ON MODEL STRUCTURE FOR COREFLECTIVE SUBCATEGORIES OF A MODEL CATEGORY

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1. Introduction

Let $\mathcal{C}$ be a coreflective subcategory of a cofibrantly generated model category $\mathcal{D}$. In this paper we show that under suitable conditions $\mathcal{C}$ admits a cofibrantly generated model structure which is left Quillen adjunct to the model structure on $\mathcal{D}$. As an application, we prove that well-known convenient categories of topological spaces, such as $k$-spaces, compactly generated spaces, and $\Delta$-generated spaces [3] (called numerically generated in [12]) admit a finitely generated model structure which is Quillen equivalent to the standard model structure on the category $\text{Top}$ of topological spaces.

2. Coreflective subcategories of a model category

Let $\mathcal{D}$ be a cofibrantly generated model category [7, 2.1.17] with generating cofibrations $I$, generating trivial cofibrations $J$ and the class of weak equivalences $W_\mathcal{D}$. If the domains and codomains of $I$ and $J$ are finite relative to $I$-cell [7, 2.1.4], then $\mathcal{D}$ is said to be finitely generated.

Recall that a subcategory $\mathcal{C}$ of $\mathcal{D}$ is said to be coreflective if the inclusion functor $i: \mathcal{C} \to \mathcal{D}$ has a right adjoint $G: \mathcal{D} \to \mathcal{C}$, so that there is a natural isomorphism $\varphi: \text{Hom}_\mathcal{D}(X, Y) \to \text{Hom}_\mathcal{C}(X, GY)$. The counit of this adjunction $\epsilon: GY \to Y$ ($Y \in \mathcal{D}$) is called the coreflexion arrow.

Theorem 2.1. Let $\mathcal{C}$ be a coreflective subcategory of a cofibrantly generated model category $\mathcal{D}$ which is complete and cocomplete. Suppose that the unit of the adjunction $\eta: X \to GX$ is a natural isomorphism, and that the classes $I$ and $J$ of cofibrations and trivial cofibrations in $\mathcal{D}$ are contained in $\mathcal{C}$. Then $\mathcal{C}$ has a cofibrantly generated model structure with $I$ as the set of generating cofibrations, $J$ as the set of generating trivial cofibrations, and $W_\mathcal{C}$ as the class of weak equivalences, where $W_\mathcal{C}$ is the class of all weak equivalences contained in $\mathcal{C}$. If $\mathcal{D}$ is finitely generated, then so is $\mathcal{C}$. Moreover, the adjunction $(i, G, \varphi): \mathcal{C} \to \mathcal{D}$ is a Quillen adjunction in the sense of [7, 1.3.1].

Proof. It suffices to show that $\mathcal{C}$ satisfies the six conditions of [7, 2.1.19] with respect to $I$, $J$ and $W_\mathcal{C}$. Clearly, the first condition holds because

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$W_C$ satisfies the two out of three property and is closed under retracts. To see that the second and the third conditions hold, let $I_C$-cell and $J_C$-cell be the collections of relative $I$-cell and $J$-cell complexes contained in $C$, respectively. Since $I_C$-cell and $J_C$-cell are subcollections of the collections of relative $I$-cell and $J$-cell complexes in $D$, respectively, the domains of $I$ and $J$ are small relative to $I_C$-cell and $J_C$-cell, respectively. The rest of the conditions are verified as follows. Let $f: X \to Y$ be a map in $C$. Since $\eta: X \to GX$ is isomorphic for $X \in D$, $f$ is $I$-injective in $C$ if and only if it is $I$-injective in $D$. Similarly, $f$ is $J$-injective in $C$ if and only if it is $J$-injective in $D$. Let $f$ be an $I$-cofibration in $D$. Then it has the left lifting property with respect to all $I$-injective maps in $C$. Hence $f$ is an $I$-cofibration in $C$. Conversely, let $f$ be an $I$-cofibration in $C$. Suppose we are given a commutative diagram

\[
\begin{array}{ccc}
X & \longrightarrow & A \\
\uparrow f & & \uparrow p \\
Y & \longrightarrow & B
\end{array}
\]

where $p$ is $I$-injective in $D$. Then there is a relative $I$-cell complex $g: X \to Z$ [7, 2.1.9] such that $f$ is a retract of $g$ by [7, 2.1.15]. Since $g$ is an $I$-cofibration in $D$, there is a lift $Z \to A$ of $g$ with respect to $p$. Then the composite

\[
Y \to Z \to A
\]

is a lift of $f$ with respect to $p$. Therefore $f$ is an $I$-cofibration in $D$. Similarly, $f$ is a $J$-cofibration in $C$ if and only if it is a $J$-cofibration in $D$. Thus we have the desired inclusions

- $J_C$-cell $\subseteq W_C \cap I_C$-cof,
- $I_C$-inj $\subseteq W_C \cap J_C$-inj, and
- either $W_C \cap I_C$-cof $\subseteq J_C$-cof or $W_C \cap J_C$-inj $\subseteq I_C$-inj.

