{6.5}$$

Because of the positivity of v_i it holds that $z_i(t) < 0$ for some positive t such that $t < t_0$. It is a contradiction. Therefore it holds that $z_i(t) > 0$ for all $t \ge 0$.

The boundedness of z_i is shown as follows.

Let $\overline{V}_i$ be the upper bound of v_i. By the third equation of (6.1) we obtain

$$\frac{\mathrm{d}z_i}{\mathrm{d}t} \le q_i\overline{V}_i - m_i z_i. \tag{6.6}$$

Then we have the followings :

$$z_i \leq \frac{q_i \overline{V}_i}{m_i} + \left(z_i(0) - \frac{q_i \overline{V}_i}{m_i} \right) e^{-m_i t}, \tag{6.7}$$

therefore it holds that

$$z_i \leq \max \left\{ z_i(0), \frac{q_i \overline{V}_i}{m_i} \right\} \text{ for all } t \geq 0, \tag{6.8}$$

and there exists a T such that $z_i \leq q_i \overline{V}_i / m_i + 1$ for all $t \geq T$.

We denote by $(x^*, \boldsymbol{v}^*, \boldsymbol{z}^*)$ the candidate of the equilibrium. At this equilibrium the right hand side of (6.1) becomes

$$\lambda - \delta x^* - \sum_{i=1}^{n} \beta_i \mu(x^*) v_i^* = 0, \tag{6.9}$$

$$\{\beta_i (r_i - \rho_i) \mu(x^*) - b_i - p_i z_i^*\} v_i^* = 0, \tag{6.10}$$

$$q_i v_i^* - m_i z_i^* = 0. \tag{6.11}$$

By (6.10), if $\beta_i(r_i - \rho_i)\mu(x^*) - b_i < 0$ then v_i^* should be zero and z_i^* becomes zero by (6.11). If $\beta_i(r_i - \rho_i)\mu(x^*) - b_i > 0$ then there can exist positive v_i^*, and if $v_i^* > 0$ then it holds that

$$\beta_i (r_i - \rho_i) \mu(x^*) - b_i - p_i z_i^* = 0. \tag{6.12}$$

Thus

$$z_i^* = \left[\frac{\beta_i (r_i - \rho_i) \mu(x^*) - b_i}{p_i} \right]_+, \tag{6.13}$$

where

$$[a]_+ = \begin{cases} a & (a \geq 0) \\ 0 & (a < 0) \end{cases}. \tag{6.14}$$

Then by (6.11)

$$v_i^* = \frac{m_i}{q_i} z_i^* = \frac{m_i}{q_i} \left(\frac{1}{p_i} [\beta_i (r_i - \rho_i) \mu(x^*) - b_i]_+ \right). \tag{6.15}$$

The values of $\hat{x}_i$ are defined by (5.8). And we define the function $h_{2,i}(x)$ as follows:

$$h_{2,i}(x) = \begin{cases} 0 & (x \leq \hat{x}_i) \\ \dfrac{\beta_i m_i}{q_i} \left(\dfrac{1}{p_i} [\beta_i (r_i - \rho_i) \mu(x) - b_i]_+ \right) & (\hat{x}_i < x) \end{cases}. \tag{6.16}$$

We consider the following two functions $h_1(x)$ and $h_{2,J}(x)$:

$$h_1(x) = \frac{\lambda - \delta x}{\mu(x)}, \quad h_{2,J}(x) = \sum_{i \in J} h_{2,i}(x). \tag{6.17}$$

Then

$$\lim_{x \to +0} h_1(x) = \infty, \quad h_1(\lambda/\delta) = 0, \tag{6.18}$$

and $h_1(x)$ is strictly decreasing. On the other hand $h_{2,J}(0) = 0$ and $h_{2,J}(x)$ is nondecreasing for positive x. Then there exists a unique x^* which satisfies $h_1(x^*) = h_{2,J}(x^*)$ with $0 < x^* \le \lambda/\delta$. This x^* determines z_i^* and v_i^* by (6.13) and (6.15) respectively. The set $(x^*, \boldsymbol{v}^*, \boldsymbol{z}^*)$ satisfies (6.9),(6.10) and (6.11).

For the initial condition $(\phi_0, \phi_1, \ldots, \phi_n, \alpha_1, \ldots, \alpha_n) \in X$ let us define the set $\mathcal{J}$ as follows :

$$\mathcal{J} = \{i \in \mathcal{S} \mid \phi_0(\theta)\phi_i(\theta) > 0 \text{ for some } \theta \le 0\}. \tag{6.19}$$

When $\mathcal{J} \neq \emptyset$ let x^* be the solution of $h_1(x) = h_{2,\mathcal{J}}(x)$ and define $K_{\mathcal{J}} = \{i \in \mathcal{J} | \hat{x}_i \le x^*\}$. The set $K = K_{\mathcal{J}}$ is the smallest set which satisfies $h_{2,K}(x) = h_{2,\mathcal{J}}(x)$ for $0 \le x \le x^*$. The equilibrium is represented by $E_{K_{\mathcal{J}}}$. If $\mathcal{J} = \emptyset$ then $K_{\mathcal{J}} = \emptyset$ and the equilibrium is $E_{\emptyset}$.

