On the construction of generalized homology-cohomology theories by using bivariant functors

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Contents

Introduction

The homology groups are the standard tools of algebraic topology. Eilenberg and Steenrod proved that homology was characterized on the category of finite simplicial complexes by seven axioms known as the Eilenberg-Steenrod axioms. They showed how to derive many of the properties of homology directly from the axioms. As is well known, a generalized homology theory is represented by a spectrum.

We call a topological space X numerically generated (or Δ-generated in [Du]) if it has the final topology with respect to its singular simplexes. Let $\mathbf{NG}$ be the category of numerically generated spaces and continuous maps. Let $\mathbf{NG_0}$ be the category of pointed numerically generated spaces and pointed continuous maps. In [SYH], we established a method for representing generalized homology-cohomology theories by bivariant functors:

$$\mathbf{NG_0^{op}} \times \mathbf{NG_0} \to \mathbf{NG_0}.$$

Let $\mathbf{NGC}_0$ be the subcategory of pointed numerically generated compact metric spaces and pointed continuous maps. As an example of a bivariant functor, in [Y], we constructed Steenrod-Čech homology-cohomology theories by bivariant functors:

$$\mathbf{NG_0^{op}} \times \mathbf{NGC_0} \to \mathbf{NG_0}.$$

This paper puts the contents of [SYH] and [Y] in order, and divided into three chapters. In the first chapter, Chapter 1, we prepare some definitions and theorems necessary for proving the Main theorem. We briefly recall basic definitions and facts about CW-complexes, generalized homology, spectra, inverse limits, direct limits, homotopy inverse limits, Čech cohomology, and Steenrod homology.

We introduce, following [SYH], the concept of enriched bifunctors and describe the passage from enriched bifunctors to generalized homology-cohomology theories. For this purpose we utilize subcategory $\mathbf{NG}$ of topological spaces which is convenient in the sense that it is complete, cocomplete, and Cartesian closed. Let $\mathbf{NG}_0$ be the full subcategory of $\mathbf{Top}_0$ consisting of numerically generated spaces. Then $\mathbf{NG}_0$ is complete, cocomplete, and is monoidally closed in the sense that there is an internal hom Z^Y satisfying a natural bijection

$$\hom_{\mathbf{NG}_0}(X \wedge Y, Z) \cong \hom_{\mathbf{NG}_0}(X, Z^Y).$$

There exist a reflector $\nu\colon \mathbf{Top}_0 \to \mathbf{NG}_0$ such that the natural map $\nu X \to X$ is a weak equivalence and a sequence of weak equivalences

$$Y^X \leftarrow \nu\,\mathbf{map}_0(X, Y) \to \mathbf{map}_0(X, Y),$$

where $\mathbf{map}_0(X, Y)$ is the set of pointed maps from X to Y equipped the compact-open topology. Thus $\mathbf{NG}_0$ is eligible, from the viewpoint of homotopy theory, as a convenient replacement for $\mathbf{Top}_0$. CW-complexes are typical examples of such numerically generated spaces.

On the category $\mathbf{Top}_0$, generalized homology and cohomology theories are usually constructed by using spectra. We present an alternative and more category-theoretical approach to homology and cohomology theories which is based on the notion of a linear enriched functor instead of a spectrum. We replace the category $\mathbf{Top}_0$ by its full subcategory $\mathbf{NG}_0$. A bifunctor

$F\colon \mathbf{NG_0^{op}} \times \mathbf{NG_0} \to \mathbf{NG_0}$ is called enriched if the map

$$F : \mathbf{map}_0(X, X') \times \mathbf{map}_0(Y, Y') \to \mathbf{map}_0(F(X', Y), F(X, Y')),$$

which assigns $F(f, g)$ to every pair (f, g), is a pointed continuous map. Note that if either f or g is constant, then so is $F(f, g)$. A bifunctor F is bilinear if two sequences

1. $F(X \cup CA, S^n) \to F(X, S^n) \to F(A, S^n)$,

2. $F(S^n, A) \to F(S^n, X) \to F(S^n, X \cup CA)$,

induced by the cofibration sequence $A \to X \to X \cup CA$, are homotopy fibrations.

A bilinear enriched functor F defines a generalized cohomology $\{h^n(X, F)\}$ and a generalized homology $\{h_n(X, F)\}$ such that

$$h_n(X, F) = \begin{cases} \pi_0 F(S^n, X) & n \geq 0 \\ \pi_0 F(S^0, \Sigma^{-n} X) & n < 0, \end{cases}$$

$$h^n(X, F) = \begin{cases} \pi_0 F(X, S^n) & n \geq 0 \\ \pi_{-n} F(X, S^0) & n < 0. \end{cases}$$

As an example, consider the bilinear enriched functor F which assigns to (X, Y) the mapping space from X to the topological free abelian group $AG(Y)$ generated by the points of Y. The Dold-Thom theorem says that if X is a CW-complex then the groups $h_n(X, F)$ and $h^n(X, F)$ are respectively isomorphic to the singular homology and cohomology groups of X. But this is not the case for general X; there exists a space X such that $h_n(X, F)$ (resp.

$h^n(X, F)$) is not isomorphic to the singular homology (resp. cohomology) group of X.

The aim of Chapter 3 is to construct a bilinear enriched functor such that for any space X the associated cohomology groups are isomorphic to the Čech cohomology groups of X. Interestingly, it turns out that the corresponding homology groups are isomorphic to the Steenrod homology groups for any compact metrizable space X. Thus we obtain a bibariant theory which ties together the Čech cohomology and the Steenrod homology theories.

Recall that the Čech cohomology group of X with coefficient group G is defined to be the colimit of the singlular cohomology groups

$$\check{H}^n(X, G) = \varinjlim_{\lambda} H^n(X_\lambda^{\check{C}}, G),$$

where λ runs through coverings of X and $X_\lambda^{\check{C}}$, is the Čech nerve corresponding to λ, i.e., $v \in X_\lambda^{\check{C}}$ is a vertex of $X_\lambda^{\check{C}}$ corresponding to an open set $V \in \lambda$. On the other hand, the Steenrod homology group of a compact metric space X is defined as follows. As X is a compact metric space, there is a sequence $\{\lambda_i\}_{i\geq 0}$ of finite open covers of X such that $\lambda_0 = \{X\}$, λ_i is a refinement of λ_{i-1}, and X is the inverse limit $\varprojlim_i X_{\lambda_i}^{\check{C}}$. According to [F], the Steenrod homology group of X with coefficients in the spectrum $\mathbb{S}$ is defined to be the group

$$H_n^{st}(X, \mathbb{S}) = \pi_n \underleftarrow{\mathrm{holim}}_{\lambda_i} (X_{\lambda_i}^{\check{C}} \wedge \mathbb{S}),$$

where $\underleftarrow{\mathrm{holim}}$ denotes the homotopy inverse limit. (See also [KKS] for the definition without using subdivisions.)

Let $\mathbf{NGC}_0$ be the subcategory of pointed numerically generated compact metric spaces and pointed continuous maps. For given a linear enriched functor $T : \mathbf{NG_0} \to \mathbf{NG_0}$, let

$$\check{\mathrm{F}} : \mathbf{NG_0^{op}} \times \mathbf{NGC_0} \to \mathbf{NG_0}$$

be a bifunctor which maps (X, Y) to the space $\varinjlim_\lambda \mathbf{map}_0(X_\lambda, \underleftarrow{\mathrm{holim}}_{\mu_i} T(Y^{\check{\mathrm{C}}}_{\mu_i}))$. Here λ runs through coverings of X, and X_λ is the Vietoris nerves corresponding to λ ([P]). The main results of the paper can be stated as follows.

Theorem 1. *The functor $\check{\mathrm{F}}$ is a bilinear enriched functor.*

Theorem 2. *Let X be a compact metrizable space. Then $h_n(X, \check{\mathrm{F}}) = H_n(X, \mathbb{S})$ is the Steenrod homology group with coefficients in the spectrum $\mathbb{S} = \{T(S^k)\}$.*

In particular, let us take AG as T, and denote by

$$\check{\mathrm{C}} : \mathbf{NG_0^{op}} \times \mathbf{NGC_0} \to \mathbf{NG_0}$$

the corresponding bifunctor $\check{\mathrm{F}}$.

Theorem 3. *For any pointed space X, $h^n(X, \check{\mathrm{C}})$ is the Čech cohomology group of X, and $h_n(X, \check{\mathrm{C}})$ is the Steenrod homology group of X if X is a compact metrlizable space.*

Recall that the Steenrod homology group H^{st} is related to the Čech homology group of X by the exact sequence

$$0 \longrightarrow \varprojlim^1_{\lambda_i} \tilde{H}_{n+1}(X^{\check{\mathrm{C}}}_{\lambda_i}) \longrightarrow H^{st}_n(X) \longrightarrow \tilde{H}_n(X) \longrightarrow 0.$$

If X is a movable compactum then we have $\varprojlim^1_{\lambda_i} \tilde{H}_{n+1}(X^{\check{\mathrm{C}}}_{\lambda_i}) = 0$, and hence the following corollary follows.

Corollary 4. *Let X be a movable compactum. Then $h_n(X, \check{\mathrm{C}})$ is the Čech homology group of X.*

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Chapter 1

Fundamental concepts

1.1 CW-complexes

The reference is [H] and [M]. First we recall the definition of CW-complex. Let X be a Hausdorff space and $\{e_\lambda\}_{\lambda\in\Lambda}$ be subsets of X satisfying the following conditions;

1. $X = \cup_{\lambda\in\Lambda} e_\lambda$.

2. For each e_λ, there is an integer $n_\lambda \geq 0$ and a map ϕ_λ from n_λ-dimensional ball D^{n_λ} to a closure set of $\overline{e}_\lambda$ such that $\phi_\lambda(S^{n_\lambda-1}) \subset \overline{e}_\lambda - e_\lambda$ and $\phi_\lambda|_{D^{n_\lambda}-S^{n_\lambda-1}} : D^{n_\lambda} - S^{n_\lambda-1} \to e_\lambda$ is a homeomorphism.

3. Let $X^q = \cup_{n_\mu \leq q} e_\mu$. For each $\lambda \in \Lambda$,

$$\overline{e}_\lambda - e_\lambda \subset X^{n_\lambda-1}.$$

Then a pair $(X, \{e_\lambda\}_{\lambda\in\Lambda})$ is called a *cell complex*. The map ϕ_λ is called a characteristic map for e_λ, and we denote dim $e_\lambda = n_\lambda$. If dim $e_\lambda = n$, then e_λ

is called an *n-cell* and denoted by e^n_λ. X^n is called an *n-skeleton* of X. A cell complex $(A, \{e_\lambda\}_{\lambda\in\Lambda'})$ is called a *subcomplex* of $(X, \{e_\lambda\}_{\lambda\in\Lambda})$ if and only if A is a subspace of X and Λ' is a subset of Λ. It is clear that for any subspace A of X, if the condition "$e_\lambda \cap A \neq \emptyset \Rightarrow \overline{e}_\lambda \subset A$" is true, $(A, \{e_\lambda\}_{\lambda\in\Lambda'})$ is a subcomplex of $(X, \{e_\lambda\}_{\lambda\in\Lambda})$. We write just a subcomplex A if there is no possibility of confusion. It is clear that X^q is a subcomplex of X.

$(X, \{e_\lambda\}_{\lambda\in\Lambda})$ is called a *finite complex* if Λ is a finite set. A cell complex X is called a *locally finite complex* if there is a finite subcomplex A such that for any $x \in X$ Int A contains x.

Since a finite subcomplex A is written as a finite union $A = \cup\overline{e}_\lambda$, obviously A is a closed set of X.

Definition 1.1.1. *Let X be a cell complex. X is a CW-complex if X has the following properties*;

(C) *for any $x \in X$, there is a finite subcomplex A such that $x \in A$.*

(W) *X has a weak topology induced by a closed covering $\{\overline{e}_\lambda\}_{\lambda\in\Lambda}$.*

It is clear that the property (C) and the next property (C′) are equivalent.

(C′) for each cell e_λ, there is a finite subcomplex A such that $e_\lambda \subset A$. Assume the property (C), then the property (W) and the next property (W′) are equivalent

(W′) X has a weak topology induced by a closed covering of $\{X\}$.

In particular, by $A \cap Y = \cup_{e_\lambda\subset A}(\overline{e}_\lambda \cap Y)$ for a finite subcomplex A, if (W) holds, so does (W′). Oppositely for each cell e_λ, there is a finite subcomplex A containing e_λ such that $\overline{e}_\lambda \cap Y = \overline{e}_\lambda \cap (A \cap Y)$. So if (W′) holds, so does (W).

Proposition 1.1.2. *A locally finite complex X is a CW-complex.*

Proof. Since X is a locally finite complex, it is clear that the property (C) is satisfied. Next let Y be a subspace of X. Assume that for any finite subcomplex A of X, and then $A \cap Y$ is a closed set of Y. For $x \in X - Y$, there is a finite subcomplex A such that $x \in \mathrm{Int}A$. The element x is contained in the open set $\mathrm{Int}A - A \cap Y$. Thus Y is a closed set, since $X - Y$ is an open set. Hence the property (W′) is satisfied. □

Proposition 1.1.3. *Let X be a CW-complex and Y a subcomplex of X. Then Y is a closed subset of X and hence a CW-complex.*

Proof. Let A be a finite subcomplex of X. Since $A \cap Y$ is a closed set of X, $A \cap Y$ is a finite subcomplex. Hence by the property (W′), Y is a closed set. It is clear that Y satisfies the property (C). Next we show that Y is satisfies the property (W′). We assume that $Z \subset Y$, and then $B \cap Z$ is a closed subset of B for any finite subcomplex B of Y. $A \cap Z$ is a closed subset of $A \cap Y$, since $A \cap Y$ is a finite subcomplex of Y. $A \cap Z$ is a closed subset of A, since Y is a closed subset of X. Hence Z is a closed subset of X. Z is a closed subset of Y, since $Z \subset Y$. Hence Y satisfies (W′). □

Proposition 1.1.4. *For any compact set K of a CW-complex X, there is a finite subcomplex A of X such that $K \subset A$.*

Proof. Let $\{e_\lambda\}_{\lambda \in \Lambda}$ be a set of cells of X. Define Λ' by $\Lambda' = \{\lambda \in \Lambda | K \cap e_\lambda \neq \emptyset\}$. Assume that Λ' is an infinite set. For any $\lambda \in \Lambda'$, we choose an element $x_\lambda \in K \cap e_\lambda$. Define Y by $Y = \{x_\lambda | \lambda \in \Lambda'\}$. Y is an infinite set, since Y corresponds to Λ'.

Let B be a subset of Y. For each cell e_λ, $\overline{e} \cap B$ is a finite set. Moreover since $\overline{e}_\lambda \cap B$ is a finite set of Hausdorff space X, this set is closed. By the condition (W), B is a closed subset of X. Hence Y is compact set, since Y is a compact subset K. Thus it contradicts the assumption, since Y is finite set. Hence Λ' is a finite set. For any $\lambda \in \Lambda'$, we can choose a finite subcomplex A_λ by (C'). Define A by $A = \cup_{\lambda \in \Lambda'} A_\lambda$. Since A is a finite subcomplex, we have $K \subset \cup_{\lambda \in \Lambda'} e_\lambda \subset A$.

$\square$

Theorem 1.1.5. *Let X be a CW-complex, A a closed subset of X and X' a topological space. Then a map $f : A \to X'$ is continuous if and only if the restriction map $f|(\overline{e}_\lambda \cap A)$ is continuous for each cell e_λ.*

Proof. X has a weak topology induced by $\{\overline{e}_\lambda\}$. The proof since A has a weak topology induced by $\{\overline{e}_\lambda \cap A\}$. $\square$

Next we consider a geometric n-simplex

$$\Delta^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | 0 \le x_i \le 1,\ 1 \le i \le n\}.$$

For any vertices $e_{i_0}, \cdots, e_{i_q} (i_0 < \cdots < i_q)$ of Δ^n, we denote the q-cell complex of characteristic map $(e_{i_0}, \cdots, e_{i_q}) : \Delta^q \to \Delta^n$ by the same symbol $(e_{i_0}, \cdots, e_{i_q})$. The set $\{(e_{i_0}, \cdots, e_{i_q})\}$ gives a triangulation of Δ^n. This triangulation is called the *standard triangulation* of Δ^n. We regard Δ^n as a CW-complex by the standard triangulation. A cell complex S is called a *finite simplicial complex* if there is a subcomplex K of Δ^n such that $K \cong S$. A CW-complex X is called a *simplicial complex*, if each subcomplex of X is a finite simplicial complex.

More generally the definition of the abstract simplicial complex is given as follows.