Here $I_C$-inj and $I_C$-cof denote, respectively, the classes of $I$-injective maps and $I$-cofibrations in $C$, and similarly for $J$-inj and $J$-cof. Therefore $C$ is a cofibrantly generated model category by [7, 2.1.19].

It is clear, by the definition, that $C$ is finitely generated if so is $C$.

Finally, to prove that $(i, G, \varphi)$ is a Quillen adjunction, it suffices to show that $G: D \to C$ is a right Quillen functor, or equivalently, $G$ preserves $J$-injective maps in $D$ by [7, 1.3.4] and [7, 2.1.17]. Let $p: X \to Y$ be a $J$-injective map in $D$. Suppose there is a commutative diagram

\[
\begin{array}{ccc}
A & \longrightarrow & GX \\
\uparrow f & & \uparrow Gp \\
B & \longrightarrow & GY
\end{array}
\]
where \( f \in J \). Then we have a commutative diagram

\[
\begin{array}{ccc}
A & \longrightarrow & GX \\
\downarrow f & & \downarrow p \\
B & \longrightarrow & GY \\
\end{array}
\]

Since \( p \) is \( J \)-injective in \( D \), there is a lift \( h: B \to X \) of \( f \). Thus we have a lift \( Gh \circ \eta: B \cong GB \to GX \) of \( f \) with respect to \( Gp \). Therefore \( Gp: GX \to GY \) is \( J \)-injective in \( C \). Similarly, we can show that \( G \) preserves \( I \)-injective maps in \( C \), and so \( G \) preserves trivial fibrations in \( C \). Hence \((i, G, \varphi)\) is a Quillen adjunction.

We turn to the case of pointed categories [7, p.4]. Let \( D_* \) be the pointed category associated with \( D \), and let \( U: D_* \to D \) be the forgetful functor. We denote by \( I_+ \) and \( J_+ \) the classes of those maps \( f: X \to Y \) in \( D_* \) such that \( Uf: UX \to UY \) belongs to \( I \) and \( J \), respectively. Then we have the following. (Compare [7, 1.1.8], [7, 1.3.5], and [7, 2.1.21].)

**Theorem 2.2.** Let \( D \) be a cofibrantly (resp. finitely) generated model category, and let \( C \) be a coreflective subcategory satisfying the conditions of Theorem 2.1. Then the pointed category \( C_* \) has a cofibrantly (resp. finitely) generated model structure, with generating cofibrations \( I_+ \) and generating trivial cofibrations \( J_+ \), such that the induced adjunction \((i_*, G_*, \varphi_*): C_* \to D_* \) is a Quillen adjunction.

We also have the following Proposition.

**Proposition 2.3.** Suppose \( C \) and \( D \) satisfy the conditions of Theorem 2.1. Suppose, further, that the coreflection arrow \( \epsilon: GY \to Y \) is a weak equivalence for any fibrant object \( Y \) in \( D \). Then the adjunctions \((i, G, \varphi): C \to D \) and \((i_*, G_*, \varphi_*): C_* \to D_* \) are Quillen equivalences.

**Proof.** Let \( X \) be a cofibrant object in \( C \) and \( Y \) a fibrant object in \( D \). Let \( f: X \to Y \) be a map in \( D \). Then we have \( \varphi f = Gf \circ \eta: X \cong GX \to GY \).

Since \( f \) coincides with the composite \( X \xrightarrow{\varphi f} GY \xrightarrow{\epsilon} Y \) and \( \epsilon \) is a weak equivalence in \( D \), \( \varphi f \) is a weak equivalence in \( C \) if and only if \( f \) is a weak equivalence in \( D \). It follows by [7, 1.3.17] that that the induced adjunction \((i_*, G_*, \varphi_*): C_* \to D_* \) is a Quillen equivalence.

**3. ON A MODEL STRUCTURE OF THE CATEGORY NG**

In [12] we introduced the notion of numerically generated spaces which turns out to be the same notion as \( \Delta \)-generated spaces introduced by Jeff Smith (cf. [3]) . Let \( X \) be a topological space. A subset \( U \) of \( X \) is numerically open if for every continuous map \( P: V \to X \), where \( V \) is an open subset of
Euclidean space, $P^{-1}(U)$ is open in $V$. Similarly, $U$ is numerically closed if for every such map $P$, $P^{-1}(U)$ is closed in $V$. A space $X$ is called a numerically generated space if every numerically open subset is open in $X$.

Let $NG$ denote the full subcategory of $\text{Top}$ consisting of numerically generated spaces. Then the category $NG$ is cartesian closed [12, 4.6]. To any $X$ we can associate the numerically generated space topology, denoted $\nu X$, by letting $U$ open in $\nu X$ if and only if $U$ is numerically open in $X$. Therefore we have a functor $\nu : \text{Top} \to NG$ which takes $X$ to $\nu X$. Clearly, the identity map $\nu X \to X$ is continuous. By the results of [7, §3] the following holds.

**Proposition 3.1.** The functor $\nu : \text{Top} \to NG$ is a right adjoint to the inclusion functor $i : NG \to \text{Top}$, so that $NG$ is a coreflective subcategory of $\text{Top}$.