When the functional W_1^{∞} is well-defined we define $U^{\mathrm{M0}}, U_i^{\mathrm{M1}}, U_i^{\mathrm{M2}}$ for $i = 1, 2, \ldots, n$ by

$$\begin{aligned}
U^{\mathrm{M0}}(x) &= \int_{x^*}^{x} \frac{\mu(\xi) - \mu(x^*)}{\mu(\xi)} \mathrm{d}\xi, \\
U_i^{\mathrm{M1}}(x, v_i, z_i) &= U_i^{\mathrm{M3}}(x, v_i, z_i) + \frac{r_i}{r_i - \rho_i} \beta_i W_1^{\infty}((\mu(x)v_i)_t; \mu(x^*)v_i^*), \\
U_i^{\mathrm{M2}}(x, v_i, z_i) &= U_i^{\mathrm{M3}}(x, v_i, z_i) + \frac{r_i}{r_i - \rho_i} \beta_i W_0^{\infty}((\mu(x)v_i)_t),
\end{aligned} \tag{6.20}$$

where

$$U_i^{\mathrm{M3}}(x, v_i, z_i) = \frac{1}{r_i - \rho_i}(v_i - v_i^* \log v_i) + \frac{p_i}{r_i - \rho_i} \int_{z_i^*}^{z_i} \frac{\tau - z_i^*}{q_i} \mathrm{d}\tau. \tag{6.21}$$

We define $U_4^M(u; K)$ as follows

$$U_4^M(u; K) = U^{\mathrm{M0}}(x) + \sum_{i \in K} U_i^{\mathrm{M1}}(x, v_i, z_i) + \sum_{i \notin K} U_i^{\mathrm{M2}}(x, v_i, z_i). \tag{6.22}$$

Let $\tilde{u} = (\tilde{x}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{z}})$ be a solution of (6.1). We can confirm that the semigroup associated with this model is asymptotically smooth as in Proposition 3.6. The ω-limit set Ω of $\tilde{u}$ is non-empty. Then there exists an entire solution u through an element $(\phi_0, \boldsymbol{\phi}, \boldsymbol{z}(0)) \in \Omega$.

Proposition 6.1. *Suppose $\mathcal{J} = \emptyset$. Let $u = (x, \boldsymbol{v}, \boldsymbol{z})$ be a solution of* (6.1) *that lies in Ω. Then the time derivative of*

$$\begin{aligned}
U_4^M(u; \emptyset) &= U^{M0}(x) + \sum_{i \in \mathbb{N}_n} U_i^{M2}(x, v_i, z_i) \\
&= \int_{\lambda/\delta}^{x} \frac{\mu(\eta) - \mu(\lambda/\delta)}{\mu(\eta)} \mathrm{d}\eta + \sum_{i=1}^{n} \frac{1}{r_i - \rho_i} v_i + \sum_{i=1}^{n} \frac{p_i}{r_i - \rho_i} \int_0^{z_i} \frac{\tau}{q_i} \mathrm{d}\tau \\
&\quad + \sum_{i=1}^{n} \frac{r_i}{r_i - \rho_i} \beta_i \int_0^{\infty} \alpha_i(\eta) \mu(x(t - \eta)) v_i(t - \eta) \mathrm{d}\eta
\end{aligned} \tag{6.23}$$

is nonpositive, where $\alpha_i(a) = \int_a^{\infty} g_i(\tau) \mathrm{d}\tau$.

Proof. By $\mathcal{J} = \emptyset$ it holds that $K_{\mathcal{J}} = \emptyset$. The time derivative of $U_4^M(u; \emptyset)$ along (6.1) becomes

$$
\begin{aligned}
\frac{\mathrm{d}U_4^M}{\mathrm{d}t} &= \left(1 - \frac{\mu(\lambda/\delta)}{\mu(x)}\right)\left(\lambda - \delta x - \sum_{i=1}^n \beta_i \mu(x) v_i\right) \\
&\quad + \sum_{i=1}^n \frac{1}{r_i - \rho_i}\left(r_i\beta_i \int_0^\infty g_i(\tau)\mu(x(t-\tau))v_i(t-\tau)\mathrm{d}\tau - \rho_i\beta_i\mu(x)v_i - b_i v_i - p_i v_i z_i\right) \\
&\quad + \sum_{i=1}^n \frac{p_i}{r_i - \rho_i}\frac{z_i}{q_i}(q_i v_i - m_i z_i) \\
&\quad + \sum_{i=1}^n \frac{r_i}{r_i - \rho_i}\beta_i \int_0^\infty g_i(\tau)\left(\mu(x(t))v_i(t) - \mu(x(t-\tau))v_i(t-\tau)\right)\mathrm{d}\tau \\
&= \lambda\left(1 - \frac{\mu(\lambda/\delta)}{\mu(x)}\right)\left(1 - \frac{x}{\lambda/\delta}\right) - \sum_{i=1}^n \beta_i\mu(x)v_i + \sum_{i=1}^n \beta_i\mu(\lambda/\delta)v_i \\
&\quad + \sum_{i=1}^n \frac{1}{r_i - \rho_i}\left(-\rho_i\beta_i\mu(x) - b_i + r_i\beta_i\mu(x)\right)v_i \\
&\quad + \sum_{i=1}^n \frac{1}{r_i - \rho_i}\left(-p_i z_i v_i + p_i z_i v_i - \frac{p_i}{q_i}m_i z_i^2\right) \\
&= \lambda\left(1 - \frac{\mu(\lambda/\delta)}{\mu(x)}\right)\left(1 - \frac{x}{\lambda/\delta}\right) + \sum_{i=1}^n \left(\beta_i\mu(\lambda/\delta) - \frac{b_i}{r_i - \rho_i}\right)v_i - \sum_{i=1}^n \frac{p_i}{r_i - \rho_i}\frac{m_i}{q_i}z_i^2.
\end{aligned}
$$

For all $i \in \mathcal{S}\backslash\mathcal{J} = \mathcal{S}$ it holds that $v_i \equiv 0$. The nonpositivity of the time derivative of $U_4^M(u;K)$ along (6.1) is shown by the monotonous increase of $\mu(x)$ and $\tilde{R}_0^i \leq 1$ for all $\in \mathbb{N}_n\backslash\mathcal{S}$. □