Definition 1.1.6. *Let X be a finite set of elements $a_0, a_1 \cdots$, called vertices, together with a collection of subsets $(a_{i_0}, a_{i_1}, \cdots, a_{i_n})$ called simplexes. X is called simplicial complex if for any simplexes σ, τ, the following two condition hold;*

1. $\sigma \cap \tau \in X$,

2. "$\sigma \cap \tau = \emptyset$" *or* "$\sigma \cap \tau \leq \sigma$ *and* $\sigma \cap \tau \leq \tau$"

A simplicial complex X is given topology as a subspace of $\mathbb{R}^n$. The topological space is called the *polyhedron* of X, denoted by $|X|$. The upper bound of dimensions of any simplex of X is called the *dimension* of X. We denote the dimension of X by $\dim X$. Then X is called $finite\ dimensionnal$ if $\dim X < \infty$. A subset Y of X is called a *subcomplex* if Y is a simplicial complex.

Proposition 1.1.7. *Let X be a simplicial complex. Then $|X|$ is a CW-complex.*

Proof. It is clear by the definition of a simplicial complex. □

Definition 1.1.8. *Let X and Y be simplicial complexes. Let*

$$V = \{v | v \in \Lambda\},$$

$$W = \{w | w \in \Omega\}$$

be the sets of verticies of X, Y respectively. A function $f : X \to Y$ is called a simplicial mapping if f satisfies the following two properties:

1. *f is a map from V to W;*

2. *if $(v_1, \cdots, v_n)$ is a simplex of X, then $(f(v_1), \cdots, f(v_n))$ span a simplex of Y.*

Assume that f is linear on each simplex, we can consider f to be the map $f : |X| \to |Y|$.

Definition 1.1.9. *X is a isomorphic of Y if there is a simplicial mapping $f : X \to Y$ such that f is a bijection.*

Definition 1.1.10. *Let X and Y be simplicial complexes. A map $f : |X| \to |Y|$ is called a piecewise-linear mapping if f satisfies the following properties; for each complex σ of X*

1. *$f(\sigma)$ is a simplex of Y,*

2. *$f|\sigma$ is a linear map.*

Definition 1.1.11. *Let f be a one to one piecewise-linear mapping $f : |X| \to |Y|$. For each simplex τ of $|Y|$, assume that there is a finite set of simplexes $\sigma_1, \cdots, \sigma_n$ of $|X|$ such that $\tau = \cup_i f(\sigma_i)$. Then a simplicial complex $\{f(\sigma)|\ \sigma \in X\}$ is called a subdivision of Y.*

1.2 Homology groups and spectra

In this section, we recall the homology theory satisfying the Eilenberg-Steenrod axioms and introduce the construction of homology theory by a spectrum.

The reference is [ES]. First we introduce the Eilenberg-Steenrod axioms as follows. We denote by H_* a collection of the next three correspondences.

1. A pair (X, A) of topological spaces corresponds to an abelian group $H_q(X, A)$ for any integer q.

2. A continuous map $f : (X, A) \to (Y, B)$ and an integer q correspond to a homomorphism
$$f_q : H_q(X, A) \to H_q(Y, B).$$

3. A pair (X, A) of topological spaces and an integer q correspond to a homomorphism
$$\partial_q : H_q(X, A) \to H_{q-1}(A).$$

The functor H_* is called a *homology theory on the category of topological pairs* if H_* satisfies the following seven axioms, where we regard X as $(X, \emptyset)$.

1. $1_* = 1$, where 1 is an identity map.

2. if $f : (X, A) \to (Y, B)$, $g : (Y, B) \to (Z, C)$, then
$$(g \circ f)_* = g_* \circ f_* : H_*(X, A) \to H_*(Z, C).$$

3. (Homotopy Axiom) Let $f, f' : (X, A) \to (Y, B)$. If f is homotopic to f', then
$$f_* = f'_* : H_*(X, A) \to H_*(Y, B).$$

4. (Exactness) Let $i : A \to X$, $j : (X, \emptyset) \to (X, A)$ be inclusion maps. Then the sequence
$$\cdots \longrightarrow H_q(A) \xrightarrow{i_q} H_q(X) \xrightarrow{j_q} H_q(X, A) \xrightarrow{\partial_q} H_{q-1}(A) \longrightarrow \cdots$$
is exact.

5. For a map $f : (X, A) \to (Y, B)$,

$$f|_A \circ \partial_q = \partial_q \circ f_q : H_q(X, A) \to H_{q-1}(B).$$

6. (Excision Axiom) If an open set U in X satisfies that $\overline{U} \subset \mathrm{Int}A$, then

$$i_* : H_*(X - U, A - U) \cong H_*(X, A),$$

   where i is an inclusion map.

7. (Dimension Axiom) $H_q(\mathrm{pt}) = 0$, $(\mathrm{q} \neq 0)$.

We call the axioms 1$\sim$7 by the Eilenberg-Steenrod axioms. $H_0(\mathrm{pt})$ is called *the coefficient group*. If one omits the Dimension axiom, then the remaining Axioms define what is called *a generalized homology theory*. Similarly we define the homology theory on the category of compact Hausdorff spaces. Then a homology theory satisfying the Eilenberg-Steenrod axioms is characterized by the coefficient group. Thus if H and H' are any two homology theories on the category of compact Hausdorff spaces satisfying the axioms, then for each homomorphism

$$\varphi : H_0(\mathrm{pt}) \to \mathrm{H}'_0(\mathrm{pt})$$

of the coefficient groups, there is a natural transformation

$$\xi : H_* \to H'_*$$

which coincides with φ in degree 0. In particular, if φ is an isomorphism, the homomorphisms

$$H_q(X, A) \to H'_q(X, A)$$

are bijective.

To construct a homology theory by a spectrum, we quote the following definitions from [Sw].

Definition 1.2.1. *A spectrum E is a collection*

$$\{(E_n, *) : n \in \mathbb{Z}\}$$

of CW-complexes such that the suspension SE_n of E_n is a subcomplex of E_{n+1} for all $n \in \mathbb{Z}$. A subspectrum $F \subset E$ consists of subcomplexes $F_n \subset E_n$ such that $SF_n \subset F_{n+1}$.

Definition 1.2.2. *A map $f : E \to F$ between spectra is a collection $\{f_n : n \in \mathbb{Z}\}$ of cellular maps $f_n : E_n \to F_n$ such that $f_{n+1}|SE_n = Sf_n$. The inclusion $i : F \to E$ of subspectrum $F \subset E$ is a function and if $g : E \to G$ is a function then $g|F = g \circ i$ is also a function.*

If $E = \{E_n\}$ is a spectrum and (X, x_0) is a CW-complex, we defined a new spectrum $E \wedge X$ as follows where $\wedge$ is the smash product. We take $(E \wedge X)_n = E_n \wedge X$ with the weak topology. Then by

$$S(E \wedge X)_n = S(E_n \wedge X) = S^1 \wedge (E_n \wedge X) \cong (S^1 \wedge E_n) \wedge X \subset E_{n+1} \wedge X,$$

$E \wedge X$ is again a spectrum.

Definition 1.2.3. *For any map $f : E \to F$ of spectra we call the sequence*

$$E \xrightarrow{f} F \xrightarrow{j} F \cup_f CE$$

a special cofibre sequence. A general cofibre sequence is any sequence

$$G \xrightarrow{g} H \xrightarrow{h} K$$

for which there is a homotopy commutative diagram

$$\begin{tikzcd} G \arrow[r, "g"] \arrow[d, "\alpha"'] & H \arrow[r, "h"] \arrow[d, "\beta"'] & K \arrow[d, "\gamma"'] \\ E \arrow[r, "f"] & F \arrow[r, "j"] & F \cup_f CE \end{tikzcd}$$

in which α, β, γ *are homotopy equivalences, i.e. the diagram commutative up to homogopy* $j \circ f \circ \alpha \simeq j \circ \beta \circ g \simeq \gamma \circ h \circ g$.

Definition 1.2.4. *A subspectrum* $F \subset E$ *is called* $cofinal$ *if each cell in* E *ultimately lies in* F.

Lemma 1.2.5. *Let* E *and* H *be spectra,* F *a subspectrum of* E *and* G *a cofinal subspectrum of* $E \wedge \{0\}^+ \cup F \wedge I^+$, *where* I^+ *is a unit interval with base point* 0. *Given a function* $g : G \to H$, *we can find a cofinal subspectrum* K *of* $E \wedge I^+$ *containing* G *and an extension of* g *to a function* $k : K \to H$. *Moreover, if* $G = E \wedge \{0\}^+ \cup F \wedge I^+$, *we can choose* $K = E \wedge I^+$.

Lemma 1.2.6. *Given a homotopy commutative diagram of functors of spectra*

$$\begin{tikzcd} G \arrow[r, "g"] \arrow[d, "\alpha"] & H \arrow[r, "h"] \arrow[d, "\beta"] & K \arrow[r, "k"] \arrow[d, dotted, "\gamma"] & G \wedge S^1 \arrow[d, "\alpha\wedge 1"] \\ G' \arrow[r, "g'"] & H' \arrow[r, "h'"] & K' \arrow[r, "k'"] & G' \wedge S^1 \end{tikzcd}$$

where the rows are cofibre sequences, we can find a functor $\gamma : K \to K'$ *such that the resulting diagram is homotopy commutative.*

Then by Lemmas 1.2.5 and 1.2.6, we have the next proposition.

Proposition 1.2.7. *If*

$$G \xrightarrow{g} H \xrightarrow{h} K$$

is a cofibre sequence, then for any spectrum E the sequences

$$[E,G] \xrightarrow[g_*]{} [E,H] \xrightarrow[h_*]{} [E,K]$$

and

$$[G,E] \xleftarrow[g^*]{} [H,E] \xleftarrow[h^*]{} [E,K]$$

are exact.

Proof. Since $h \circ g \simeq 0$, it follows $h_* \circ g_* = 0$. Suppose that $f : E \to H$ satisfies $h_*[f] = 0$. We apply Lemma 1.2.6 to the diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d, "f"'] & E \wedge I \arrow[r] \arrow[d, "\overline{h}"'] & E \wedge S^1 \arrow[r, "1"] \arrow[d, "k"'] & E \wedge S^1 \arrow[d, "f\wedge 1"'] \\ H \arrow[r, "h"] & K \arrow[r] & G \wedge S^1 \arrow[r, "g\wedge 1"] & H \wedge S^1 \end{tikzcd}$$

where $\overline{h} : E \wedge I \to K$ is a null homotopy of $h \circ f$. We obtain a map $k : E \wedge S^1 \to G \wedge S^1$ such that $(g \wedge 1) \circ k \simeq f \wedge 1$. From the natural equivalence

$$\sigma : [E,G] \to [E \wedge S^1, G \wedge S^1]$$

we get a map $k' : E \to G$ such that $k \simeq k' \wedge 1$. Then we have

$$(g \circ k') \wedge 1 = (g \wedge 1) \circ (k' \wedge 1) \simeq (g \wedge 1) \circ k \simeq f \wedge 1.$$

Since $\sigma : [E,H] \to [E \wedge S^1, H \wedge S^1]$ is injective, it follows that

$$g \circ k' \simeq f.$$

Further $g^* \circ h^* = 0$ follows from $h \circ g \simeq 0$. Suppose given $f : H \to E$ such that $g^*[f] = 0$. We apply Lemma 1.2.6 to the diagram

$$\begin{tikzcd} G \arrow[r, "g"] & H \arrow[r, "h"] \arrow[d, "f"'] & K \arrow[r] \arrow[d, "f'"'] & G \wedge S^1 \\ 0 \arrow[r] & E \arrow[r, "1"] & E \arrow[r] & 0 \end{tikzcd}$$

to obtain a map $f' : K \to E$ such that $f' \circ h \simeq f'$; that is, $h^*[f'] = [f]$. $\square$

For any spectrum $E = \{E_n\}$ we can define ΣE to be the spectrum with $\Sigma E_n = E_{n+1}$, $n \in \mathbb{Z}$. We can iterate $\Sigma^n = \Sigma \circ \Sigma^{n-1}$, $n \geq 2$, but Σ also has an inverse Σ^{-1} defined by $(\Sigma^{-1}E)_n = E_{n-1}$. We can define the homology and cohomology theories associated with any spectrum E.

Proposition 1.2.8. *For each CW-complexs pair (X, x_0) and $n \in \mathbb{Z}$,*

$$E_n(X) = \pi_0(E \wedge X) = [\Sigma^n S^0, E \wedge X],$$

$$E^n(X) = [E(X), \Sigma^n E] \cong [\Sigma^{-1} S^0 \wedge X, E]$$

are homology, cohomology theories respectively.

Proof. For $f : (X, x_0) \to (Y, y_0)$ we take

$$E_n(f) = (1 \wedge f)_*,$$

$$E^n(X)(f) = E(f)^*.$$

We define $\sigma_n : E_n(X) \to E_{n+1}(SX)$ by the composite

$$E_n(X) = [\Sigma^n S^0, E \wedge X] \xrightarrow[\cong]{\Sigma} [\Sigma^{n+1} S^0, \Sigma E \wedge X]$$

$$\longrightarrow [\Sigma^{n+1} S^0, E \wedge S^1 \wedge X] = E_{n+1}(SX).$$

Then σ_n is clearly a natural equivalence. We define $\sigma^n : E^{n+1}(SX) \to E^n(X)$ by the composite

$$E^{n+1}(SX) = [E(SX), \Sigma^{n+1} E] \xleftarrow[i^*]{\cong} [\Sigma E(X), \Sigma^{n+1} E]$$

$$\xrightarrow[\cong]{\Sigma^{-1}} [E(X), \Sigma^n(X)] = E^n(X).$$

Thus we have used the fact that $E(SX)$ is a cofinal subspectrum of $\Sigma E(X)$, and hence the inclusion $i : E(SX) \to \Sigma E(X)$ induces an isomorphism i^*. σ^n is a natural equivalence.

Let (X, A, x_0) be any pointed CW-complexes pair. Since

$$E_n \wedge (X \cup CA) \cong (E_n \wedge X) \cup C(E_n \wedge A),$$

for any $n \in \mathbb{Z}$, we see that

$$E \wedge A \xrightarrow{1\wedge i} E \wedge X \xrightarrow{1\wedge j} E \wedge (X \cup CA)$$

is a cofibre sequence. It follows from Lemma 1.2.6 that

$$[\Sigma^n S^0, E \wedge A] \xrightarrow{(1\wedge i)_*} [\Sigma^n S^0, E \wedge X] \xrightarrow{(1\wedge j)_*} [\Sigma^n s^0, E \wedge (X \cup CA)]$$

is exact. But this is just the sequence

$$E_n(A) \xrightarrow{i_*} E_n(X) \xrightarrow{j_*} E_n(X \cup CA).$$

Thus E_* is a homology theory.

Since $S^n(X \cup CA) \cong S^n X \cup C(S^n A)$, for any $n \in \mathbb{Z}$, we see that

$$E(A) \xrightarrow{E(j)} E(X) \xrightarrow{E(j)} E(X \cup CA)$$

is a cofibre sequence. Hence it follows from Lemma 1.2.6 that

$$[E(A), \Sigma^n E] \xleftarrow[E(i)^*]{} [E(X), \Sigma^n E] \xleftarrow[E(j)^*]{} [E(X \cup CA), \Sigma^n E]$$

is exact. But this is just the sequence

$$E^n(A) \xleftarrow{i^*} E^n(X) \xleftarrow{j^*} E^n(X \cup CA).$$

Thus E^* is a cohomology theory. $\square$

1.3 The Vietoris and Čech nerves

In this section we introduce the two nerves, of Vietoris type and of Čech type. First we define the Čech nerve. The reference is [K]. Let $\lambda = \{U_\alpha : \alpha \in \Gamma\}$ be an open covering of a topological space X. We define the Čech nerve $X_\lambda^{\check{C}}$ of an open covering λ as follows. To every open set U_α, we associate a vertex α. If $U_\alpha \cap U_\beta$ is nonempty, we connected the vertices α and β with an edge. If $U_\alpha \cap U_\beta \cap U_\gamma$ is nonempty, we fill in the face of the triangle $\alpha\beta\gamma$. Repeating this procedure for all finite intersections gives the nerve of λ. Let Ω be a subset of Γ. The nerve $X_{\lambda'}^{\check{C}}$ of $\lambda' = \{V_\beta : \beta \in \Omega\}$ is a subcomplex $X_\lambda^{\check{C}}$. If Γ is a finite set, $X_\lambda^{\check{C}}$ is a finite simplicial complex. If λ is star finite, $X_\lambda^{\check{C}}$ is locally finite.