A continuous map $f : X \to Y$ between topological spaces is called a weak homotopy equivalence in $\text{Top}$ if it induces an isomorphism of homotopy groups

$$f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$$

for all $n > 0$ and $x \in X$. Let $I$ be the set of boundary inclusions $S^{n-1} \to D^n$, $n \geq 0$, $J$ the set of inclusions $D^n \times \{0\} \to D^n \times I$, and $W_{\text{Top}}$ the class of weak homotopy equivalences. The standard model structure on $\text{Top}$ can be described as follows.

**Theorem 3.2 ([7, 2.4.19]).** There is a finitely generated model structure on $\text{Top}$ with $I$ as the set of generating cofibrations, $J$ as the set of generating trivial cofibrations, and $W_{\text{Top}}$ as the class of weak equivalences.

The category $NG$ is complete and cocomplete by [12, 3.4]. A space $X$ is numerically generated if and only if $\nu X = X$ holds. Thus the unit of the adjunction $\eta : X \to \nu X$ is a natural homeomorphism. Moreover, since CW-complexes are numerically generated spaces by [12, 4.4], the classes $I$ and $J$ are contained in $NG$. Let $W_{NG}$ be the class of maps $f : X \to Y$ in $NG$ which is a weak equivalence in $\text{Top}$. Since the coreflection arrow $\nu Y \to Y$, given by the identity of $Y \in \text{Top}$, is a weak equivalence (cf. [12, 5.4]), we have the following by Theorem 2.1 and Proposition 2.3.

**Theorem 3.3.** The category $NG$ has a finitely generated model structure with $I$ as the set of generating cofibrations, $J$ as the set of generating trivial cofibrations, and $W_{NG}$ as the class of weak equivalences. Moreover the adjunction $(i, \nu, \varphi) : NG \to \text{Top}$ is a Quillen equivalence.

We turn to the case of pointed spaces. Let $\text{Top}_*$ be the category of pointed topological spaces. By [7, 2.4.20], there is a finitely generated model structure on the category $\text{Top}_*$, with generating cofibrations $I_+$ and generating
trivial cofibrations $J_+$. Then we have the following by Theorem 2.2 and Proposition 2.3.

**Corollary 3.4.** There is a finitely generated model structure on the category $\text{NG}_*$ of pointed numerically generated spaces, with generating cofibrations $I_+$ and generating trivial cofibrations $J_+$. Moreover, the inclusion functor $i_*: \text{NG}_* \rightarrow \text{Top}_*$ is a Quillen equivalence.

**Remark.** (1) The argument of Theorem 3.3 can be applied to the subcategories $K$ of $k$-spaces and $T$ of compactly generated spaces. Similarly, the argument of Corollary 3.4 can be applied to the pointed categories $K_*$ and $T_*$. Compare [2.4.28], [2.4.25], [2.4.26] of [7].

(2) Let $\text{Diff}$ be the category of diffeological spaces (cf. [8]). In [12] we introduced a pair of functors $T: \text{Diff} \rightarrow \text{Top}$ and $D: \text{Diff} \rightarrow \text{Top}$, where $T$ is a left adjoint to $D$, and showed that the composite $TD$ coincides with $\nu: \text{Top} \rightarrow \text{NG}$. Thus $\text{NG}$ can be embedded as a full subcategory into $\text{Diff}$. It is natural to ask whether $\text{Diff}$ has a model category structure with respect to which the pair $(T, D)$ gives a Quillen adjunction between $\text{Top}$ and $\text{Diff}$.

Let $I$ be the unit interval, and let $\lambda: \mathbb{R} \rightarrow I$ be the smashing function, that is, a smooth function such that $\lambda(t) = 0$ for $t \leq 0$ while $\lambda(t) = 1$ for $t \geq 1$. Let $\tilde{I}$ denote the unit interval equipped with the quotient diffeology $\lambda_*(D_{\mathbb{R}})$, where $D_{\mathbb{R}}$ is the standard diffeology of $\mathbb{R}$. In [5] we introduce a finitely generated model category structure on $\text{Diff}$ with the boundary inclusions $\partial \tilde{I}^{n-1} \rightarrow \tilde{I}^n$ as generating cofibrations, and with the inclusions $\partial \tilde{I}^{n-1} \times \tilde{I} \cup \tilde{I}^n \times \{0\} \rightarrow \tilde{I}^n \times \tilde{I}$ as generating trivial cofibrations. Its class of weak equivalences consists of those smooth maps $f: X \rightarrow Y$ inducing an isomorphism $f_*: \pi_n(X, x_0) \rightarrow \pi_n(Y, f(x_0))$ for every $n \geq 0$ and $x_0 \in X$. Here, the homotopy set $\pi_n(X, x_0)$ is defined to be the set of smooth homotopy classes of smooth maps $(\tilde{I}^n, \partial \tilde{I}^n) \rightarrow (X, x_0)$.

It is expected that with respect to the model structure on $\text{Diff}$ described above, the pair $(T, D)$ induces a Quillen adjunction between $\text{Top}$ and $\text{Diff}$.

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