Theorem 6.2. *If $\mathcal{J} = \emptyset$, then all solutions converge to the infection free equilibrium $E_\emptyset$.*

Proof. We can show that the maximal invariant set M in

$$\left\{(x_0, \boldsymbol{v}_0, \boldsymbol{z}(0)) \in \Omega \;\middle|\; \frac{\mathrm{d}}{\mathrm{d}t}U_4^M(x_t, \boldsymbol{v}_t, \boldsymbol{z})|_{t=0} = 0\right\} \tag{6.24}$$

is the singleton $\{(\lambda/\delta, 0, \cdots, 0)\}$. Let u be a solution of (6.1) in Ω. Then the ω-limit set and α-limit set are contained in M. They are equal to $\{E_\emptyset\}$. Since $U_4^M(u;\emptyset)$ is nonincreasing along the solution u, u must be equal to $E_\emptyset$ identically. Then the ω-limit set Ω is equal to $\{E_\emptyset\}$. It follows that all solutions of equation (6.1) for which no strain in $\mathcal{S}$ is *present* converge to $E_\emptyset$. □

When it holds that $\mathcal{J} \neq \emptyset$, we define the sets N and ∂N as follows :

$$
\begin{aligned}
N &= \{(x_t, \boldsymbol{v}_t, \boldsymbol{z}(t)) \in X^{\mathcal{S}\backslash\mathcal{J}} \mid \phi_0(\theta)\phi_s(\theta) > 0 \text{ for some } \theta \leq 0 \ \forall i \in K_\mathcal{J}\}, \\
\partial N &= \{(x_t, \boldsymbol{v}_t, \boldsymbol{z}(t)) \in X^{\mathcal{S}\backslash\mathcal{J}} \mid \phi_0(\theta)\phi_s(\theta) = 0 \text{ for all } \theta \leq 0 \ \forall i \in K_\mathcal{J}\}.
\end{aligned} \tag{6.25}
$$

6.1 Coexistence of strains

We represent the equilibrium $E_{K_\mathcal{J}} = (x^*, \boldsymbol{v}^*, \boldsymbol{z}^*)$. It holds that $v_i^* > 0, z_i^* > 0$ for $i \in K_\mathcal{J}$ and $v_i^* = z_i^* = 0$ for $i \notin K_\mathcal{J}$.

Theorem 6.3. *If $\mathcal{J} \neq \emptyset$ then every solution on $X^{\mathcal{S}\setminus\mathcal{J}}$ converges to the equilibrium $E_{K_{\mathcal{J}}}$ under the condition :*

$$1 - \sum_{i \in K_{\mathcal{J}}} \frac{\rho_i}{r_i - \rho_i} \cdot \frac{\beta_i \mu(x^*) v_i^*}{\delta x^*} \geq 0, \tag{6.26}$$

where $\mathcal{J} \neq \emptyset$ and the equilibrium is $(x^*, \boldsymbol{v}^*, \boldsymbol{z}^*)$.

We will show this Theorem 6.3 by mathematical induction on $\#\mathcal{J}$.

6.1.1 Case $\#\mathcal{J} = 1$

Let $\mathcal{J} = \{s\}$. Other strains are *not present* or it holds that $\tilde{R}_0^i \leq 1$ for them. If the s-th strain is *present*, then the solution converges to the equilibrium $E_{K_{\mathcal{J}}} = E_{\{s\}}$. Else if s-th strain is *not present*, then the solution converges to the equilibrium $E_{\emptyset}$.

When the s-th strain is *present* then we will show that the solution will not converge to the equilibrium $E_{\emptyset}$ same as in 5.1.1. Let define the function $V_s(t)$ by (5.27). If the solution converges to $E_{\emptyset}$ then $v_s(t) \to 0, x(t) \to \lambda/\delta$ and $z_s \to 0$ as $t \to \infty$. Therefore it holds that $V_s(t) \to 0$ as $t \to \infty$. On the other hand the time derivative of $V_s(t)$ becomes as follows :

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V_s(t) &= \{(r_s - \rho_s)\beta_s \mu(x(t)) - b_s - p_s z_s\} v_s(t) \\ &= \left\{ \frac{b_s}{\mu(\hat{x}_s)} (\mu(x(t)) - \mu(\hat{x}_s)) - p_s z_s \right\} v_s(t). \end{aligned} \tag{6.27}$$

By $s \in \mathcal{S}$ it holds that $\hat{x}_s < \lambda/\delta$, that is $\mu(\hat{x}_s) < \mu(\lambda/\delta)$. Then there exist $\epsilon > 0$ and $T > 0$ such that $\mu(x(t)) - \mu(\hat{x}_s) \geq \epsilon$ and z_s is sufficiently small for all $t \geq T$. Thus it holds that $\frac{\mathrm{d}}{\mathrm{d}t} V_s(t) > 0$ for all $t \geq T$, it is a contradiction.

Therefore we can apply Theorem 4.2 in [6], and the infection is persistent. The similar argument in 5.1.1 will show that the ω-limit set Ω of $(\tilde{x}, \tilde{v}_1, \cdots \tilde{v}_n, \tilde{\boldsymbol{z}})$ is non-empty, compact and invariant. It follows that Ω is the union of the entire orbits of the equation (6.1). That is, if $(\phi_0, \boldsymbol{\phi}, \boldsymbol{z}(0)) \in Y_\Delta \times Y_\Delta^n \times \mathbb{R}_+^n$ is a point in Ω, then there exists an entire solution through $(\phi_0, \boldsymbol{\phi}, \boldsymbol{z}(0))$ such that every point on the solution is in Ω. For the solution $u = (x, \boldsymbol{v}, \boldsymbol{z})$ that lies in Ω, combined with Proposition 5.1 and Proposition 3.10, there exist $\epsilon > 0$ and $M > 0$ such that

$$\epsilon \leq x(t) \leq M, \quad \epsilon \leq v_s(t) \leq M \text{ for all } t \in \mathbb{R}. \tag{6.28}$$

Then the functional W_1^∞ defined by (5.29) is well defined for every solution of (6.1) that lies in Ω.