Let λ be an open covering $\{U_\alpha : \alpha \in \Gamma\}$ of X. For any subset A of X, let $\Gamma_A = \{\beta : \beta \in \Gamma, U_\beta \cap A\}$, and then $\lambda_A = \{U_\beta \cap A : \beta \in \Gamma_A\}$ is open covering of A. Then $A_{\lambda_A}^{\check{C}}$ is a subcomplex of $X_\lambda^{\check{C}}$. Let $\mu = \{V_\omega : \omega \in \Omega\}$ be an open covering of X which is a refinement of λ. Define Ω_A, τ_A by $\Omega_A = \{\delta : \delta \in \Omega, V_\delta \cap A \neq \emptyset\}$, $\tau_A = \{V_\delta \cap A : \delta \in \Omega\}$ respectively. Let $u = \{u_\alpha \in \Gamma\}$, $v = \{v_\omega : \omega \in \Omega\}$ be the sets of vertices of $X_\lambda^{\check{C}}$, $X_\tau^{\check{C}}$ respectively. For $V_\omega \in \tau$ we chose an open set $U_\alpha \in \lambda$ such that $V_\omega \subset U_\alpha$. We define a map $\pi_u^v : v \to u$ by $\pi_u^v(v_\omega) = u_\alpha$. Let $v_{r_0}, \cdots, v_{r_n}$ of X_τ be vertices of $X_\tau^{\check{C}}$ contained in one simplex. By the property $\emptyset \neq \cap_{i=0}^n V_{r_i} \subset \cap_{i=0}^n U_{\alpha_i}$, we see that $u_{\alpha_i} = \pi_u^v(v_{\tau_i})$ for $i = 0, \cdots, n$ are contained in one simplex of $X_\lambda^{\check{C}}$. By the property $\pi_u^v|_{A_{\tau_A}} : A_{\tau_A} \to A_{\lambda_A}$, the map π_u^v is $\pi_u^v : (X_\tau, A_{\tau_A}) \to (X_\lambda, A_{\lambda_A})$, which is called a *projection*.

Lemma 1.3.1. *If $X_\lambda^{\check{C}}$ and $X_\tau^{\check{C}}$ have weak topology, the map $\pi_u^v : (X_\tau^{\check{C}}, A_{\tau_A}^{\check{C}}) \to$*

$(X^{\check{C}}_{\lambda}, A^{\check{C}}_{\lambda_A})$ *is a continuous map.*

Definition 1.3.2. *Let X be a topological space and K a simplicial complex. Let f_0, f_1 be maps $X \to K$. We say that f_0 is contiguous to f_1 if a set $f_0(x) \cup f_1(x)$ is contained in a simplex of K.*

Theorem 1.3.3. *Give a simplicial complex K a metric topology. If f_0 is a contiguous to f_1, then f_0 is homotopic to f_1.*

Proof. Let Λ be a set of vertices of K. We define a map $F : X \times I \to K$ as follows. For $(x, t) \in X \times I$, we chose a simplex σ containing $f_0(x) \cup f_1(x)$. The interval $[f_0(x), f_1(x)]$ is contained in σ. Define $F(x, t)$ is as the point to divide $[f_0(x), f_1(x)]$ into $t : (1 - t)$. $F(x, t)$ has coordinates $((1 - tv(f_0(x)) + tv(f_1(x)) : v \in \Lambda)$. Let $F(x_0, t_0) = y_0$. For $\epsilon > 0$, let

$$U_i = \{y : y \in K, \rho((f_i(x_0), y) < \frac{\epsilon}{3}\},\ i = 1, 2,$$

where ρ is a metric of K. Then the set $W = f_0^{-1}(U_0) \cap f_1^{-1}(U_1)$ is a neighborhood of x_0. Let

$$I_0 = I \cap (t_0 - \frac{\epsilon}{3}, t_0 + \frac{\epsilon}{3}).$$

Consider the neighborhood $V = W \times I_0$ of (x_0, y_0). We show that

$$\rho(F(x', t'), y_0) < \epsilon \ for\ any\ (x', t') \in V.$$

Let $y' = F(x', t')$. We have

$$y' = ((1 - t')v(f_0(x') + t'v(f_1(x') : v \in \Lambda).$$

Let

$$y'' = ((1 - t')v \times (f_0(x_0)) + t'v(f_1(x_0)) : v \in \Lambda).$$

We have

$$\begin{aligned}\rho(y'', y') &= \Sigma_{v\in\Lambda}|(1-t')\{v(f_0(x_0)) - v(f_1(x_0)) + t'\{v(f_1(x_0)) - v(f_1(x'))\}\}| \\ &\leq (1-t')\rho(f_0(x_0), f_0(x')) + t'\rho(f_1(x_0), f_1(x') \\ &< \frac{\epsilon}{3}.\end{aligned}$$

Hence

$$\rho(y_0, y') \leq \rho(y_0, y'') + \rho(y'', y') < \frac{2}{3}\epsilon + \frac{\epsilon}{3} = \epsilon.$$

Thus F is continuous. □

Theorem 1.3.4. *Let M and K be simpicial complexes with weak topology. Let f_0 and f_1 be maps from M to K. If f_0 is contiguous to f_1, then f_0 is homotopic to f_1.*

Proof. Since $M\times I$ has weak topology, we define the homotopy $F : M\times I \to I$ in a similar way to Theorem 1.3.3. For any simplex σ of M, the restriction $F|_{\sigma\times I}$ is continuous. By Theorem 1.1.5, we set that F is continuous. □

Corollary 1.3.5. *Let λ be an open covering of X. Let τ be a refinement of λ. For any π^v_u, π'^v_u, the map π^v_u is homotopic to π'^v_u.*

Proof. If v_{β_i} for $i = 0, \cdots n$ are vertices of a simplex σ of Čech nerve $X^{\check{C}}_\tau$ of the open covering τ, we have $\cap^n_{i=0} V_{\beta_i} \neq \emptyset$. Put $\pi^v_u(v_{\beta_i}) = u_{\alpha_i}$ and $\pi^{v'}_u(v_{\beta_i}) = u_{\alpha_i}$ for $i = 0, 1\cdots, n$, and then we have $\cap^n_{i=0}(U_{\alpha_i} \cap U_{\alpha'_i}) \subset \cap^n_{i=0} V_{\beta_i} \neq \emptyset$. Hence $u_{\alpha_i}, u_{\alpha'_i}$ are contained in a simplex ω of $X^{\check{C}}_\tau$. For any element x of σ, a set $\pi^v_u(x) \cup \pi^{v'}_u(x)$ belongs to ω. Thus π^v_n is contiguous to $\pi^{v'}_u$. By Theorems 1.3.3 and 1.3.4, π^v_n is homotopic to $\pi^{v'}_u$. □

For each $X \in \mathbf{NG_0}$, let λ be an open covering of X. According to [P], the Vietoris nerve of λ is a simplicial set in which an n-simplex is an ordered

$(n+1)$-tuple $(x_0, x_1, \cdots, x_n)$ of points contained in an open set $U \in \lambda$. Face and degeneracy operators are respectively given by

$$d_i(x_0, \cdots, x_n) = (x_0, x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$$

and

$$s_i(x_0, x_1, \cdots x_n) = (x_0, x_1, \cdots x_{i-1}, x_i, x_i, x_{i+1}, \cdots, x_n),\ 0 \le i \le n.$$

We denote the realization of the Vietoris nerve of λ by X_λ. If λ is a refinement of μ, then there is a canonical map $\pi^\lambda_\mu : X_\lambda \to X_\mu$ induced by the identity map of X.

The relation between the Vietoris and the Čech nerves is given by the following proposition due to Dowker.

Proposition 1.3.6. ([*Dow*]) *The Čech nerve* $X^{\check{C}}_\lambda$ *and the Vietoris nerve* X_λ *have the same homotopy type.*

According to [Dow], for arbitrary topological space, the Vietoris and Čech homology groups are isomorphic and the Alexander-Spanier and Čech cohomology groups are isomorphic.

1.4 Inverse limits and direct limits

We introduce Inverse limits and direct limits. The reference is [Ma] and [K].

Definition 1.4.1. *Let* Λ *be a directed set* $(\Lambda, \le)$. *Let* G *be a family of sets* $\{G_\lambda : \lambda \in \Lambda\}$. X *is called a projective system if it satisfies the following properties;*

1. if $\lambda \leq \mu$ *for any* $\lambda, \mu \in \Lambda$, *there is a correspondence* $\pi_\mu^\lambda : G_\lambda \to G_\mu$,

2. $\pi_\lambda^\lambda = 1$. If $\lambda \leq \mu \leq \nu$, *we have* $\pi_\nu^\lambda = \pi_\nu^\mu \circ \pi_\mu^\lambda$.

We denote the projective system by $\{G_\lambda; \pi_\mu^\lambda\}$. The correspondence π_μ^λ is called a *projection*. We write just $\{G_\lambda\}$ if there is no possibility of confusion.

Let a projective system $\{G_\lambda : \pi_\mu^\lambda\}$ be a family of groups $\{G_\lambda\}$ and a family of homomorphisms $\{\pi_\mu^\lambda : X_\lambda \to X_\mu\}$. Let G be the direct sum $G = \Sigma_{\lambda\in\Lambda} G_\lambda$. Let a subgroup G' of G be generated by

$$\{g_\lambda - \pi_\mu^\lambda g_\lambda |\ \lambda \leq \mu,\ \lambda, \mu \in \Lambda\}.$$

A quotient group G/G' is called a *direct limit* of groups denoted by $\varinjlim\{G_\lambda\}$. For $\lambda \in \Lambda$, the projection : $G \to \varinjlim\{G_\lambda\}$ induces a homomorphism π_λ : $G_\lambda \to \varinjlim\{G_\lambda\}$. Clearly we have

$$\pi_\mu \circ \pi_\mu^\lambda = \pi^\lambda : G_\lambda \to \varinjlim\{G_\lambda\}.$$

Proposition 1.4.2. *For any element* $g_\infty \in \varinjlim\{G_\lambda\}$, *there is an element* g_μ *of* G_μ *such that* $\pi_\mu g_\mu = g_\infty$.

Proof. Let g_∞ be a coset $[\Sigma_{i=1}^n g_{\lambda_i}]$. Since Λ is a directed set, there is a $\mu \in \Lambda$ such that $\lambda_i \leq \mu$ for any i. Let $g_\mu = \Sigma_{i=1}^k \pi_\mu^{\lambda_i} g_{\lambda_i}$. Since $g_\mu \in G_\mu$, we have

$$\Sigma_{i=1}^k g_{\lambda_i} - g_\mu = \Sigma_{i=1}^k (g_{\lambda_i} - \pi_\mu^{\lambda_i'} g_\lambda).$$

Hence we have $\pi_\mu g_\mu = g_\infty$. □

Proposition 1.4.3. *If* $\pi_\mu^\lambda g_\lambda = 0$ *for* $g_\lambda \in G_\lambda$, *there is* $\mu \in \Lambda$ *such that* $\lambda \leq \mu$ *and* $\pi_\mu^\lambda g_\lambda = 0$.

Proof. Assume that $\pi_\lambda g_\lambda = 0$. By the assumption, we see that $g_\lambda = \Sigma_{i=1}^k (g_{\lambda_i} - \pi_{\mu_i}^{\lambda_i} g_{\mu_i})$. There is $\mu \in \Lambda$ such that $\lambda \le \mu$, $\mu_i \le \mu$ for $i = 1, \cdots, k$. Hence we have $\pi_\mu^\lambda g_\lambda = 0$. $\square$

Definition 1.4.4. *A map of the direct system $\{G_\lambda\}$ into the direct system $\{G'_\mu\}$ consists of an order preserving map $f : \Lambda \to \Lambda'$ and a homomorphism*

$$f_\lambda : \{G_\lambda\} \to \{G'_{f(\lambda)}\} \ for\ each\ \lambda \in \Lambda$$

subject to the following property: if $\lambda < \mu$, then the following diagram is commutative:

$$\begin{tikzcd} G_\lambda \arrow[r, "f_\lambda"] \arrow[d, "\pi_\mu^\lambda"'] & G'_{f(\lambda)} \arrow[d, "\pi_{f(\mu)}^{f(\lambda)}"] \\ G_\mu \arrow[r, "f_\mu"'] & G'_{f(\mu)} \end{tikzcd}$$

Then f induce a homomorphism from Direct limits $\varinjlim\{G_\lambda\}$ to $\varinjlim\{G'_\mu\}$. By **Proposition 1.4.2**, for any g_∞ there is a $g_\mu \in G_\mu$ such that $\pi_\mu g_\mu = g_\infty$. We define a homomorphism $\varinjlim f_\infty : \varinjlim\{G_\lambda\} \to \varinjlim\{G'_\mu\}$ by $\varinjlim f_\infty(g_\infty) = \pi_\mu f_\mu(g_\mu)$ where π_μ is a projection $\pi_\mu : G'_\mu \to \varinjlim\{G'_\mu\},\ \mu \in \Lambda'$. We denote this homomorphism simply by $\varinjlim f$ if there is no possibility of confusion.

Definition 1.4.5. *Let Λ be a directed set. A subset $\Lambda' \subset \Lambda$ is called a $cofinal$ subset if for any $\lambda \in \Lambda$ there is $\mu \in \Lambda'$ such that $\lambda < \mu$.*

Note that a cofinal subset is also a directed set with respect to the given order relation. Let us recall that properties of direct limits of exact sequence of modules. Let R be a ring with unit. Suppose that $\{G_\lambda\}$ is a direct system of R-modules defined on a direct system Λ. Let Λ' be a cofinal subset of Λ. Let G'_λ be a direct system defined as follows. For any $\lambda \in \Lambda'$, $G'_\lambda = G_\lambda$.

if $\lambda,\ \mu \in \Lambda'$ and $\lambda < \mu$, then $\pi'^{\lambda}_{\mu} = \pi^{\lambda}_{\mu}$. The inclusion map $\Lambda' \to \Lambda$ is an order-preserving map. Hence for each $\lambda \in \Lambda'$, the identity map $G'_\lambda \to G_\lambda$ defines a map of the directed system $\{G'_\lambda\}$ into the directed system $\{G_\lambda\}$. Obviously the map induces an isomorphism $\varinjlim_\lambda G'_\lambda \to \varinjlim_\lambda G_\lambda$.

Theorem 1.4.6. *Suppose that $\{G_\lambda\}$, $\{G'_\lambda\}$ and $\{G''_\lambda\}$ are direct systems of modules, all defined on the same directed set Λ. For each $\lambda \in \Lambda$, assume given homomorphisms $f_\lambda : G_\lambda \to G'_\lambda$ and $g_\lambda : G'_\lambda \to G''_\lambda$ such that the following sequence*

$$G'_\lambda \xrightarrow{f_\lambda} G_\lambda \xrightarrow{g_\lambda} G''_\lambda$$

is exact. Then the limit sequence

$$\varinjlim_\lambda G'_\lambda \xrightarrow{\varinjlim f_\lambda} \varinjlim_\lambda G_\lambda \xrightarrow{\varinjlim g_\lambda} \varinjlim_\lambda G''_\lambda$$

is exact.

Proof. Since $g_\lambda f_\lambda = 0$, for each λ, it follows that $\varinjlim g_\lambda \circ \varinjlim f_\lambda = 0$. We prove that kernel $\varinjlim g_\lambda \subset$ image $\varinjlim f_\lambda$. Let $a \in \varinjlim G_\lambda$ be such that

$$\varinjlim\ g_\lambda(a) = 0.$$

There exists an index $\lambda \in \Lambda$ and an element $a_\lambda \in G_\lambda$ such that

$$\pi_\lambda(a_\lambda) = a,$$

where $\pi_\lambda : G_\lambda \to \varinjlim G$, $\lambda \in \Lambda$, are the maps defining $\varinjlim\ G$ as a direct limit of $\varinjlim\ G$. Then

$$0 = \varinjlim\ g_\lambda(a) = (\varinjlim\ g_\lambda \circ \pi_\lambda)(a_\lambda) = (\pi''_\lambda \circ g_\lambda)(a_\lambda).$$

Since $(\pi''_\lambda \circ g_\lambda)(a_\lambda) = 0$, there exists an index $\mu > \lambda$ such that

$$({\pi''}^\lambda_\mu \circ g_\lambda)(a_\lambda) = 0.$$

Therefore

$$(g_\mu \circ \pi^\lambda_\mu)(a_\lambda) = ({\pi''}^\lambda_\mu \circ g_\lambda)(a_\lambda) = 0,$$

when by exactness there exists an element $a'_\lambda \in G'_\mu$ such that

$$f_\mu(a'_\mu) = \pi^\lambda_\mu(a_\lambda).$$

Then we have

$$\begin{aligned}(\varinjlim f \circ {\pi'}_\mu)(a_\lambda) &= (\pi_\mu \circ f_\mu)(a'_\mu) \\ &= (\pi_\mu \circ \pi^\lambda_\mu)(a_\lambda) \\ &= \pi_\lambda(a_\lambda) = a\end{aligned}$$

as required. □

Corollary 1.4.7. *If g_λ is a monomorphism for each $\lambda \in \Lambda$, then so is $\varinjlim g_\lambda$.*

Corollary 1.4.8. *If f_λ is an epimorphism for each $\lambda \in \Lambda$, so is $\varinjlim f_\lambda$.*

Next we recall the definition of *inverse limit*. Inverse limits are dual to direct limits.

Definition 1.4.9. *Let Λ be a directed poset $(\Lambda, \leq)$. Let G be a family of sets $\{G_\lambda : \lambda \in \Lambda\}$. G is called an inverse system if satisfies the following properties satisfied;*

1. if $\lambda \leq \mu$ for any $\lambda, \mu \in \Lambda$, there is a correspondence $\pi^\lambda_\mu : X_\mu \to X_\lambda$,

2. $\pi^\lambda_\lambda = 1$. If $\lambda \leq \mu \leq \nu$, we have $\pi^\lambda_\nu = \pi^\mu_\nu \circ \pi^\lambda_\mu$.