Proposition 6.4. *Let $u = (x, \boldsymbol{v}, \boldsymbol{z})$ be an entire solution of* (6.1) *that lies in Ω. Then* *the time derivative of*

$$
\begin{aligned}
U_4^M(u;\{s\}) &= U^{M0}(x) + U_s^{M1}(x,v_s,z_s) + \sum_{i\neq s} U_i^{M2}(x,v_i,z_i) \\
&= \int_{x^*}^{x} \frac{\mu(\xi)-\mu(x^*)}{\mu(\xi)}\mathrm{d}\xi + \frac{1}{r_s-\rho_s}(v_s - v_s^*\log v_s) + \frac{p_s}{r_s-\rho_s}\int_{z_s^*}^{z_s}\frac{\tau - z_s^*}{q_s}\mathrm{d}\tau \\
&+ \sum_{i\neq s}\frac{1}{r_i-\rho_i}v_i + \sum_{i\neq s}\frac{p_i}{r_i-\rho_i}\int_0^{z_i}\frac{\tau}{q_i}\mathrm{d}\tau \\
&+ \frac{r_s}{r_s-\rho_s}\beta_s\mu(x^*)v_s^*\int_0^\infty \alpha_s(\tau) H\left(\frac{\mu(x(t-\tau))v_s(t-\tau)}{\mu(x^*)v_s^*}\right)\mathrm{d}\tau \\
&+ \sum_{i\neq s}\frac{r_i}{r_i-\rho_i}\beta_i\int_0^\infty \alpha_i(\tau)\mu(x(t-\tau))v_i(t-\tau)\mathrm{d}\tau
\end{aligned}
\tag{6.29}
$$

is nonpositive under the following condition

$$
1 - \frac{\rho_s}{r_s-\rho_s}\frac{\beta_s\mu(x^*)v_s^*}{\delta x^*} \geq 0, \tag{6.30}
$$

where $\alpha_i(\tau) = \int_a^\infty g_i(a)\mathrm{d}a$ *.*

Proof. The time derivative of $U_4^M(u;\{s\})$ along (6.1) becomes

$$
\begin{aligned}
\frac{\mathrm{d}U_4^M}{\mathrm{d}t} &= \left(1 - \frac{\mu(x^*)}{\mu(x)}\right)\left(\lambda - \delta x - \sum_{i=1}^{n}\beta_i\mu(x)v_i\right) \\
&+ \frac{1}{r_s-\rho_s}\left(1 - \frac{v_s^*}{v_s}\right)\left(r_s\beta_s\int_0^\infty g_s(\tau)\mu(x(t-\tau))v_s(t-\tau)\mathrm{d}\tau - \rho_s\beta_s\mu(x)v_s - b_s v_s - p_s z_s v_s\right) \\
&+ \frac{p_s}{r_s-\rho_s}\frac{z_s - z_s^*}{q_s}(q_s v_s - m_s z_s) \\
&+ \sum_{i\neq s}\frac{1}{r_i-\rho_i}\left(r_i\beta_i\int_0^\infty g_i(\tau)\mu(x(t-\tau))v_i(t-\tau)\mathrm{d}\tau - \rho_i\beta_i\mu(x(t))v_i(t) - b_i v_i - p_i z_i v_i\right) \\
&+ \sum_{i\neq s}\frac{p_i}{r_i-\rho_i}\frac{z_i}{q_i}(q_i v_i - m_i z_i) \\
&+ \frac{r_s\beta_s}{r_s-\rho_s}\int_0^\infty g_s(\tau)\left(\mu(x)v_s - \mu(x(t-\tau))v_s(t-\tau) + \mu(x^*)v_s^*\log\frac{\mu(x(t-\tau))v_s(t-\tau)}{\mu(x)v_s}\right)\mathrm{d}\tau \\
&+ \sum_{i\neq s}\frac{r_i}{r_i-\rho_i}\beta_i\int_0^\infty g_i(\tau)\left(\mu(x(t))v_i(t) - \mu(x(t-\tau))v_i(t-\tau)\right)\mathrm{d}\tau
\end{aligned}
$$

$$
\begin{aligned}
&= \left(1-\frac{\mu(x^*)}{\mu(x)}\right)\left(\delta x^* + \beta_s\mu(x^*)v_s^* - \delta x - \sum_{i=1}^n \beta_i\mu(x)v_i\right)\\
&\quad + \frac{1}{r_s-\rho_s}\left(r_s\beta_s\int_0^\infty g_s(\tau)\mu(x(t-\tau))v_s(t-\tau)\mathrm{d}\tau - \rho_s\beta_s\mu(x(t))v_s(t) - b_sv_s - p_sz_sv_s\right)\\
&\quad + \frac{1}{r_s-\rho_s}\left(-\frac{v_s^*}{v_s}r_s\beta_s\int_0^\infty g_s(\tau)\mu(x(t-\tau))v_s(t-\tau)\mathrm{d}\tau + \rho_s\beta_s\mu(x(t))v_s^* + b_sv_s^* + p_sz_sv_s^*\right)\\
&\quad + \frac{r_s\beta_s}{r_s-\rho_s}\int_0^\infty g_s(\tau)\left(\mu(x)v_s - \mu(x(t-\tau))v_s(t-\tau) + \mu(x^*)v_s^*\log\frac{\mu(x(t-\tau))v_s(t-\tau)}{\mu(x)v_s}\right)\mathrm{d}\tau\\
&\quad + \frac{p_s}{r_s-\rho_s}\left(v_sz_s - v_sz_s^* - \frac{m_sz_s^2 - m_sz_sz_s^*}{q_s}\right)\\
&\quad + \sum_{i\neq s}\frac{1}{r_i-\rho_i}\left(r_i\beta_i\mu(x)v_i - \rho_i\beta_i\mu(x(t))v_i(t) - b_iv_i - p_iz_iv_i\right)\\
&\quad + \sum_{i\neq s}\frac{p_i}{r_i-\rho_i}\frac{z_i}{q_i}(q_iv_i - m_iz_i)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(1-\frac{\mu(x^*)}{\mu(x)}\right)\delta\left(x^*-x\right) + \beta_s\mu(x^*)v_s^* - \beta_s\mu(x)v_s - \frac{\mu(x^*)}{\mu(x)}\beta_s\mu(x^*)v_s^* + \beta_s\mu(x^*)v_s\\
&\quad - \sum_{i\neq s}\beta_i\mu(x)v_i + \sum_{i\neq s}\beta_i\mu(x^*)v_i\\
&\quad + \frac{r_s\beta_s\mu(x^*)v_s^*}{r_s-\rho_s}\int_0^\infty g_s(\tau)\left(-\frac{\mu(x(t-\tau))v_s(t-\tau)}{\mu(x^*)v_s} + \log\frac{\mu(x(t-\tau))v_s(t-\tau)}{\mu(x)v_s}\right)\mathrm{d}\tau\\
&\quad + \frac{\rho_s\beta_s}{r_s-\rho_s}\mu(x(t))(v_s^*-v_s) + \frac{(r_s\beta_s-\rho_s\beta_s)\mu(x^*) - p_sz_s^*}{r_s-\rho_s}(v_s^*-v_s) + \frac{r_s\beta_s}{r_s-\rho_s}\mu(x)v_s\\
&\quad + \frac{p_s}{r_s-\rho_s}\left(v_s^*z_s - v_sz_s^* - \frac{v_s^*z_s^2}{z_s^*} + v_s^*z_s\right)\\
&\quad + \sum_{i\neq s}\frac{1}{r_i-\rho_i}\left(r_i\beta_i\mu(x)v_i - \rho_i\beta_i\mu(x(t))v_i(t) - b_iv_i - p_iz_iv_i\right)\\
&\quad + \sum_{i\neq s}\frac{p_i}{r_i-\rho_i}\frac{z_i}{q_i}(q_iv_i - m_iz_i)\\
&= \delta x^*\left(1-\frac{\mu(x^*)}{\mu(x)}\right)\left(1-\frac{x}{x^*}\right) + \sum_{i\neq s}\left(\beta_i\mu(x^*) - \frac{b_i}{r_i-\rho_i}\right)v_i - \sum_{i\neq s}\frac{p_i}{r_i-\rho_i}\frac{m_i}{q_i}z_i^2.\\
&+ \frac{r_s\beta_s\mu(x^*)v_s^*}{r_s-\rho_s}\int_0^\infty g(\tau)\left(2-\frac{\mu(x^*)}{\mu(x)} - \frac{\mu(x(t-\tau))v_s(t-\tau)}{\mu(x^*)v_s} + \log\frac{\mu(x(t-\tau))v_s(t-\tau)}{\mu(x)v_s}\right)\mathrm{d}\tau\\
&+ \frac{p_sv_s^*z_s}{r_s-\rho_s}\left(2-\frac{z_s}{z_s^*} - \frac{z_s^*}{z_s}\right) - \frac{\rho_s}{r_s-\rho_s}\beta_s\mu(x^*)v_s^*\left(2-\frac{\mu(x^*)}{\mu(x)} - \frac{\mu(x)}{\mu(x^*)}\right). \qquad (6.31)
\end{aligned}
$$

The monotone nonicreasing of $\mu(x)/x$ leads

$$
\frac{-\rho_s}{r_s-\rho_s}\beta_s\mu(x^*)v_s^*\left(2-\frac{\mu(x^*)}{\mu(x)} - \frac{\mu(x)}{\mu(x^*)}\right) \le \frac{-\rho_s}{r_s-\rho_s}\beta_s\mu(x^*)v_s^*\left(1-\frac{\mu(x^*)}{\mu(x)}\right)\left(1-\frac{x}{x^*}\right).
$$

Then it holds that

$$
\begin{aligned}
\frac{\mathrm{d}U_4^M}{\mathrm{d}t} &\le \delta x^* \left(1 - \frac{\rho_s}{r_s-\rho_s}\frac{\beta_s\mu(x^*)v_s^*}{\delta x^*}\right)\left(1-\frac{\mu(x^*)}{\mu(x)}\right)\left(1-\frac{x}{x^*}\right) \\
&+ \frac{r_s\beta_s\mu(x^*)v_s^*}{r_s-\rho_s}\int_0^\infty g(\tau)\left(2-\frac{\mu(x^*)}{\mu(x)}-\frac{\mu(x(t-\tau))v_s(t-\tau)}{\mu(x^*)v_s}+\log\frac{\mu(x(t-\tau))v_s(t-\tau)}{\mu(x)v_s}\right)\mathrm{d}\tau \\
&+ \frac{p_s v_s^* z_s}{r_s-\rho_s}\left(2-\frac{z_s}{z_s^*}-\frac{z_s^*}{z_s}\right)+\sum_{i\neq s}\left(\beta_i\mu(x^*)-\frac{b_i}{r_i-\rho_i}\right)v_i-\sum_{i\neq s}\frac{p_i}{r_i-\rho_i}\frac{m_i}{q_i}z_i^2. \qquad (6.32)
\end{aligned}
$$

For all $i \in \mathcal{S}\backslash\{s\}$ it holds that $v_i \equiv 0$. The nonpositivity of the time derivative of $U_4^M(u;\{s\})$ along (6.1) is shown by the monotonous increase of $\mu(x)$, the arithmetic-geometric mean inequality, the extension of arithmetic-geometric mean inequality and $\tilde{R}_0^i \le 1$ for all $\in \mathbb{N}_n\backslash\mathcal{S}$ under the condition $1 - \frac{\rho_s}{r_s-\rho_s}\frac{\beta_s\mu(x^*)v_s^*}{\delta x^*} \ge 0$. □

For the boundary equilibria there exists no homoclinic solution. It is shown by Theorem 4.9.