We denote the inverse system by $\{G_\lambda;\pi^\lambda_\mu\}$. A correspondence π^λ_ν is called a *projection.*

Let $\{G_\lambda:\pi^\lambda_\mu\}$ be consisting of an inverse system a family of groups $\{G_\lambda\}$ and a family of homomorphisms $\{\pi^\lambda_\mu : G_\mu \to G_\lambda\}$. Put

$$G' = \{(x_\lambda) \in \prod_{\lambda\in\Lambda} G_\lambda \mid \pi^\lambda_\mu(x_\mu) = \lambda,\ for\ all\ \lambda \leq \mu\}.$$

The subspace G' is called an *inverse limit* of groups denoted by $\varprojlim_\lambda\{G_\lambda\}$.

A homomorphism $\pi_\lambda : \varprojlim_\lambda\{G_\lambda\} \to G_\lambda$ is induced by projection : $\prod_{\lambda\in\Lambda} G_\lambda \to G_\lambda$. It satisfies that

$$\pi^\lambda_\mu \circ \pi_\mu = \pi_\lambda : \varprojlim_\lambda\{G_\lambda\} \to G_\lambda,\ \lambda \leq \mu,\ \lambda,\ \mu \in \Lambda.$$

Definition 1.4.10. *Let Λ be a directed set consisting of the positive integers with their usual ordering. Let G_i be a R-modules for all $i \in \Lambda$. Suppose that there is a homomorphism $\varphi_i : G_{i+1} \to G_i$ for all $i \in \Lambda$. Let G be the inverse system $\{G_i;\pi_i\}$. Then G is represented by a diagram*

$$G_1 \xleftarrow{\pi_1} G_2 \xleftarrow{\pi_2} G_3 \xleftarrow{\pi_3} \cdots$$

and is called a tower of R-modules.

Definition 1.4.11. *Let*

$$G_1 \xleftarrow{\pi_1} G_2 \xleftarrow{\pi_2} G_3 \xleftarrow{\pi_3} \cdots$$

be a tower left R-modules. The inverse limit is a submodule of the cartesian product:

$$G' = \prod_{n=1}^{\infty} G_n$$

which is specified as follows. We define a homomorphism

$$d : G' \to G'$$

by the formula

$$d(x_1, x_2, x_3 \cdots) = (x_1 - \varphi_2(x_2), x_2 - \varphi_3(x_3), x_4 - \varphi_3(x_3), \cdots).$$

Then the kernel of d is the inverse limit of the sequence. The cokernel of d is called the $first$ $derived$ $functor$ of the inverse limit. We denote it by $\lim^1 G_n$.

Observe that $\lim^1$ has the following property. If the diagram

$$\begin{tikzcd}
G_1 \arrow[d, "\Phi_1"] & G_2 \arrow[l, "\pi_1"'] \arrow[d, "\Phi_2"] & G_3 \arrow[l, "\pi_2"'] \arrow[d, "\Phi_3"] & \cdots \arrow[l, "\pi_3"'] \\
G_1' & G_2' \arrow[l, "\pi_1'"'] & G_3' \arrow[l, "\pi_2'"'] & \cdots \arrow[l, "\pi_3'"']
\end{tikzcd}$$

is commutative, there is an induced homomorphism

$$\lim{}^1\Phi_n : \lim{}^1 G_n \to \lim{}^1 G_n'.$$

Thus $\lim^1$ is a covariant functor from the category of inverse sequences of R-modules to the category of R-modules.

Proposition 1.4.12. *Given a short exact sequence of towers*

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & G_2' \arrow[r, "\Phi_2"] \arrow[d] & G_2 \arrow[r, "\Psi_2"] \arrow[d] & G_2'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G_1' \arrow[r, "\Phi_1"] & G_1 \arrow[r, "\Phi_1"] & G_1'' \arrow[r] & 0
\end{tikzcd}$$

Such a short exact sequence gives rise to the following long exact sequence

$$0 \longrightarrow \varprojlim G'_n \longrightarrow \varprojlim G_n \longrightarrow \varprojlim G''_n$$

$$\xrightarrow{\Delta} \lim^1 G'_n \longrightarrow \lim^1 G_n \longrightarrow \lim^1 G''_n \longrightarrow 0$$

Proof. We consider the homomorphism

$$d : \prod : G_n \to \prod G_n$$

which defines $\varprojlim G_n$ and $\lim^1 G_n$ as is defined a cochain complex $\{C^n, \delta_n\}$ with $C^0 = C^1 = \prod M_n$, $C^n = 0$ for $n \neq 0$ or 1, $\delta_0 = d$, and $\delta_n = 0$ for $n \neq 0$ for $n \neq 0$. Then

$$\varprojlim G_n = H^0(C)$$

$$\lim{}^1 G_n = H^1(C),$$

and $H^n(C) = 0$ for $n \neq 0$ or 1. The short exact sequence of towers gives rise to a short exact sequence of such cochain complexes. The corresponding long exact cohomology sequence is the exact sequence whose existence is asserted in the statement of the theorem. $\square$

Definition 1.4.13. *An inverse sequence of modules*

$$G_1 \longleftarrow G_2 \longleftarrow G_3 \longleftarrow \cdots$$

is said to satisfy the $Mittag-Leffler$ condition if for every integer n there exists an integer $m \geq n$ such that for any $i \geq m$,

$$\text{Image}(G_i \to G_m) = \text{Image}(G_m \to G_n).$$

Proposition 1.4.14. *Let*

$$G_1 \longleftarrow G_2 \longleftarrow G_3 \longleftarrow \cdots$$

be a tower of R-modules which satisfies the Mittag-Leffler condition. Then

$$\varinjlim_n{}^1 G_n = 0.$$

Proof. For each integer n, let G'_n be a submodule of G_n which is the image of $G_i \to G_n$ for a sufficiently large integer $i > n$. By the Mittag-Leffler condition, the homomorphism $G_{n+1} \to G_n$ maps G'_{n+1} onto G_n. We have an induced tower

$$G'_1 \longleftarrow G'_2 \longleftarrow G'_3 \longleftarrow \cdots$$

such that all homomorphisms are onto. Hence by the definition of $\lim^1$, we see that$\lim^1 G'_n = 0$. Consider the tower of quotient modules

$$\frac{G_1}{G'_1} \longleftarrow \frac{G_2}{G'_2} \longleftarrow \frac{G_3}{G'_3} \longleftarrow \cdots .$$

There is an integer $m > n$ such that the homomorphism

$$\frac{G_m}{G'_m} \to \frac{G_n}{G'_n}$$

is zero. This fact implies that $\lim^1 G_n/G'_n = 0$. Then it is sufficient to apply the exact sequence of Proposition 1.4.12 to the short exact sequence

$$0 \to \{G'_n\} \to \{G_n\} \to \{G_n/G'_n\} \to 0.$$

$\square$

We recall the properties of direct and inverse limits of modules. After that we recall the properties of limit of topological spaces.

Definition 1.4.15. *Let Λ be a directed set. Let X_λ be a topological space for all $\lambda \in \Lambda$. Suppose that there is a continuous map $\pi^\lambda_\mu : X_\mu \to X_\lambda$, $\lambda \leq \mu$. The inverse limit system $\{X_\lambda, \pi\}$ is called an inverse system of topological spaces.*

Proposition 1.4.16. *Let $\{X_\lambda; \pi^\lambda_\mu\}$ be an inverse system of topological spaces. If X_λ is a Hausdorff space for each $\lambda \in \Lambda$, then $\varprojlim_\lambda X_\lambda$ is a closed subset of $\prod_{\lambda\in\Lambda}$.*

Proof. Let $y = (x_\lambda : \lambda \in \Lambda)$ be an element of $\prod_\lambda \backslash \varprojlim X_\lambda$. It is sufficient to prove that there is an open neighborhood W of y such that $W \cap \varprojlim X_\lambda$. There is an open covering $\lambda \geq \mu$ such that $\pi^\lambda_\mu x_\mu \neq x_\lambda$. Since X_λ is Hausdorff, there is an open neighborhood V_λ, U_λ of $\pi^\lambda_\mu x_\mu$, x_λ in X_λ respectively such that $V_\lambda \cap U_\lambda \neq \emptyset$. Let

$$V_\mu = (\pi^\lambda_\mu)^{-1} V_\lambda.$$

Then V_μ is an open neighborhood in X_μ. Let $W = \{(x_\lambda) : x_\lambda \in U_\lambda, x\mu \in V_\mu\}$ be an open set containing an element y. Then $W \cap \varprojlim X_\lambda = \emptyset$. □

Proposition 1.4.17. *If X_λ is compact for each $\lambda \in \Lambda$, then $\varprojlim X_\lambda$ is compact.*

Proof. For each $\mu \in \Lambda$ we define a subspace

$$Y_\mu = \{(x_\lambda) \in \prod X_\lambda | \pi^\lambda_\mu(x_\mu) = x_\lambda,\ \lambda \geq \mu\}.$$

By the argument of Proposition 1.4.16, we see that Y_μ is a closed subset $\prod X_\lambda$. Since $\prod_\lambda X_\lambda$ is compact, Y_μ is compact for all μ. Hence we often the following

$$\varprojlim X_\lambda = \cap_{\mu\in\Lambda} Y_\mu \neq \emptyset,$$

since Λ is a ordered set. Thus it is clear that $\varprojlim X_\lambda$ is compact. □

Proposition 1.4.18. *Let $\mathcal{U}$ be an open covering of $\varprojlim X_\lambda$. There are an index $\lambda_0 \in \Lambda$ and an open covering $\mathcal{V}$ of X_{λ_0} such that $(\pi_{\lambda_0})^{-1}\mathcal{V}$ is a refinement of $\mathcal{U}$.*

Proof. For any element $(x_\lambda) \in \varprojlim X_\lambda$, we choose finite elements $\mu_1 \cdots \mu_k$ of Λ and an open neighborhood V_{μ_i} containing x_{μ_i} such that an open neighborhood

$$W_{(x'_\lambda)} = \{(x'_\lambda) : (x'_\lambda) \in \varprojlim X_\lambda, x_{\mu_i} \in V_{\mu_i},\ i = 1, \cdots, k\}$$

is contained in an open set in $\mathcal{U}$. Then $W = \{W_{(x'_\lambda)}\}$ is an open covering of $\varprojlim X_\lambda$. Since $\varprojlim X_\lambda$ is compact, there is an open covering $\{W^j_{(x_\lambda)} \in W | j = 1, \cdots, k\}$ of $\varprojlim X_\lambda$. There are some elements μ_i corresponding to $W^j_{(x_\lambda)}$. We denote this set μ_i by Γ_j. Observe that $\cup_j \Gamma_j$ is a finite set of Λ. Since Λ is an ordered set, there is an element λ_0 such that $\mu \leq \lambda_0$ for any $\mu \in \Lambda$. Let

$$V_j = \cap\{(\pi^\mu_{\lambda_0})^{-1} V_\mu | \mu \in \Gamma_j\}.$$

Then V_j is an open set of X_{λ_0} and $(\pi_{\lambda_0})^{-1} V_j$ is contained in an element of $\mathcal{U}$. Put

$$V_0 = \{X_{\lambda_0} - \pi_{\lambda_0} \varprojlim X_\lambda\}.$$

Since $\varprojlim X_\lambda$ is compact, $\pi_{\lambda_0} \varprojlim X_\lambda$ is an open set of X_λ. Let $\mathcal{V} = \{V_j | j = 0, \cdots, k\}$. Then $\mathcal{V}$ is an open covering of X_{λ_0}.

□

1.5 Homotopy inverse limits

We introduce the homotopy inverse limit following [V] and [BK]. Let $\mathbf{Top}_0$ be a category of based topological spaces and based maps. Let $\mathbf{I}$ be a small category and $D : \mathbf{I} \to \mathbf{Top}_0$ a $\mathbf{I}$-diagram in $\mathbf{Top}_0$.

Let $n > 0$ and put

$$\mathbf{I}_n(A,B) = \{(f_n, \cdots, f_1) \in (\mathrm{mor}\,\mathbf{I})^n \mid f_n \circ \cdots \circ f_1 : A \to B \textit{ is defined in } \mathbf{I}\,.A, B \in \mathbf{I}\},$$

$$\mathbf{I}_0(A,A) = \{(\mathrm{id}_A)\},\ \mathbf{I}_0(A,B) = \emptyset \textit{ for } A \neq B.$$

Definition 1.5.1. *For any topological spaces X, Y, let $F(X,Y)$ denote the space of maps $X \to Y$ with compact open topology. Then the homotopy inverse limit of D,*

$$\underleftarrow{\mathrm{holim}} D \subset (\prod_{A,B\in\mathbf{I}} F(\coprod_{n=0}^{\infty} \mathbf{I}_n(A,B) \times I^n, D(B)))$$

is the subset of all elements $\{\alpha_{A,B} : \coprod_{n=0}^{\infty} I_n(A,B) \times I^n \to D(B) |\ A, B \in I\}$ where I is the unit interval, satisfying

$\alpha_{A,B}(f_n, t_n, \cdots, f_1, t_1) =$

$$\begin{cases} \alpha_{A,B}(f_n, t_n, \cdots, f_{i+1}, t_{i+1}, f_{i-1}, \cdots, t_1) & f_1 = \mathrm{id} \ \ for\ i < n \\ \alpha_{A,B}(f_{n-1}, t_{n-1}, \cdots, t_1) & f_n = \mathrm{id}, \\ \alpha_{A,B}(f_n, t_n, \cdots, t_{i+1}, f_i \circ f_{i-1}, t_{i-1}, \cdots t_1) & t_i = 1,\ i > 1 \\ \alpha_{C,B}(f_n, t_n, \cdots, t_2) & t_1 = 1 \\ D(B)(f_n, t_n, \cdots, f_i; \alpha(A,E)(f_{i-1}, t_{i-1}, \circ \cdots \circ t_1)(x)) & t_i = 0 \\ * & x = \textit{base point} \end{cases}$$

where $C =$ range (f_1) and E $=$ range (f_{i-1}).

The *homotopy limit* of D is defined dually.

Let $\mathbf{Top}_0^{\mathbf{I}}$ be a set of covariant functors $\mathbf{I} \to \mathbf{Top}_0$. For $X \in \mathbf{Top}_0^{\mathbf{I}}$, the map $\mathbf{I}/i$ induces a natural map

$$\underleftarrow{\lim}\ X_i \to \underleftarrow{\mathrm{holim}}\ X_i$$

which, in general, is not a weak equivalence. We give some examples.

Proposition 1.5.2. *The following are examples for which the natural map* $\varprojlim X_i \to \operatorname{holim}_{\leftarrow} X_i$ *is a weak equivalence and in which each map* $X_i \to \mathrm{pt}$ *is assumed to be fibration:*

1. $\mathbf{I}$ *is discrete.*

2. $\mathbf{I}$ *contains only two objects and one map between them; then the homotopy inverse limit reduces to the usual mapping path space.*

3. $\mathbf{I}$ *has an initial object* $i_0 \in \mathbf{I}$, *i.e. for each* $i \in \mathbf{I}$, *there is exactly one map* $i_0 \to i \in \mathbf{I}$; *in this case, the natural map*

$$\operatorname{holim}_{\leftarrow} X_i \to X_{i_0}$$

is also a weak equivalence.

4. *Every diagram in* $\mathbf{I}$ *of the form*

$$X' \longrightarrow X \longleftarrow X''$$

in which at least one of the maps is a fibration.

5. *Every tower of fibrations*

$$\cdots \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0.$$

We decompose the homotopy groups of the inverse limit of a tower of fibrations into a $\varprojlim$-part and a $\varprojlim^1$-part.