6.1.2 Case $\#\mathcal{S} \ge 2$ and $2 \le \#\mathcal{J} \le \#\mathcal{S}$

In this section we assume that $\#\mathcal{S} \ge 2$ and $\#\mathcal{J} \ge 2$ for a subset $\mathcal{J} \subset \mathcal{S}$. Our induction hypothesis is concerned with the validity of Theorem 6.3 for each subset $\mathcal{J}' \subset \mathcal{S}$ such that $1 \le \#\mathcal{J}' < \#\mathcal{J}$.

We define the sets N and ∂N as (6.25).

$$
\begin{aligned}
N &= \{(x_t, \boldsymbol{v}_t, \boldsymbol{z}(t)) \in X^{\mathcal{S}\backslash\mathcal{J}} \,|\, \text{the } i \text{ -th strain is } \textit{present} \text{ for all } i \in K_{\mathcal{J}}\}, \\
\partial N &= \{(x_t, \boldsymbol{v}_t, \boldsymbol{z}(t)) \in X^{\mathcal{S}\backslash\mathcal{J}} \,|\, \text{the } i \text{ -th strain is not } \textit{present} \text{ for some } i \in K_{\mathcal{J}}\}.
\end{aligned}
$$

Let us define $\mathcal{J}' \subset \mathcal{J}$ by

$$
\mathcal{J}' = \{i \in \mathcal{J} \;|\phi_0(\theta)\phi_i(\theta) > 0 \text{ for some } \theta \le 0\}. \qquad (6.33)
$$

where $(\mathcal{J}\backslash\mathcal{J}') \cap K_{\mathcal{J}} \neq \emptyset$, and define $\mathcal{K}$ as follows :

$$
\mathcal{K} = \{K_{\mathcal{J}'} | \mathcal{J}' \subset \mathcal{J} \text{ and } (\mathcal{J}\backslash\mathcal{J}') \cap K_{\mathcal{J}} \neq \emptyset\}. \qquad (6.34)
$$

The boundary equilibrium E_K corresponds to the set $K \in \mathcal{K}$. It holds that $x^*_{K_{\mathcal{J}}} < x^*_K$. Because $\#\mathcal{J}' < \#\mathcal{J}$ we can use inductive hypothesis for $\#\mathcal{J}' \ge 1$. That is if $\mathcal{J}' \neq \emptyset$ then every solution converges to the equilibrium $E_{K_{\mathcal{J}'}}$. If $\mathcal{J}' = \emptyset$ then by Theorem 6.2 every solution converges to $E_\emptyset$. Therefore the attractor of the boundary semiflow is $\bigcup_{K\in\mathcal{K}} E_K$.

Now we will show that if all strains of $K_{\mathcal{S}}$ are *present* the solution does not converge to any equilibrium on ∂N. We define the function $V_i(t)$ by (5.27).

$$
V_i(t) = v_i(t) + r_i\beta_i \int_0^\infty \alpha_i(a)\mu(x(t-a))v_i(t-a)\mathrm{d}a.
$$

The time derivative of this function is

$$
\frac{\mathrm{d}}{\mathrm{d}t}V_i(t) = \left\{\frac{b_i}{\mu(\hat{x}_i)}\left(\mu(x(t)) - \mu(\hat{x}_i)\right) - p_i z_i\right\} v_i(t).
$$

By the assumption $K_{\mathcal{J}} \backslash K \neq \emptyset$ for every $K \in \mathcal{K}$, there exists an i such that $i \in K_{\mathcal{J}} \backslash K$ and $\hat{x}_i < x^*_{K_{\mathcal{J}}} < x^*_K$. Therefore it holds that $\mu(\hat{x}_i) < \mu(x^*_K)$. Because it holds that $u \to E_K$ and $x \to x^*_K$ as $t \to \infty$, it is true that $v_i \to 0$, $V_i \to 0$ and $z_i \to 0$ for every $i \notin K$.

Thus for $\epsilon > 0$ there exists $T > 0$ such that $\mu(x(t)) - \mu(\hat{x}_i) \geq \epsilon$ for all $t \geq T$. Therefore it holds that $\frac{\mathrm{d}}{\mathrm{d}t} V_i(t) > 0$ for all $t \geq T$ and it is a contradiction.

Proposition 6.5. *For all $K_1, K_2 \in \mathcal{K}$, if $K_1 \neq K_2$ and $x^*_{K_1} \leq x^*_{K_2}$ then there exists an i such that $i \in K_1 \backslash K_2$.*

Proof. Suppose $K_1 \backslash K_2 = \emptyset$, that is $K_1 \subset K_2$. Then $h_{2,K_1}(x^*_{K_2}) \leq h_{2,K_2}(x^*_{K_2})$.

When $x^*_{K_1} < x^*_{K_2}$, $h_1(x^*_{K_1}) = h_{2,K_1}(x^*_{K_1}) < h_{2,K_1}(x^*_{K_2})$ and $h_1(x^*_{K_1}) > h_1(x^*_{K_2}) = h_{2,K_2}(x^*_{K_2})$ leads $h_{2,K_1}(x^*_{K_2}) > h_{2,K_2}(x^*_{K_2})$. It is a contradiction.

When $x^*_{K_1} = x^*_{K_2}$ it holds that $h_{2,K_1}(x^*_{K_1}) = h_{2,K_2}(x^*_{K_1})$. On the other hand, by $K_1 \neq K_2$ there exists an $i \in K_2 \backslash K_1$. Then for $\hat{x}_i < x \leq x^*_{K_1}$ it holds that $h_{2,K_1}(x) < h_{2,K_2}(x)$. It is a contradiction. □

Proposition 6.6. *For all $K_1, K_2 \in \mathcal{K}$, let $u(t)$ be a nontrivial (nonconstant) entire solution of* (6.1) *such that*

$$\lim_{t \to -\infty} u(t) = E_{K_1} \text{ and } \lim_{t \to \infty} u(t) = E_{K_2},$$

then $x^*_{K_1} \geq x^*_{K_2}$.

Proof. Let us define the function $V_i(t)$ by (5.27).

$$V_i(t) = v_i(t) + r_i \beta_i \int_0^\infty \alpha_i(a) \mu(x(t-a)) v_i(t-a) \mathrm{d}a.$$

The time derivative of this function is

$$\frac{\mathrm{d}}{\mathrm{d}t} V_i(t) = \left\{ \frac{b_i}{\mu(\hat{x}_i)} \left(\mu(x(t)) - \mu(\hat{x}_i) \right) - p_i z_i \right\} v_i(t).$$

If $x^*_{K_1} < x^*_{K_2}$ then $K_1 \neq K_2$ and Proposition 6.5 leads that there exists an i such that $i \in K_1 \backslash K_2$ and $\hat{x}_i < x^*_{K_1}$. Therefore it holds that $\mu(\hat{x}_i) < \mu(x^*_{K_1})$. If $u \to E_{K_2}$ as $t \to \infty$ then $x \to x^*_{K_2}$ and it holds that $v_i \to 0$ for $i \notin K_2$. Thus for $\epsilon > 0$ there exists $T > 0$ such that $\mu(x(t)) - \mu(\hat{x}_i) \geq \epsilon$ for all $t \geq T$. Therefore it holds that $\frac{\mathrm{d}}{\mathrm{d}t} V_i(t) > 0$ for all $t \geq T$. It is a contradiction. Consequently if $u \to E_{K_2}$ then $x^*_{K_1} \geq x^*_{K_2}$. □

Proposition 6.7. *For some $K \in \mathcal{K}$, let $u(t)$ be an entire solution of* (6.1) *such that*

$$\lim_{t \to -\infty} u(t) = E_K \text{ and } \lim_{t \to \infty} u(t) = E_K, \tag{6.35}$$

then $u(t)$ is a constant function under the following condition :

$$1 - \sum_{i \in K} \frac{\rho_i}{r_i - \rho_i} \frac{\beta_i \mu(x^*_K) v^*_i}{\delta x^*_K} \geq 0, \tag{6.36}$$

where $E_K = (x^*_K, \boldsymbol{v}^*, \boldsymbol{z}^*), v^*_i z^*_i > 0$ for $i \in K$ and $v^*_i = z^*_i = 0$ for $i \notin K$.

Proof. We consider the functional $U_4^M(u;K)$ for $K \in \mathcal{K}$ defined by (6.22).

$$
\begin{aligned}
&U_4^M(u;K)\\
&= \int_{x_K^*}^{x} \frac{\mu(\xi)-\mu(x_K^*)}{\mu(\xi)} \mathrm{d}\xi + \sum_{i\in K} \frac{1}{r_i-\rho_i}(v_i - v_i^* \log v_i) + \sum_{i\in K} \frac{p_i}{r_i-\rho_i} \int_{z_i^*}^{z_i} \frac{\tau - z_i^*}{q_i} \mathrm{d}\tau\\
&\quad + \sum_{i\in K} \frac{r_i}{r_i-\rho_i} \beta_i \mu(x_K^*) v_i^* \int_0^\infty \alpha_i(\tau) H\left(\frac{\mu(x(t-\tau))v_i(t-\tau)}{\mu(x_K^*)v_i^*}\right) \mathrm{d}\tau\\
&\quad + \sum_{i\notin K} \frac{1}{r_i-\rho_i} v_i + \sum_{i\notin K} \frac{p_i}{r_i-\rho_i} \int_0^{z_i} \frac{\tau}{q_i} \mathrm{d}\tau\\
&\quad + \sum_{i\notin K} \frac{r_i}{r_i-\rho_i} \beta_i \int_0^\infty \alpha_i(\tau) \mu(x(t-\tau)) v_i(t-\tau) \mathrm{d}\tau
\end{aligned}
\tag{6.37}
$$

The integral of the second line is well defined by the hypothesis of induction because $K = K_{\mathcal{J}'}$ and $\#\mathcal{J}' < \#\mathcal{J}$. The time derivative of $U_4^M(u;K)$ becomes

$$
\begin{aligned}
\frac{\mathrm{d}U_4^M}{\mathrm{d}t} &\le \delta x^* \left(1 - \sum_{i\in K} \frac{\rho_i}{r_i-\rho_i} \frac{\beta_i \mu(x_K^*) v_i^*}{\delta x_k^*}\right)\left(1 - \frac{\mu(x_K^*)}{\mu(x)}\right)\left(1 - \frac{x}{x_K^*}\right)\\
&+ \sum_{i\in K} \frac{r_i \beta_i \mu(x_K^*) v_i^*}{r_i-\rho_i} \int_0^\infty g(\tau)\left(2 - \frac{\mu(x_K^*)}{\mu(x)} - \frac{\mu(x(t-\tau))v_i(t-\tau)}{\mu(x_K^*)v_i} + \log \frac{\mu(x(t-\tau))v_i(t-\tau)}{\mu(x)v_i}\right) \mathrm{d}\tau\\
&+ \sum_{i\in K} \frac{p_i v_i^* z_i}{r_i-\rho_i}\left(2 - \frac{z_i}{z_i^*} - \frac{z_i^*}{z_i}\right) + \sum_{i\notin K}\left(\beta_i \mu(x_K^*) - \frac{b_i}{r_i-\rho_i}\right) v_i - \sum_{i\notin K} \frac{p_i}{r_i-\rho_i} \frac{m_i}{q_i} z_i^2.
\end{aligned}
\tag{6.38}
$$

When the inequality (6.36) holds, it holds that $\mathrm{d}U_4^M/\mathrm{d}t \le 0$ for all t.