Theorem 1.5.3. ([*BK*]) *Let* $X = \varprojlim X_n$, *where*

$$\cdots \longrightarrow X_n \xrightarrow{p} X_{n-1} \longrightarrow \cdots \longrightarrow X_{-1} = *$$

is a tower of fibrations. Then there is, for every $i \geq 0$, a natural short exact sequence

$$* \longrightarrow \varprojlim_i^1 \pi_{n+1}(X_i) \longrightarrow \pi_n(\underleftarrow{\mathrm{holim}}_i X) \longrightarrow \varprojlim_i \pi_n(X_i) \longrightarrow *.$$

1.6 Čech cohomology and homology groups

Let X be a topological space. Let

$$\Lambda = \{\lambda \mid \lambda : \textit{an open covering of } X\}.$$

If μ is a refinement of λ for $\lambda,\ \mu \in \Lambda$, we denote it by $\lambda \leq \mu$. So the set Λ is an ordered set. Since we choose finite elements λ_i, for $i = 1, 2, \cdots, k$ in Λ, an open covering

$$\mu = \{\cap U_i \mid U_i \in \lambda_i,\ i = 1, \cdots, k\}$$

of X is a refinement of λ_i for any i. Thus $\lambda_i \leq \mu$, for $i = 1, \cdots, k$. We denote the nerve of λ by N_λ. If $\lambda \leq \mu$, there is a natural map

$$\pi_\mu^\lambda : X_\mu \to X_\lambda.$$

Let $\pi_{\mu *}^\lambda$ be a homomorphism

$$\pi_{\mu *}^\lambda : H_n(X_\mu, G) \to H_n(X_\lambda, G),\ n = 0, 1, \cdots$$

induced by π_μ^λ, where $H_n(X_\lambda, G)$ is a singular homology with coeffcient G. Remark that $\pi_{\mu *}^\lambda$ is decided without any relation to π_μ^λ. Let $\pi_\mu^{\lambda'}$ be a natural map. Since π_μ^λ and $\pi_\mu^{\lambda'}$ are contiguous, $\pi_{\mu *}^\lambda = \pi_{\mu *}^{\lambda'}$ by Theorem 1.3.3. If $\lambda \leq \mu \leq \nu$,we have $\pi_{\lambda *}^\mu \circ \pi_{\nu *}^\lambda = \pi_{\nu *}^\lambda$. For $n \in \mathbb{N}$, we have an inverse system

$$\{\check{\mathrm{H}}(X_\lambda, G); \pi_{\mu *}^\lambda\}$$

of modules. We define the Čech homology group with coefficient G for X by

$$\check{\mathrm{H}}_n(X : G) = \varprojlim_\lambda H_n(X_\lambda : G).$$

Then π^λ_μ induces a homomorphism

$$\pi^{\lambda *}_\mu : H^n(X_\lambda : G) \to H^n(X_\mu : G).$$

For $n \in \mathbb{N}$, we have a direct system

$$\{\check{\mathrm{H}}(X_\lambda, G); \pi^{\lambda *}_\mu\}.$$

We define the Čech cohomology group with coefficient G for X by

$$\check{\mathrm{H}}^n(X : G) = \varinjlim_\lambda H_n(X_\lambda : G).$$

Next let A be a closed subset of X. For any $\lambda \in \Lambda$, let

$$\lambda_A = \{U \cap A \mid U \cap A \neq \emptyset,\ U \in \lambda\}.$$

Then λ_A is an open covering of A. If $\lambda \leq \mu$, for $\lambda, \mu \in \Lambda$, there is a natural map

$$\pi^\lambda_\mu : (X_\mu, A_{\mu_A}) \to (X_\lambda, A_{\lambda_A}).$$

Thus we have an inverse system

$$\{H_n(X_\lambda, A_{\lambda_A} : G); \pi^\lambda_{\mu *}\}$$

and a direct system

$$\{H^n(X_\lambda, A_{\lambda_A} : G); \pi^{\lambda *}_\mu\}$$

of modules. We define the Čech homology group with coefficient G for a pair (X, A) by

$$\check{\mathrm{H}}_n(X, A : G) = \varprojlim_\lambda H_n(X_\lambda, A_{\lambda_A} : G)$$

and the Čech cohomology group with coefficient G for a pair (X, A) by

$$\check{\mathrm{H}}^n(X, A : G) = \varinjlim_{\lambda} H^n(X_\lambda, A_{\lambda_A} : G).$$

Let Y be a topological space and B a closed subset of Y. For any continuous map $f : (X, A) \to (Y, B)$, we define homomorphisms

$$f_* : \check{\mathrm{H}}_n(X, A) \to \check{\mathrm{H}}_n(Y, B)$$

and

$$f^* : \check{\mathrm{H}}^n(Y, B) \to \check{\mathrm{H}}^n(X, A)$$

as follows. Let $\Lambda = \{\lambda\}$ and $\Gamma = \{\rho\}$ be the collection of open coverings of X and Y respectively. For any $\rho \in \Gamma$, $f^{-1}\rho = \{f^{-1}(V) \mid V \in \rho\}$ is an open covering of X. Since there is $\lambda \in \Lambda$ such that $f^{-1}\rho = \lambda$, we define a correspondence $\xi : \Gamma \to \Lambda$ by $\xi\rho = \lambda$. If $\rho \leq \tau$, we have $\xi\rho \leq \xi\tau$. So ξ preserves the order. For each $\rho \in \Gamma$, let

$$\rho_B = \{V \cap B \mid V \cap B \neq \emptyset,\ V \in \rho\}.$$

Let Y_ρ, and B_{ρ_B} be the nerves of ρ, and ρ_B respectively. Since for each $V \in \rho$ there is an open set $U \in \lambda$ such that $f^{-1}(U) = V$, we define a simplicial map

$$f_\rho : X_{\xi\rho} \to Y_\rho$$

by the correspondence $U \to V$ where $X_{\xi\rho}$ is the nerve of $\xi\rho = \lambda$. Let $A_{\xi\rho_B}$ be a nerve of λ_A. Since $f_\rho : A_{\xi\rho_B} \to B_{\rho_B}$, we have

$$f_\rho : (X_{\xi\rho}, A_{\xi\rho_B}) \to (Y_\rho, B_{\rho_B}).$$

Hence f_ρ induces homomorphisms

$$f_{\rho*} : H_n(X_{\xi\rho}, A_{\xi\rho_B} : G) \to H_n(Y_\rho, B_{\rho_B} : G)$$

and

$$f_\rho^* : H^n(Y_\rho, B_{\rho_B} : G) \to H^n(X_{\xi\rho}, A_{\xi\rho_B} : G).$$

Moreover if $\rho \le \tau$ for $\rho,\ \tau \in \Gamma$, the two diagrams

$$\begin{tikzcd}
H^n(Y_\rho, B_{\rho_B} : G) \arrow[r, "f_\rho^*"] \arrow[d, "\mu_\tau^{\rho*}"'] & H^n(X_{\xi\rho}, A_{\xi\rho_B} : G) \arrow[d, "\pi_{\xi\tau}^{\xi\rho*}"] \\
H^n(Y_\rho, B_{\rho_B} : G) \arrow[r, "f_\xi^*"'] & H^n(X_{\xi\rho}, A_{\xi\rho_B} : G)
\end{tikzcd}$$

and

$$\begin{tikzcd}
H_n(X_{\xi\rho}, A_{\xi\rho_B} : G) \arrow[r, "f_{\rho*}"] \arrow[d, "\pi_{\xi\tau*}^{\xi\rho}"'] & H_n(Y_\tau, B_{\tau_B} : G) \arrow[d, "\mu_{\tau*}^{\rho}"] \\
H_n(X_{\xi\rho}, A_{\xi\rho_B} : G) \arrow[r, "f_{\xi*}"'] & H_n(Y_\rho, B_{\rho_B} : G).
\end{tikzcd}$$

are commutative, where $\mu_\tau^\rho,\ \pi_{\xi\tau}^{\xi\rho}$ are simplicial maps induced by the inclusions. Thus there is a homomorphism induced by f such that

$$\varprojlim f_{\rho*} : \check{\mathrm{H}}_n(X, A; G) \to \check{\mathrm{H}}_n(Y, B; G)$$

and

$$\varinjlim f_\rho^* : \check{\mathrm{H}}^n(Y, B; G) \to \check{\mathrm{H}}^n(X, A; G).$$

We denote $\varprojlim f_{\rho*}$ and $\varinjlim f_\rho^*$ by f_* and f^* respectively. By the definition of $\varprojlim f_{\rho*}$ and $\varinjlim f_\rho^*$, the following proposition is clear.

Proposition 1.6.1. *Let $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (Z, C)$ be maps. Then*

$$g_* f_* = (gf)_* : \check{\mathrm{H}}_n(X, A; G) \to \check{\mathrm{H}}_n(Z, C; G)$$

and

$$f^* g^* = (gf)^* : \check{\mathrm{H}}^n(Z, C; G) \to \check{\mathrm{H}}^n(X, A; G).$$

1.7 Steenrod homology

In this section, we recall the definition of the Steenrod homology group.

In [St], Steenrod constructed a homology theory H_*^{st} for compact pairs of metric spaces as follows. From p.834 of the same paper [St], we recall

Definition 1.7.1. *A regular map of complex K in X is a function f defined over the vertices of K with values in X such that, for any ε_0, all but a finite number of simplices have their vertices imaging onto sets of diameter$<$ ε.*

Definition 1.7.2. *A regular q-chain of X is a set of three objects*: *a complex A,a regular map f of A in X, and a q-chain C^q of A. If C^q is a q-cycle, (A, f, C^q) is called a regular q- cycle.*

Definition 1.7.3. *Two regular q-cycles $(A_1, f_1.C_1^q)$ and (A_2, f_2, C_1^q) of X are homologous if there exists a $(q+1)$-chain (A, f, C^{q+1}) such that A_1 and A_2 are closed (not necessarily disjoint) subcomplexes of A, f agrees with f_1 on A_1 and f_2 on A_2, and $\partial C^{q+1} = C_1^q - C_2^q$.*

We define a Steerod homology group $H_*^s(X)$ by the homology group associated the regular q-cycles of X. Steenrod shows that this homology theory satisfies the Eilenberg-Steenrod axioms for all compact pairs. In [Mi], Milnor proves that Steenrod homology theory satisfies two extra axioms as follows.

1. Invariance under relative homeomorphism.

2. (Cluster axiom) If X is a union of countably many compact subsets $X_1, X_2, \cdots$ which intersect pointwise at a single point b, and which

have diameters tending to zero, then $H_q(X, b)$ is naturally isomorphic to the direct product of the groups $H_q(X_i, b)$.

Moreover Milnor characterized the Steenrod homology group by a coefficient group.

Theorem 1.7.4. [*Mi*] *There exists one and only one homology theory $H_*(\ )$ defined for pairs of compact metric spaces which satisfies the two extra Axioms as well as the seven Eilenberg-Steenrod Axioms and which satisfies $H_0(\mathrm{pt}) = \mathrm{G}$.*

In addition, there is a method of defining Steenrod homology theories by a spectrum. In [EH] and [KKS], Steenrod homology theories with coefficients are defined by a spectrum as follows. Let S be a spectrum and X a compact metric space. There is a nerve N_i as in Section 1, 3. Then $N_i \wedge S$ is a spectrum. The Steenrod homology theory is defined by

$$H_k(X, S) = \pi_k \underleftarrow{\mathrm{holim}}_i (N_i, D).$$

Chapter 2

Homology and cohomology associated with a bibariant functor

2.1 Diffeological spaces

To derive the properties of the category of numerically generated spaces we use an adjunction between the categories of topological and diffeological spaces, respectively. Recall from [I] that a diffeological space consists of a set X together with a family D of maps from open subsets of Euclidean spaces into X satisfying the following conditions:

Covering Any constant parametrization $\mathbf{R}^n \to X$ belongs to D.

Locality A parametrization $\sigma\colon U \to X$ belongs to D if every point u of U has a neighborhood W such that $\sigma|W\colon W \to X$ belongs to D.

Smooth compatibility If $\sigma\colon U \to X$ belongs to D, then so does the composite $\sigma f\colon V \to X$ for any smooth map $f\colon V \to U$ between open subsets of Euclidean spaces.

We call D a *diffeology* of X, and each member of D a *plot* of X.

A map between diffeological spaces $f\colon X \to Y$ is called smooth if for any plot $\sigma\colon U \to X$ of X the composite $f\sigma\colon U \to Y$ is a plot of Y. In particular, if D and D' are diffeologies on a set X then the identity map $(X, D) \to (X, D')$ is smooth if and only if D is contained in D'. In that case, we say that D is finer than D', or D' is coarser than D. Clearly, the class of diffeological spaces and smooth maps form a category **Diff**.

Theorem 2.1.1. *The category* **Diff** *is complete, cocomplete, and cartesian closed.*

A category is complete if it has equalizers and small products, and is cocomplete if it has coequalizers and small coproducts. Therefore, the theorem follows from the basic constructions given below.

Products Given diffeological spaces X_j, $j \in J$, their product is given by a pair $(\prod_{j\in J} X_j, D)$, where D is the set of parametrizations $\sigma\colon U \to \prod_{j\in J} X_j$ such that every its component $\sigma_j\colon U \to X_j$ is a plot of X_j.

Coproducts The coproduct of X_j, $j \in J$, is given by $(\coprod_{j\in J} X_j, D)$, where D is the set of parametrizations $\sigma\colon U \to \coprod_{j\in J} X_j$ which can be written locally as the composite of the inclusion $X_j \to \coprod_{j\in J} X_j$ with a plot of X_j.

Subspaces Any subset A of a diffeological space X is itself a diffeological space with plots given by those parametrizations $\sigma\colon U \to A$ such that the post composition with the inclusion $A \to X$ is a plot of X.

Quotients Let $p\colon X \to Y$ be a surjection from a diffeological space X to a set Y. Then Y inherits from X a diffeology consisting of those parametrizations $\sigma\colon U \to Y$ which lifts locally, at every point $u \in U$, along p.

Exponentials Given diffeological spaces X and Y, the set $\hom_{\mathbf{Diff}}(X, Y)$ has a diffeology $D_{X,Y}$ consisting of those $\sigma\colon U \to \hom_{\mathbf{Diff}}(X, Y)$ such that for every plot $\tau\colon V \to X$ of X, the composite

$$U \times V \xrightarrow{(\sigma,\tau)} \hom_{\mathbf{Diff}}(X, Y) \times X \xrightarrow{\mathrm{ev}} Y$$

is a plot of Y. Putting it differently, $D_{X,Y}$ is the coarsest diffeology such that the evaluation map $\mathrm{ev} : \hom_{\mathbf{Diff}}(\mathrm{X}, \mathrm{Y}) \times \mathrm{X} \to \mathrm{Y}$ is smooth.

Let us denote by $\mathrm{C}^\infty(X, Y)$ the diffeological space $(\hom_{\mathbf{Diff}}(X, Y), D_{X,Y})$. Then there is a natural map $\alpha\colon \mathrm{C}^\infty(X \times Y, Z) \to \mathrm{C}^\infty(X, \mathrm{C}^\infty(Y, Z))$ given by the formula: $\alpha(f)(x)(y) = f(x, y)$ for $x \in X$ and $y \in Y$. The following exponential law implies the cartesian closedness of **Diff**.

Theorem 2.1.2. $[I, 1.60]$ *The map α induces a smooth isomorphism*

$$\mathrm{C}^\infty(X \times Y, Z) \cong \mathrm{C}^\infty(X, \mathrm{C}^\infty(Y, Z)).$$

2.2 Numerically generated spaces

Given a topological space X, let DX be the diffeological space with the same underlying set as X and with all continuous maps from open subsets of Euclidean spaces into X as plots. Clearly, a continuous map $f\colon X \to Y$ induces a smooth map $DX \to DY$. Hence there is a functor $D\colon\ \mathbf{Top} \to \mathbf{Diff}$ which maps a topological space X to the diffeological space DX.

On the contrary, any diffeological space X determines a topological space TX having the same underlying set as X and is equipped with the final topology with respect to the plots of X. Any smooth map $f\colon X \to Y$ induces a continuous map $TX \to TY$, hence we have a functor $T\colon\ \mathbf{Diff} \to \mathbf{Top}$.

Proposition 2.2.1. *The functor T is a left adjoint to D.*

Proof. Let X be a diffeological space and Y a topological space. Then a map $f\colon TX \to Y$ is continuous if and only if the composite $f \circ \sigma$ is continuous for every plot σ of X. But this is equivalent to say that $f\colon X \to DY$ is smooth. Thus the natural map

$$\hom_{\mathbf{Top}}(TX, Y) \to \hom_{\mathbf{Diff}}(X, DY)$$

is bijective for every $X \in \mathbf{Diff}$ and $Y \in \mathbf{Top}$. □

Proposition 2.2.2. *A topological space X is numerically generated if and only if the counit of the adjunction $TDX \to X$ is a homeomorphism.*

Proof. The condition $TDX = X$ holds if and only if X has the final topology with respect to all the continuous maps from an open subset of a Euclidean space into X. But this is equivalent to say that X has the final topology with respect to the singular simplexes of X. □

Let us write $\nu = TD$, so that X is numerically generated if and only if $\nu X = X$ holds.

Lemma 2.2.3. *For any topological space X we have $\nu(\nu X) = \nu X$.*

Proof. Every plot $\sigma\colon U \to X$ of X lifts to a plot of νX, since $\nu U = U$ holds for any open subset U of $\mathbf{R}^n$. Thus νX has the same plots as X, and hence $\nu(\nu X)$ has the same topology as νX. $\square$

Let $\mathbf{NG}$ be the category of numerically generated spaces and continuous maps. It follows that $\mathbf{NG}$ is reflective in $\mathbf{Top}$, and the correspondence $X \mapsto \nu X$ induces a reflector $\nu\colon \mathbf{Top} \to \mathbf{NG}$.