The assumption (6.35) leads $u(t) \equiv E_K$. □

Therefore we can apply Theorem 4.2 in [6] and the flow $U_{\mathcal{S}\setminus\mathcal{J}}$ is persistent on $(N, \partial N)$. Let $\tilde{u} = (\tilde{x}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{z}})$ be a solution of equation (6.1) with $(\tilde{x}_0, \tilde{\boldsymbol{v}}_0, \tilde{\boldsymbol{z}}(0)) \in N$. Then same as in Section 3, it follows that the ω-limit set Ω of $(\tilde{x}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{z}})$ is non-empty, compact and invariant. It follows that Ω is the union of orbits of equation (6.1). That is, if $(\phi_0, \boldsymbol{\phi}, \boldsymbol{\alpha}) \in Y_\Delta \times Y_\Delta^n \times \mathbb{R}_+^n$ is an omega limit point in Ω, then there exists an entire solution through $(\phi_0, \boldsymbol{\phi}, \boldsymbol{\alpha})$ such that every point on the solution is in Ω. For the solution $u = (x, \boldsymbol{v}, \boldsymbol{z})$ that lies in Ω, the functional $U_4^M(u; K_{\mathcal{J}})$ is well defined.

Proposition 6.8. *Let Ω be the ω-limit set of the solution of* (6.1) *and let u be a solution that lies in Ω. Then the time derivative of*

$$
\begin{aligned}
U_4^M(u;K) &= \int_{x_K^*}^{x} \frac{\mu(\xi)-\mu(x_K^*)}{\mu(\xi)} \mathrm{d}\xi + \sum_{i\in K} \frac{1}{r_i-\rho_i}(v_i - v_i^* \log v_i) + \sum_{i\in K} \frac{p_i}{r_i-\rho_i} \int_{z_i^*}^{z_i} \frac{\tau - z_i^*}{q_i} \mathrm{d}\tau\\
&\quad + \sum_{i\in K} \frac{r_i}{r_i-\rho_i} \beta_i \mu(x_K^*) v_i^* \int_0^\infty \alpha_i(\tau) H\left(\frac{\mu(x(t-\tau))v_i(t-\tau)}{\mu(x_K^*)v_i^*}\right) \mathrm{d}\tau\\
&\quad + \sum_{i\notin K} \frac{1}{r_i-\rho_i} v_i + \sum_{i\notin K} \frac{p_i}{r_i-\rho_i} \int_0^{z_i} \frac{\tau}{q_i} \mathrm{d}\tau\\
&\quad + \sum_{i\notin K} \frac{r_i}{r_i-\rho_i} \beta_i \int_0^\infty \alpha_i(\tau) \mu(x(t-\tau)) v_i(t-\tau) \mathrm{d}\tau
\end{aligned}
\tag{6.39}
$$

is nonpositive under the condition (6.36)*.*

Theorem 6.9. *All solutions of equation* (6.1) *for which all* i*-th strain* ($i \in K_{\mathcal{J}}$) *is present converge to* $E_{K_{\mathcal{J}}}$ *under the condition* (6.36)*.*

Proof. We can show that the maximal invariant set M in

$$\left\{(x_0, \boldsymbol{v}_0, \boldsymbol{z}(0)) \in \Omega \;\middle|\; \frac{\mathrm{d}}{\mathrm{d}t} U_4^M(x_t, \boldsymbol{v}_t, \boldsymbol{z}(t))|_{t=0} = 0\right\}, \tag{6.40}$$

is the singleton $\{E_{K_{\mathcal{J}}}\}$. Let $u = (x, \boldsymbol{v}, \boldsymbol{z})$ be a solution of (6.1) in Ω. Then the ω-limit set and α-limit set are contained in M. They are equal to $\{E_{K_{\mathcal{J}}}\}$. Since $U_4^M(u; K_{\mathcal{J}})$ is nonincreasing along the solution u, u must be equal to $E_{K_{\mathcal{J}}}$ identically. Then the ω-limit set Ω is equal to $\{E_{K_{\mathcal{J}}}\}$. □

Thus we have that every solution on $X^{\mathcal{S}\setminus\mathcal{J}}$ converges to $E_{K_{\mathcal{J}}}$. Therefore when $\mathcal{J} = \mathcal{S}$ every solution on $X^{\mathcal{S}\setminus\mathcal{S}}$ converges to the equilibrium E_{K_S}.

7 Conclusion

We used constructive method to formulate Lyapunov functionals from the simple one-strain models to the multistrain models. We have made mathematically rigorous description. These methods and techniques stated in this paper are applicable to the other various models.

The conditions that we can construct Lyapunov functionals for the models are slightly strong sufficient conditions. More appropriate conditions are desirable. In this thesis we does not consider the multistrain model with immune variable such that the activation of immunity is represented by $q_i v_i z_i$ for every i-th strain. This model includes much mathematical difficulty compared to other models. The analysis of this model will be an important study.

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