Proposition 2.2.4. *The category $\mathbf{NG}$ is complete and cocomplete. For every small diagram $F\colon J \to \mathbf{NG}$, we have*

$$\lim_J F \cong T(\lim_J DF) \cong \nu(\lim_J IF)$$

$$\operatorname{colim}_J F \cong T(\operatorname{colim}_J DF) \cong \operatorname{colim}_J IF$$

where I denotes the inclusion functor $\mathbf{NG} \to \mathbf{Top}$.

Proof. Since $\mathbf{Diff}$ is complete, the diagram $DF\colon J \to \mathbf{Diff}$ has a limiting cone $\{\phi_j\colon \lim_J DF \to DF(j)\}$. We shall show that the cone

$$\{T\phi_j\colon T(\lim_J DF) \to TDF(j) = F(j)\}$$

is a limiting cone to F. Let $\{\psi_j\colon X \to F(j)\}$ be an arbitrary cone to F. Then $\{D\psi_j\colon DX \to DF(j)\}$ is a cone to DF, and hence there is a unique morphism $u\colon DX \to \lim_J DF$ such that $D\psi_j = \phi_j \circ u$ holds. But then $Tu\colon X = TDX \to T(\lim_J DF)$ is a unique morphism such that $\psi_j = T\phi_j \circ Tu$

holds. Hence $\{T\phi_j\}$ is a limiting cone to F. Since the right adjoint functor D preserves limits, we have $T(\lim_J DF) \cong TD(\lim_J IF) = \nu(\lim_J IF)$.

Similar argument shows that $T(\operatorname{colim}_J DF)$ is a colimit of F. But in this case we have $T(\operatorname{colim}_J DF) \cong \operatorname{colim}_J \nu F = \operatorname{colim}_J IF$, since the left adjoint functor T preserves colimits. $\square$

2.3 Exponentials in NG

A map $f\colon X \to Y$ between topological spaces is said to be numerically continuous if the composite $f \circ \sigma\colon \Delta^n \to Y$ is continuous for every singular simplex $\sigma\colon \Delta^n \to X$. Clearly, we have the following.

Proposition 2.3.1. *Let $f\colon X \to Y$ be a map between topological spaces. Then the following conditions are equivalent:*

1. *$f\colon X \to Y$ is numerically continuous.*
2. *$f \circ \sigma\colon U \to Y$ is continuous for any continuous map $\sigma\colon U \to X$ from an open subset U of a Euclidean space into X.*
3. *$f\colon \nu X \to Y$ is continuous.*
4. *$f\colon DX \to DY$ is smooth.*

Let us denote by $\mathbf{map}(X, Y)$ the set of continuous maps from X to Y equipped with compact-open topology, and let $\mathbf{smap}(X, Y)$ be the set of numerically continuous maps equipped with the initial topology with respect to the maps

$$\sigma^*\colon \mathbf{smap}(X, Y) \to \mathbf{map}(\Delta^n, Y), \quad \sigma^*(f) = f \circ \sigma,$$

where $\sigma\colon \Delta^n \to X$ runs through singular simplexes of X. More explicitly, the space $\mathbf{map}(X, Y)$ has a subbase consisting of those subsets

$$W(K, U) = \{f \mid f(K) \subset U\},$$

where K is a compact subset of X and U is an open subset of Y. On the other hand, $\mathbf{smap}(X, Y)$ has a subbase consisting of those subsets

$$W(\sigma, L, U) = \{f \mid f(\sigma(L)) \subset U\},$$

where $\sigma\colon \Delta^n \to X$ is a singular simplex, L a compact subset of Δ^n, and U an open subset of Y. As we have

$$W(\sigma, L, U) \cap \mathbf{map}(X, Y) = W(\sigma(L), U),$$

the inclusion map $\mathbf{map}(X, Y) \to \mathbf{smap}(X, Y)$ is continuous.

Proposition 2.3.2. *The inclusion map* $\mathbf{map}(X, Y) \to \mathbf{smap}(X, Y)$ *is bijective for all* Y *if and only if* X *is numerically generated.*

Proof. If X is numerically generated then a numerically continuous map $f\colon X \to Y$ is automatically continuous. Hence $\mathbf{map}(X, Y) \to \mathbf{smap}(X, Y)$ is surjective for all Y. Conversely, if $\mathbf{map}(X, \nu X) \to \mathbf{smap}(X, \nu X)$ is surjective, then the unit of the adjunction $X \to \nu X$ is continuous, implying that X is numerically generated. □

Proposition 2.3.3. *Suppose that* X *is a CW-complex. Then the inclusion map* : $\mathbf{map}(X, Y) \to \mathbf{smap}(X, Y)$ *is a homeomorphism for any* Y.

Proof. Since X has weak topology with respect to the family of closed cells, a map f from X to Y is continuous if and only if it is numerically continuous.

Hence $\mathbf{map}(X,Y) \to \mathbf{smap}(X,Y)$ is a bijection. To prove the continuity of its inverse, we have to show that every subset of the form $W(K,U)$ is open in $\mathbf{smap}(X,Y)$. Since X is closure-finite, K is contained in a finite complex, say A. Let $\{e_1,\dots,e_k\}$ be the set of cells of A, and let $L_i = \psi_i^{-1}(\overline{e_i} \cap K) \subset \Delta^{n_i}$, where $\psi_i \colon \Delta^{n_i} \to X$ is a characteristic map for e_i. Then we have

$$W(K,U) = W(\psi_1, L_1, U) \cap \cdots \cap W(\psi_k, L_k, U).$$

Hence $W(K,U)$ is open in $\mathbf{smap}(X,Y)$. $\square$

The proposition above implies that any CW-complex X is numerically generated. Thus we have the following.

Corollary 2.3.4. *The category* **NG** *contains all CW-complexes.*

For numerically generated spaces X and Y, let us denote

$$Y^X = \nu\, \mathbf{smap}(X,Y).$$

Then there is a map $\alpha \colon Z^{X\times Y} \to (Z^Y)^X$ which assigns to $f \colon X \times Y \to Z$ the map $\alpha(f) \colon X \to Z^Y$ given by the formula $\alpha(f)(x)(y) = f(x,y)$ for $x \in X$ and $y \in Y$.

Theorem 2.3.5. *The natural map $\alpha \colon Z^{X\times Y} \to (Z^Y)^X$ is a homeomorphism.*

This clearly implies the following.

Corollary 2.3.6. *The category* **NG** *is a cartesian closed category.*

To prove the theorem, we use the relationship between Y^X and the exponentials in **Diff**.

Proposition 2.3.7. *For any topological spaces X and Y, we have*

$$D\,\mathbf{smap}(X,Y) = \mathrm{C}^\infty(DX, DY).$$

Proof. Let $\sigma\colon U \to \mathbf{smap}(X,Y)$ be a map from an open subset $U \subset \mathbf{R}^n$. Then σ is a plot of $D\,\mathbf{smap}(X,Y)$ if and only if the composite

$$U \xrightarrow{\sigma} \mathbf{smap}(X,Y) \xrightarrow{\tau^*} \mathbf{map}(\Delta^m, Y)$$

is continuous for every singular simplex $\tau\colon \Delta^m \to X$. But $\tau^*\sigma$ corresponds to the the composite

$$U \times \Delta^m \xrightarrow{\sigma\times\tau} \mathbf{smap}(X,Y) \times X \xrightarrow{\mathrm{ev}} Y,$$

under the homeomorphism

$$\mathbf{map}(U \times \Delta^m, Y) \cong \mathbf{map}(U, \mathbf{map}(\Delta^m, Y)).$$

Thus σ is a plot of $D\,\mathbf{smap}(X,Y)$ if and only if $\mathrm{ev}(\sigma,\tau)$ is continuous for every τ. which is equivalent to say that σ is a plot of $\mathrm{C}^\infty(DX, DY)$. □

We are now ready to prove Theorem 2.3.5.

Proof of Theorem 2.3.5. The map α is a homeomorphism since it coincides with the composite of homeomorphisms

$$\begin{aligned}
Z^{X\times Y} &= \nu\,\mathbf{smap}(X\times Y, Z)\\
&= T\,\mathrm{C}^\infty(D(X\times Y), DZ)\\
&= T\,\mathrm{C}^\infty(DX\times DY, DZ)\\
&\cong T\,\mathrm{C}^\infty(DX, \mathrm{C}^\infty(DY, DZ)) && (1)\\
&= T\,\mathrm{C}^\infty(DX, D\,\mathbf{smap}(Y, Z))\\
&= \nu\,\mathbf{smap}(X, \mathbf{smap}(Y, Z))\\
&\cong \nu\,\mathbf{smap}(X, \nu\,\mathbf{smap}(Y, Z)) = (Z^Y)^X && (2)
\end{aligned}$$

in which (1) follows from Theorem 2.1.2, and (2) is induced by the numerical isomorphism $\mathbf{smap}(Y, Z) \to \nu\,\mathbf{smap}(Y, Z)$. $\square$

2.4 The space of basepoint preserving maps

Let $\mathbf{Top}_0$ and $\mathbf{Diff}_0$ be the categories of pointed objects in **Top** and **Diff**, respectively. Evidently, the adjunction (T, D) between **Top** and **Diff** induces an adjunction (T_0, D_0) between $\mathbf{Top}_0$ and $\mathbf{Diff}_0$. Thus the category $\mathbf{NG}_0$ of pointed objects in **NG** can be identified with a full subcategory of $\mathbf{Top}_0$ consisting of those pointed spaces (X, x_0) such that $\nu X = X$ holds. Clearly, $\mathbf{NG}_0$ is complete and cocomplete.

Given pointed spaces X and Y, let $\mathbf{map}_0(X, Y)$ and $\mathbf{smap}_0(X, Y)$ denote, respectively, the subspace of $\mathbf{map}(X, Y)$ and $\mathbf{smap}(X, Y)$ consisting of basepoint preserving maps. Then Proposition 2.3.3 implies the following.

Proposition 2.4.1. *If X is a pointed CW-complex, then the inclusion map $\mathbf{map}_0(X, Y) \to \mathbf{smap}_0(X, Y)$ is a homeomorphism for any pointed space Y.*

As before Section 2.3, let us denote $Y^X = \nu\,\mathbf{smap}_0(X,Y)$. By taking the constant map as basepoint, Y^X is regarded as an object of $\mathbf{NG}_0$. Recall that the smash product $X \wedge Y$ of pointed spaces $X = (X, x_0)$ and $Y = (Y, y_0)$ is defined to be the quotient of $X \times Y$ by its subspace $X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y$. We now define a pointed map

$$\alpha_0 \colon Z^{X \wedge Y} \to (Z^Y)^X$$

to be the composite $Z^{X\wedge Y} \xrightarrow{p^*} (Z, z_0)^{(X\times Y, X\vee Y)} \xrightarrow{\alpha'} (Z^Y)^X$, where the middle term $(Z, z_0)^{(X\times Y, X\vee Y)}$ denotes the subspace of $Z^{X\times Y}$ consisting of those maps $f \colon X\times Y \to Z$ such that $f(X\vee Y) = \{z_0\}$ holds, p^* is induced by the natural map $p \colon X \times Y \to X \wedge Y$, and α' is the restriction of the homeomorphism

$$\nu\,\mathbf{smap}(X \times Y, Z) \cong \nu\,\mathbf{smap}(X, \nu\,\mathbf{smap}(Y, Z)).$$

Since $p \colon X\times Y \to X\wedge Y$ is universal among continuous maps $f \colon X\times Y \to Z$ satisfying $f(X\vee Y) = \{z_0\}$, the induced map p^* is bijective, whence so is α_0.

Proposition 2.4.2. *The map $\alpha_0 \colon Z^{X\wedge Y} \to (Z^Y)^X$ is a homeomorphism for any $X,\ Y,\ Z \in \mathbf{NG}_0$.*

Proof. Given pairs of topological spaces (X, A), (Y, B), we have

$$(Y, B)^{(X,A)} = T\,\mathrm{C}^\infty((DX, DA), (DY, DB)),$$

where $\mathrm{C}^\infty((DX, DA), (DY, DB))$ is the subspace of $\mathrm{C}^\infty(DX, DY)$ consisting of those smooth maps $f \colon DX \to DY$ satisfying $f(DA) \subset DB$. Thus, to prove that α_0 is a homeomorphism, we need only show that for any pointed diffeological spaces A, B and C, the bijection

$$p^* \colon\ \mathrm{C}^\infty((A\wedge B, *), (C, c_0)) \to \mathrm{C}^\infty((A\times B, A\vee B), (C, c_0))$$

induced by the natural map $p\colon A \times B \to A \wedge B = A \times B/A \vee B$ is a smooth isomorphism. Here $* = p(A \vee B)$ is a basepoint of $A \wedge B$, and c_0 is a basepoint of C. To see that $(p^*)^{-1}$ is smooth, let us take a plot $\sigma\colon U \to \mathrm{C}^\infty((A \times B, A \wedge B), (C, c_0))$ and show that $\widetilde{\sigma} = (p^*)^{-1} \cdot \sigma$ is a plot of $\mathrm{C}^\infty((A \wedge B, *), (C, c_0))$. By definition, this is the case if for any plot $\tau\colon V \to A \wedge B$ the composite

$$U \times V \xrightarrow{\widetilde{\sigma} \times \tau} \mathrm{C}^\infty((A \wedge B, *), (C, c_0)) \times (A \wedge B) \xrightarrow{\mathrm{ev}} C$$

is a plot of C. But for any $v \in V$ there exist a neighborhood W of v and a plot $\widetilde{\tau}\colon W \to A \times B$ such that $p\widetilde{\tau} = \tau|W$ hold. Therefore, the composite $\mathrm{ev}(\widetilde{\sigma} \times \tau)$ coincides, on $U \times W$, with the plot

$$U \times W \xrightarrow{\widetilde{\sigma} \times \widetilde{\tau}} \mathrm{C}^\infty((A \times B, A \vee B), (C, c_0)) \times (A \times B) \xrightarrow{\mathrm{ev}} C.$$

This implies that $\mathrm{ev}(\widetilde{\sigma} \times \tau)$ is locally, hence globally, a plot of C. Thus $\widetilde{\sigma} = (p^*)^{-1} \cdot \sigma$ is a plot of $\mathrm{C}^\infty((A \wedge B, *), (C, c_0))$. □

The proposition implies a natural bijection

$$\hom_{\mathbf{NG}_0}(X \wedge Y, Z) \cong \hom_{\mathbf{NG}_0}(X, Z^Y).$$

Hence we have the following.

Corollary 2.4.3. *The category $\mathbf{NG}_0$ is a symmetric monoidal closed category with tensor product $\wedge$ and internal hom of the form Z^Y.*

Proposition 2.4.4. (1) *For every pointed space X, the counit of the adjunction $\varepsilon\colon \nu X \to X$ is a weak homotopy equivalence.*

(2) *If $X \in \mathbf{NG}_0$, then the bijection $\iota\colon \mathbf{map}_0(X, Y) \to \mathbf{smap}_0(X, Y)$ is a weak homotopy equivalence.*

Proof. (1) Since S^n is a CW-complex, we have

$$\pi_n(X, x) = \pi_0\,\mathbf{map}((S^n, e), (X, x)) \cong \pi_0\,\mathbf{smap}((S^n, e), (X, x))$$

for every $x \in X$. Therefore, the numerically continuous map $\eta\colon X \to \nu X$ induces the inverse to $\varepsilon_*\colon \pi_n(\nu X, x) \to \pi_n(X, x)$.

(2) We have a commutative diagram

$$\begin{tikzcd}
\mathbf{map}_0(S^n, \mathbf{map}_0(X, Y)) \arrow[r, "\iota_*"] & \mathbf{map}_0(S^n, \mathbf{smap}_0(X, Y)) \arrow[d, equal] \\
 & \mathbf{smap}_0(S^n, \mathbf{smap}_0(X, Y)) \\
\mathbf{map}(S^n \wedge X, Y) \arrow[uu, "\alpha_0"] \arrow[r, "\cong"] & \mathbf{smap}(S^n \wedge X, Y), \arrow[u, "\cong"']
\end{tikzcd}$$

which shows that

$$\iota_*\colon\ \mathbf{map}_0(S^n, \mathbf{map}_0(X, Y)) \to \mathbf{map}_0(S^n, \mathbf{smap}_0(X, Y))$$

is bijective. Thus $\mathbf{smap}_0(X, Y)$ has the same n-loops as $\mathbf{map}_0(X, Y)$. Moreover, a similar diagram as above, but S^n replaced by $I_+ \wedge S^n$, shows that $\mathbf{smap}_0(X, Y)$ has the same homotopy classes of n-loops as $\mathbf{map}_0(X, Y)$. It follows that

$$\iota_*\colon \pi_n(\mathbf{map}_0(X, Y), f) \cong \pi_n(\mathbf{smap}_0(X, Y), f)$$

is an isomorphism for every $f \in \mathbf{map}_0(X, Y)$ and $n \geq 0$. $\square$

Corollary 2.4.5. *For all X, $Y \in \mathbf{NG}_0$, the space Y^X is weakly equivalent to the space of maps $\mathbf{map}_0(X, Y)$ equipped the compact-open topology.*

2.5 Homology theories via enriched functors

Let $\mathbf{C}_0$ be a full subcategory of $\mathbf{NG}_0$. Then $\mathbf{C}_0$ is an enriched category over $\mathbf{NG}_0$ with hom-objects $F_0(X,Y) = Y^X$. A covariant functor T from $\mathbf{C}_0$ to $\mathbf{NG}_0$ is called enriched if the correspondence $F_0(X,Y) \to F_0(TX,TY)$, which maps f to Tf, is a morphism in $\mathbf{NG}_0$, that is, a basepoint preserving continuous map. Similarly, a contravariant functor T from $\mathbf{C}_0$ to $\mathbf{NG}_0$ is called enriched if the map $F_0(X,Y) \to F_0(TY,TX)$ is a morphisms in $\mathbf{NG}_0$.

Let $T\colon \mathbf{C}_0 \to \mathbf{NG}_0$ be an enriched functor. Then for any pointed map $f\colon X \wedge Y \to Z$ such that Y and Z are objects of $\mathbf{C}_0$ the composite

$$X \xrightarrow{\alpha_0(f)} F_0(Y,Z) \xrightarrow{T} F_0(TY,TZ)$$

induces, by adjunction, a pointed map $X \wedge TY \to TZ$. Similarly, an enriched cofunctor, $T\colon \mathbf{C}_0^{\mathrm{op}} \to \mathbf{NG}_0$ assigns $X \wedge TZ \to TY$ as an adjunct to the composite $X \to F_0(Y,Z) \to F_0(TZ,TY)$.

Proposition 2.5.1. *Enriched functors and cofunctors from $\mathbf{C}_0$ to $\mathbf{NG}_0$ preserve homotopies.*

Proof. Let $h\colon I_+ \wedge X \to Y$ be a pointed homotopy between h_0 and h_1. Then an enriched functor $T\colon \mathbf{C}_0 \to \mathbf{NG}_0$ induces a homotopy $I_+ \wedge TX \to TY$ between Th_0 and Th_1. Similarly, an enriched cofunctor T induces a homotopy $I_+ \wedge TY \to TX$ between Th_0 and Th_1. □

Corollary 2.5.2. *If $T\colon \mathbf{C}_0 \to \mathbf{NG}_0$ is an enriched functor then a homotopy equivalence $f\colon X \to Y$ induces isomorphisms $Tf_*\colon \pi_n TX \cong \pi_n TY$ for $n \geq 0$. Similarly, if T is an enriched cofunctor then f induces isomorphisms $Tf_*\colon \pi_n TY \cong \pi_n TX$ for $n \geq 0$.*

From now on, we assume that $\mathbf{C}_0$ satisfies the following conditions: (i) $\mathbf{C}_0$ contains all finite CW-complexes. (ii) $\mathbf{C}_0$ is closed under finite wedge sum. (iii) If $A \subset X$ is an inclusion of objects in $\mathbf{C}_0$ then its cofiber $X \cup CA$, belongs to $\mathbf{C}_0$; in particular, $\mathbf{C}_0$ is closed under the suspension functor $X \mapsto \Sigma X$. The category $\mathbf{FCW}_0$ of finite CW-complexes is a typical example of such a category.

Given a continuous map $f\colon X \to Y$, let

$$E(f) = \{(x, l) \in X \times \mathbf{map}(I, Y) \mid f(x) = l(0)\}$$

be the mapping track of f. Then the map $p\colon E(f) \to Y$, $p(x, l) = l(1)$, has the homotopy lifting property for all spaces, and hence induces a bijection $p_*\colon \pi_{n+1}(E(f), F(f)) \to \pi_{n+1}Y$ for all $n \geq 0$, where $F(f)$ denotes the fiber of p at the basepoint of Y. A sequence of pointed maps $Z \xrightarrow{i} X \xrightarrow{f} Y$ is called a homotopy fibration sequence if there is a homotopy of pointed maps from $f \circ i$ to the constant map such that the induced map $Z \to F(f)$ is a weak homotopy equivalence.

Definition 2.5.3. *An enriched functor $T\colon\ \mathbf{C}_0 \to \mathbf{NG}_0$ is called linear if for every pair of objects (X, A) with $A \subset X$, the sequence*

$$TA \to TX \to T(X \cup CA),$$

induced by the cofibration sequence $A \subset X \subset X \cup CA$, is a homotopy fibration sequence with respect to the null homotopy of $TA \to T(X \cup CA)$ coming from the contraction of A within the reduced cone CA. Likewise, an enriched cofunctor $T\colon\ \mathbf{C}_0^{\mathrm{op}} \to \mathbf{NG}_0$ is called linear if the induced sequence

$$T(X \cup CA) \to TX \to TA$$

is a homotopy fibration sequence.

If T is a linear functor, then every pair (X, A) gives rise to an exact sequence of pointed sets

$$\cdots \to \pi_{n+1}T(X \cup CA) \xrightarrow{\Delta} \pi_n TA \xrightarrow{Ti_*} \pi_n TX \xrightarrow{Tj_*} \pi_n T(X \cup CA) \to \cdots$$

terminated at $\pi_0 T(X \cup CA)$. Here i and j denote the inclusions $A \subset X$ and $X \subset X \cup CA$, respectively, and Δ is the composite

$$\pi_{n+1}T(X \cup CA) \xrightarrow{p_*^{-1}} \pi_{n+1}(E(Tj), F(Tj)) \xrightarrow{\partial} \pi_n F(Tj) \xrightarrow{\nu_*^{-1}} \pi_n TA.$$

Similarly, a linear cofunctor T induces an exact sequence

$$\cdots \to \pi_{n+1}TA \xrightarrow{\Delta} \pi_n T(X \cup CA) \xrightarrow{Tj_*} \pi_n TX \xrightarrow{Ti_*} \pi_n TA \to \cdots$$

terminated at $\pi_0 TA$.

Theorem 2.5.4. *For every linear functor $T\colon \mathbf{C}_0 \to \mathbf{NG}_0$, there exists a generalized homology theory $X \mapsto \{h_n(X;T)\}$ defined on $\mathbf{C}_0$ such that $h_n(X;T)$ is isomorphic to $\pi_n TX$ if $n \geq 0$, and to $\pi_0 T(\Sigma^{-n}X)$ otherwise.*

Proof. We first show that the map $T(X \vee Y) \to TX \times TY$ induced by the projections of $X \vee Y$ onto X and Y is a weak equivalence. This means that the functor $\Gamma \to \mathbf{NG}_0$, which maps a pointed finite set $\mathbf{k} = \{0, 1, \ldots, k\}$ to $T(X \wedge \mathbf{k}) = T(X \vee \cdots \vee X)$, is special in the sense that the natural map $T(X \wedge \mathbf{k}) \to T(X)^k$ is a weak equivalence for all $k \geq 0$. Hence $\pi_n TX$ is an abelian monoid with respect to the multiplication

$$\pi_n TX \times \pi_n TX \cong \pi_n T(X \vee X) \xrightarrow{T\nabla_*} \pi_n TX$$

induced by the folding map $\nabla\colon X \vee X \to X$. This multiplication coincides with the standard multiplication of $\pi_n TX$ since they are compatible with each other. In particular, $\pi_1 TX$ is an abelian group for all X. Moreover, any pointed map $f\colon X \to Y$ induces a natural transformation $T(X\wedge \mathbf{k}) \to T(Y \wedge \mathbf{k})$, whence it is a homomorphism of abelian monoids $Tf_*\colon \pi_n TX \to \pi_n TY$ for all $n \geq 0$.

To see that $T(X \vee Y) \to TX \times TY$ is a weak equivalence, it suffices to show that the sequence $TX \to T(X \vee Y) \to TY$, induced by the inclusion $X \to X\vee Y$ and the projection $X\vee Y \to Y$, is a homotopy fibration sequence. But $TX \to T(X \vee Y) \to TY$ is homotopy equivalent to the homotopy fibration sequence $TX \to T(X \vee Y) \to T(CX \vee Y)$ through the homotopy equivalence $T(CX \vee Y) \simeq TY$ induced by the retraction $CX \vee Y \to Y$.

Next consider the homotopy exact sequence associated with the sequence $TX \to T(CX) \to T(\Sigma X)$. As $T(CX)$ is weakly contractible, we obtain for every $n \geq 0$ a short exact sequence of abelian monoids

$$0 \to \pi_{n+1}T(\Sigma X) \xrightarrow{\varDelta} \pi_n TX \to 0.$$

Since $\pi_{n+1}T(\Sigma X)$ is an abelian group, the homomorphism $\varDelta$ is injective, whence it is an isomorphism. But this in turn means that $\pi_n TX$ is an abelian group for $n \geq 0$. Therefore, $h_n(X;T)$ is an abelian group for all $n \in \mathbf{Z}$.

For a pointed map $f\colon X \to Y$, we define

$$h_n(f)\colon h_n(X;T) \to h_n(Y;T)$$

to be the homomorphism induced by $Tf\colon TX \to TY$ for $n \geq 0$, and $T(\Sigma^{-n}f)$ for $n < 0$. It is easy to see that the functor $X \mapsto \{h_n(X;T)\}$ together with a natural isomorphism $h_n(X;T) \cong h_{n+1}(\Sigma X;T)$, given by Δ^{-1} for $n \geq 0$ and the identity for $n < 0$, satisfies the homotopy and exactness axioms. $\square$

If $T\colon \mathbf{C}_0 \to \mathbf{NG}_0$ is an enriched functor, then for every $X \in \mathbf{C}_0$ the spaces $T(\Sigma^n X)$ together with the maps $S^1 \wedge T(\Sigma^n X) \to T(\Sigma^{n+1} X)$ induced by the homeomorphism $S^1 \wedge \Sigma^n X \cong \Sigma^{n+1} X$ form a prespectrum ∂TX. Following Goodwillie [G], we call ∂TX the derivative of T at X. Let $L(\partial TX)$ be the spectrification of ∂TX (cf. [L]). Then its zeroth space $L(\partial TX)_0$ is an infinite loop space and the correspondence $X \mapsto L(\partial TX)_0$ defines an enriched functor $LT\colon \mathbf{C}_0 \to \mathbf{NG}_0$. If, moreover, T is linear, then Theorem 2.5.4 implies that the natural map $TX \to LTX$ is a weak equivalence for every X; hence LT defines the same homology theory as T.

An enriched functor T is called stable if the natural map $TX \to LTX$ is a homeomorphism for every X. In particular, LT is stable for any T. Let **SLEF** be the category of stable linear enriched functors $\mathbf{C}_0 \to \mathbf{NG}_0$ with enriched natural transformations as morphisms. Then there is a functor D from **SLEF** to the category **Spec** of spectra which maps a stable functor T to its derivative ∂TS^0 at S^0. We have shown that the homology theory $h_\bullet(-;T)$ induced by a linear enriched functor T is represented by the spectrum $D(LT) = L(\partial TS^0)$.

Conversely, any homology theory represented by a spectrum comes from a linear enriched functor. In fact, D has a left adjoint $I\colon$ **Spec** $\to$ **SLEF** defined as follows: For a spectrum $E = \{E_n\}$, IE is the enriched functor which maps X to the zeroth space $L(E \wedge X)_0$ of the spectrification of the prespectrum $E \wedge X = \{E_n \wedge X\}$. The unit $E \to DIE$ and the counit

$IDT \to T$ of the adjunction are weak equivalences given by the maps

$$E_n \to \Omega^n(E_n \wedge S^n) \to \Omega^\infty(E_\infty \wedge S^n) = L(E \wedge S^n)_0 = IE(S^n) = DIE_n,$$
$$IDTX = L(\partial TS^0 \wedge X)_0 \xrightarrow{L\mu} L(\partial TX)_0 = LTX \cong TX,$$

where $\mu\colon \partial TS^0 \wedge X \to \partial TX$ is a map of prespectra consisting of the maps $T(S^n) \wedge X \to T(\Sigma^n X)$ induced by the identity of $S^n \wedge X$.

Let us regard **Spec** and **SLEF** as a model category with respect to the classes of weak equivalences, fibrations, and cofibrations consist, respectively, of level weak equivalences, level fibrations, and morphisms that have the left lifting property with respect to the class of trivial fibrations. Then we have the following.

Proposition 2.5.5. *The functor D: **SLEF** $\to$ **Spec** is a right Quillen equivalence, and hence induces an equivalence between the homotopy categories.*

Corollary 2.5.6. *The homotopy category of **SLEF** is equivalent to the stable category.*

2.6 Bivariant homology-cohomology theories

We now introduce the notion of a bilinear functor, and describe a passage from bilinear functors to generalized cohomology theories. In fact, we shall show that a bilinear functor gives rise to a pair of generalized homology and cohomology theories, or in other words, a bivariant homology-cohomology theory.

Let $F\colon \mathbf{C}_0{}^{\mathrm{op}} \times \mathbf{C}_0 \to \mathbf{NG}_0$ be a bivariant functor which is contravariant with respect to the first argument, and is covariant with respect to the second argument. We say that F is enriched (over $\mathbf{NG}_0$) if for all pointed spaces X, X', Y, and Y', the map

$$F_0(X', X) \times F_0(Y, Y') \to F_0(F(X, Y), F(X', Y')), \quad (f, g) \mapsto F(f, g)$$

is continuous and is pointed in the sense that if either f or g is constant then so is $F(f, g)$.

Definition 2.6.1. *An enriched bifunctor $F\colon {\mathbf{C}_0}^{\mathrm{op}} \times \mathbf{C}_0 \to \mathbf{NG}_0$ is called a bilinear functor if for all (X, A) and (Y, B) the sequences*

$$F(X \cup CA, Y) \to F(X, Y) \to F(A, Y),$$
$$F(X, B) \to F(X, Y) \to F(X, Y \cup CB)$$

are homotopy fibration sequences.

Theorem 2.6.2. *For every bilinear functor $F\colon {\mathbf{C}_0}^{\mathrm{op}} \times \mathbf{C}_0 \to \mathbf{NG}_0$, there exist a generalized homology theory $X \mapsto \{h_n(X; F)\}$ and a generalized cohomology theory $X \mapsto \{h^n(X; F)\}$ such that*

$$h_n(X; F) \cong \pi_0 F(S^{n+k}, \Sigma^k X), \quad h^n(X; F) \cong \pi_0 F(\Sigma^k X, S^{n+k}) \tag{2.1}$$

hold whenever k, $n + k \geq 0$. Moreover, $h_n(X; F)$ is naturally isomorphic to the n-th homology group $h_n(X; T)$ given by the covariant part T of F.

Proof. Since $F(X, Y)$ is linear with respect to Y, $\pi_n F(X, Y)$ is an abelian group for all X, Y and $n \geq 0$. Clearly, this abelian group structure is natural with respect to both X and Y. Moreover, the bilinearity of F implies natural isomorphisms

$$\pi_n F(X, Y) \cong \pi_{n+1} F(X, \Sigma Y), \quad \pi_n F(\Sigma X, Y) \cong \pi_{n+1} F(X, Y)$$

Consequently, there is a natural isomorphism $\pi_0 F(X,Y) \cong \pi_0 F(\Sigma X, \Sigma Y)$, called the suspension isomorphism.

For every pointed space X and every integer n, let us define

$$h_n(X;F) = \operatorname*{colim}_{k\to\infty} \pi_0 F(S^{n+k}, \Sigma^k X),\ \ h^n(X;F) = \operatorname*{colim}_{k\to\infty} \pi_0 F(\Sigma^k X, S^{n+k})$$

where the colimits are taken with respect to the suspension isomorphisms. Clearly (2.1) holds, and we have $h_n(X;F) \cong h_n(X;T)$ where T is the covariant part of F. Thus the functor $X \mapsto \{h_n(X;F)\}$ together with the evident natural isomorphism $h_n(X;F) \cong h_{n+1}(\Sigma X;F)$ defines a generalized homology theory. Similarly, the covariant functor $X \mapsto \{h^n(X;F)\}$ together with the natural isomorphism $h^{n+1}(\Sigma X;F) \cong h^n(X;F)$ defines a generalized cohomology theory, since it satisfies the homotopy and exactness axioms. $\square$

Proposition 2.6.3. *([SYH]) If X is a CW-complex, we have $h_n(X,F) = H_n(X,\mathbb{S})$ and $h^n(X,F) = H^n(X,\mathbb{S})$, the generalized homology and cohomology groups with coefficients in the spectrum $\mathbb{S} = \{F(S^0,S^n) \mid n \geq 0\}$.*

Chapter 3

An enriched bifunctor representing the Čech cohomology group

3.1 The Čech cohomology and the Steenrod homology

We recall that the Čech cohomology group of X with coefficient group G is defined to be the colimit of the singlular cohomology groups

$$\check{H}^n(X, G) = \varinjlim_{\lambda} H^n(X_\lambda^{\check{C}}, G),$$

where λ runs through coverings of X and $X_\lambda^{\check{C}}$ is the Čech nerve corresponding to λ, i.e., $v \in X_\lambda^{\check{C}}$ is a vertex of $X_\lambda^{\check{C}}$ corresponding to an open set $V \in \lambda$. On the other hand, the Steenrod homology group of a compact metric space X

is defined as follows. As X is a compact metric space, there is a sequence $\{\lambda_i\}_{i\geq 0}$ of finite open covering of X such that $\lambda_0 = \{X\}$, λ_i is a refinement of λ_{i-1}, and X is the inverse limit $\varprojlim_i X^{\check{\mathrm{C}}}_{\lambda_i}$. According to [F], the Steenrod homology group of X with coefficient in a spectrum $\mathbb{S}$ is defined to be the group

$$H^{st}_n(X, \mathbb{S}) = \pi_n \underleftarrow{\mathrm{holim}}_{\lambda_i}(X^{\check{\mathrm{C}}}_{\lambda_i} \wedge \mathbb{S}),$$

where $\underleftarrow{\mathrm{holim}}$ denotes the homotopy inverse limit. (See also [KKS] for the definition without using subdivisions.)

3.2 Proof of Theorem 1

Let T be a linear enriched functor. We define a bifunctor $\check{\mathrm{F}} : \mathbf{NG_0^{\mathrm{op}}} \times \mathbf{NGC_0} \to \mathbf{NG_0}$ as follows. For $X \in \mathbf{NG_0}$ and $Y \in \mathbf{NGC}_0$, we put

$$\check{\mathrm{F}}(X, Y) = \varinjlim_{\lambda} \mathbf{map}_0(X_\lambda,\ \underleftarrow{\mathrm{holim}}_{\mu_i}\, T(Y^{\check{\mathrm{C}}}_{\mu_i})),$$

where λ is an open covering of X and $\{\mu_i\}_{i\geq 0}$ is a set of finite open coverings of Y such that $\mu_0 = \{Y\}$, μ_i is a refinement of μ_{i-1}, and Y is the inverse limit $\varprojlim_i Y^{\check{\mathrm{C}}}_{\mu_i}$.

Given based maps $f : X \to X'$ and $g : Y \to Y'$, we define a map

$$\check{\mathrm{F}}(f, g) \in \mathbf{map}_0(\check{\mathrm{F}}(X', Y), \check{\mathrm{F}}(X, Y'))$$

as follows. Let ν and γ be open coverings of X' and Y' respectively, and let $f^{\#}\nu = \{f^{-1}(U) \mid U \in \nu\}$ and $g^{\#}\gamma = \{g^{-1}(V) \mid V \in \gamma\}$. Then $f^{\#}\nu$ and $g^{\#}\gamma$ are open coverings of X and Y respectively. By the definition of the nerve, there are natural maps $f_\nu : X_{f\#\nu} \to X'_\nu$ and $g_\gamma : Y^{\check{\mathrm{C}}}_{g\#\gamma} \to (Y')^{\check{\mathrm{C}}}_{\gamma}$. Hence we have the map

$$T(g_\gamma)^{f_\nu} : T(Y^{\check{\mathrm{C}}}_{g\#\gamma})^{X'_\nu} \to T((Y')^{\check{\mathrm{C}}}_{\gamma})^{X_{f\#\nu}}$$

induced by f_ν and g_γ. Thus we can define

$$\check{\mathrm{F}}(f,g) = \varinjlim_\nu \underleftarrow{\mathrm{holim}}_\gamma T(g_\gamma)^{f_\nu} : \check{\mathrm{F}}(X',Y) \to \check{\mathrm{F}}(X,Y').$$

Theorem 1. The functor $\check{\mathrm{F}}$ is a bilinear enriched functor.

First we prove that the sequence

$$\check{\mathrm{F}}(X \cup CA, Z) \to \check{\mathrm{F}}(X,Z) \to \check{\mathrm{F}}(A,Z)$$

induced by the sequence $A \to X \to X \cup CA$, is a homotopy fibration sequence. Let λ be an open covering of $X \cup CA$, and let λ_X, λ_{CA} and λ_A be the coverings of X, CA and A consisting of those $U \in \lambda$ such that U intersects with X, CA, and A, respectively. We need the following lemma.

Lemma 3.2.1. *We have a homotopy equivalence*

$$(X \cup CA)^{\check{\mathrm{C}}}_{\lambda} \simeq X^{\check{\mathrm{C}}}_{\lambda_X} \cup C(A^{\check{\mathrm{C}}}_{\lambda_A}).$$

Proof. By the definition of the Čceh nerve, we have $(X \cup CA)^{\check{\mathrm{C}}}_{\lambda} = X^{\check{\mathrm{C}}}_{\lambda_X} \cup (CA)^{\check{\mathrm{C}}}_{\lambda_{CA}}$. By the homotopy equivalence

$$A^{\check{\mathrm{C}}}_{\lambda_A} = A^{\check{\mathrm{C}}}_{\lambda_A} \times \{0\} \simeq A^{\check{\mathrm{C}}}_{\lambda_A} \times I,$$

where I is the unit interval, we have

$$X^{\check{\mathrm{C}}}_{\lambda_X} \cup (CA)^{\check{\mathrm{C}}}_{\lambda_{CA}} \simeq X^{\check{\mathrm{C}}}_{\lambda_X} \cup A^{\check{\mathrm{C}}}_{\lambda_A} \times I \cup (CA)^{\check{\mathrm{C}}}_{\lambda_{CA}}.$$

Since $(CA)^{\check{C}}_{\lambda_{CA}} \simeq *$, we have

$$X^{\check{C}}_{\lambda_X} \cup A^{\check{C}}_{\lambda_A} \times I \cup (CA)^{\check{C}}_{\lambda_{CA}} \simeq X^{\check{C}}_{\lambda_X} \cup C(A^{\check{C}}_{\lambda_A}).$$

Hence we have $(X \cup CA)_\lambda \simeq X^{\check{C}}_{\lambda_X} \cup C(A^{\check{C}}_{\lambda_A})$. □

By Proposition 1.3.6 and Lemma 3.2.1, we see that the sequence

$$A_{\lambda_A} \to X_{\lambda_X} \to (X \cup CA)_\lambda$$

is a homotopy cofibration sequence. Hence the sequence

$$[(X \cup CA)_\lambda, Z] \to [X_{\lambda_X}, Z] \to [A_{\lambda_A}, Z]$$

is an exact sequence for any λ. Since the nerves of the form λ_X (resp. λ_A) are cofinal in the set of nerves of X (resp. A), we conclude that the sequence

$$\check{\mathrm{F}}(X \cup CA, Z) \to \check{\mathrm{F}}(X, Z) \to \check{\mathrm{F}}(A, Z)$$

is a homotopy fibration sequence.

Now we show that the sequence $\check{\mathrm{F}}(Z, A) \to \check{\mathrm{F}}(Z, X) \to \check{\mathrm{F}}(Z, X \cup CA)$ is a homotopy fibration sequence. The linearity of T implies that the sequence

$$T(A^{\check{C}}_{\lambda_A}) \to T(X^{\check{C}}_{\lambda_X}) \to T((X \cup CA)^{\check{C}}_\lambda)$$

is a homotopy fibration sequence. Since the fibre $T(A^{\check{C}}_{\lambda_A})$ is homeomorphic to the inverse limit

$$\varprojlim \, (* \to T((X \cup CA)^{\check{C}}_\lambda) \leftarrow T(X^{\check{C}}_{\lambda_X})),$$

we have

$$\begin{aligned}
&\varprojlim \, (* \to \underleftarrow{\mathrm{holim}}_\lambda \, T((X \cup CA)^{\check{C}}_\lambda) \leftarrow \underleftarrow{\mathrm{holim}}_{\lambda_X} \, T(X^{\check{C}}_{\lambda_X})) \\
&\simeq \varprojlim \, \underleftarrow{\mathrm{holim}}_\lambda (* \to T((X \cup CA)^{\check{C}}_\lambda) \leftarrow T(X^{\check{C}}_{\lambda_X})) \\
&\simeq \underleftarrow{\mathrm{holim}}_\lambda \varprojlim (* \to T((X \cup CA)^{\check{C}}_\lambda) \ \leftarrow T(X^{\check{C}}_{\lambda_X})) \\
&\simeq \underleftarrow{\mathrm{holim}}_\lambda \, T(A^{\check{C}}_{\lambda_A}).
\end{aligned}$$

This implies that the sequence

$$\underleftarrow{\operatorname{holim}}_{\lambda_A} T(A^{\check{\mathrm{C}}}_{\lambda_A}) \to \underleftarrow{\operatorname{holim}}_{\lambda_X} T(X^{\check{\mathrm{C}}}_{\lambda_X}) \to \underleftarrow{\operatorname{holim}}_{\lambda} T((X \cup CA)^{\check{\mathrm{C}}}_{\lambda})$$

is a homotopy fibration sequence, and hence so is $\check{\mathrm{F}}(Z, A) \to \check{\mathrm{F}}(Z, X) \to \check{\mathrm{F}}(Z, X \cup CA)$.

Next we prove the continuity of $\check{\mathrm{F}}$. Let $F(X, Y) = \mathbf{map}_0(X, \underleftarrow{\operatorname{holim}}_{\mu_i} T(Y^{\check{\mathrm{C}}}_{\mu_i}))$, so that we have $\check{\mathrm{F}}(X, Y) = \underrightarrow{\lim}_{\lambda} F(X_\lambda, Y)$. We need the following lemma.

Lemma 3.2.2. *The functor F is an enriched bifunctor.*

Proof. Let $F_1(Y) = \underleftarrow{\operatorname{holim}}_{\mu_i} T(Y^{\check{\mathrm{C}}}_{\mu_i})$ and $F_2(X, Z) = \mathbf{map}_0(X, Z)$, so that we have $F(X, Y) = F_2(X, F_1(Y))$. Clearly F_2 is continuous.

Let G_1 be the functor which maps Y to $\underleftarrow{\operatorname{holim}}_{\mu_i} Y^{\check{\mathrm{C}}}_{\mu_i}$. Since T is enriched, F_1 is continuous if so is G_1. It suffices to show that the map $G_1' \colon \mathbf{map}_0(Y, Y') \times \underleftarrow{\operatorname{holim}}_{\mu_i} Y^{\check{\mathrm{C}}}_{\mu_i} \to \underleftarrow{\operatorname{holim}}_{\lambda_j} (Y')^{\check{\mathrm{C}}}_{\lambda_j}$, adjoint to G_1, is continuous for any Y and Y'. Given an open covering λ of Y', let p^n_λ be the natural map $\underleftarrow{\operatorname{holim}}_{\lambda} (Y')^{\check{\mathrm{C}}}_{\lambda} \to \mathbf{map}_0(\Delta^n, (Y')^{\check{\mathrm{C}}}_{\lambda})$. Then G_1' is continuous if so is the composite

$$p^n_\lambda \circ G_1' \colon \mathbf{map}_0(Y, Y') \times \underleftarrow{\operatorname{holim}}_{\mu_i} Y^{\check{\mathrm{C}}}_{\mu_i} \to \mathbf{map}_0(\Delta^n, (Y')^{\check{\mathrm{C}}}_{\lambda})$$

for every $\lambda \in \operatorname{Cov}(Y')$ and every n. Here we may assume by [SYH, Proposition 4.3] that $\mathbf{map}_0(\Delta^n, (Y')^{\check{\mathrm{C}}}_{\lambda})$ is equipped with compact-open topology. Let $(g, \alpha) \in \mathbf{map}_0(Y, Y') \times \underleftarrow{\operatorname{holim}}_{\mu_i} Y^{\check{\mathrm{C}}}_{\mu_i}$, and let $W_{K,U} \subset \mathbf{map}_0(\Delta^n, (Y')^{\check{\mathrm{C}}}_{\lambda})$ be an open neighborhood of $p^n_\lambda(G_1'(g, \alpha))$, where K is a compact set of Δ^n and U is an open set of $(Y')^{\check{\mathrm{C}}}_{\lambda}$.

Let us choose simplices σ of $Y^{\check{\mathrm{C}}}_{g^\sharp\lambda}$ with vertices $g^{-1}(U(\sigma, k))$, where $U(\sigma, k) \in \lambda$ for $0 \le k \le \dim\sigma$. Let

$$O(\sigma) = \bigcap\nolimits_{0 \le k \le \dim\sigma} U(\sigma, k) \subset Y'.$$

Let us choose a point $y_\sigma \in \bigcap_{0\le k\le \dim\sigma} g^{-1}(U(\sigma,k))$, and then $g(y_\sigma) \in O(\sigma)$. Let W_1 be the intersection of all $W_{y_\sigma, O(\sigma)}$.

There is an integer l such that

$$\mu_l > \overline{\mu}_l > g^{\#}\lambda$$

where $\overline{\mu_l}$ is the closed covering $\{\overline{V} | V \in \mu_l\}$ of Y. Thus for any $U \in \mu_l$, there is an open set $V_U \in g^{\#}\lambda$ such that $\overline{U} \subset g^{-1}(V_U)$. Since Y is a compact set, $\overline{U}$ is compact. Let W_2 be the intersection of $W_{\overline{U}, V_U}$, and let $W = W_1 \cap W_2$.

Since $\mu_l > g^{\#}\lambda$, we have

$$p^n_\lambda(G'_1(g,\alpha)) = (g_\lambda)_*(\pi^{\mu_l}_{g^{\#}\lambda})_* p^n_{\mu_l}\alpha.$$

where $(g_\lambda)_*$ and $(\pi^{\mu_l}_{g^{\#}\lambda})_*$ are induced by $g_\lambda : Y^{\check{\mathrm{C}}}_{g^{\#}\lambda} \to (Y')^{\check{\mathrm{C}}}_\lambda$ and $\pi^{\mu_l}_{g^{\#}\lambda} : Y^{\check{\mathrm{C}}}_{\mu_l} \to Y^{\check{\mathrm{C}}}_{g^{\#}\lambda}$, respectively. Let

$$W' = (p^n_{\mu_l})^{-1}(W_{K,(\pi^{\mu_l}_{g^{\#}\lambda})^{-1}(g_\lambda)^{-1}(U)}).$$

Then $W \times W'$ is a neighborhood of (g,α) in $\mathbf{map}_0(Y,Y') \times \underleftarrow{\mathrm{holim}}_{\mu_i} Y_{\mu_i}$. To see that $p_\lambda \circ G'_1$ is continuous at (g,α), we need only show that $W \times W'$ is contained in $(p_\lambda \circ G'_1)^{-1}(U)$. Suppose that (h,β) belongs to $W \times W'$. Since W is contained in W_1, we have

$$y_\sigma \in h^{-1}(O(\sigma)) \subset \bigcap_{0\le k\le \dim\sigma} h^{-1}(U(\sigma,k)).$$

This means that the vertices $h^{-1}(U(\sigma,k)) \in h^\sharp\lambda$, $0 \le k \le \dim\sigma$, determine simplices σ' of $Y_{h^\sharp\lambda}$ corresponding to each $\sigma \subset Y_{g^\sharp\lambda}$. Thus we have an isomorphism

$$s : Y^{\check{\mathrm{C}}}_{h^\sharp\lambda} \to Y^{\check{\mathrm{C}}}_{g^\sharp\lambda},\ h^{-1}(U(\sigma,k)) \mapsto g^{-1}(U(\sigma,k)).$$

Moreover since W is contained in W_2, we have $\overline{\mu_l} > h^{\#}\lambda$.

Since the diagram

$$\begin{tikzcd} Y^{\check{C}}_{\mu_l} \arrow[r] \arrow[dr] & Y^{\check{C}}_{g^{\#}\lambda} \arrow[r, "g_\lambda"] & (Y')^{\check{C}}_{\lambda} \\ & Y^{\check{C}}_{h^{\#}\lambda} \arrow[u, "s"] \arrow[ur, "h_\lambda"'] & \end{tikzcd}$$

is commutative, we have

$$p^n_\lambda \circ G'_1(h,\beta)(K) = h_\lambda \pi^{\mu_l}_{h^{\#}\lambda}(\beta)(K) = g_\lambda \pi^{\mu_l}_{g^{\#}\lambda}(\beta)(K).$$

Since $g_\lambda \pi^{\mu_l}_{g^{\#}\lambda}(\beta)(K)$ is continued in U, so is $p^n_\lambda \circ G'_1(h,\beta)(K)$.

Thus $p^n_\lambda \circ G'_1$ is continuous for all $\lambda \in \mathrm{Cov}(Y')$, and hence so is

$$G'_1\colon\ \mathbf{map}_0(Y,Y') \times \underleftarrow{\mathrm{holim}}_{\mu_i} Y^{\check{C}}_{\mu_i} \to \underleftarrow{\mathrm{holim}}_{\lambda_j} (Y')^{\check{C}}_{\lambda_j}.$$

$\square$

We are now ready to prove Theorem 1. For given pointed spaces X, Y and a covering μ of X, let i_μ denote the natural map $F(X_\mu, Y) \to \varinjlim_\mu F(X_\mu, Y)$. To prove the theorem, it suffices to show that the map

$$\begin{aligned} \check{\mathrm{F}}' \circ (1 \times i_\lambda)\colon\ \mathbf{map}_0(X,X') \times F(X'_\lambda, Y) &\to \mathbf{map}_0(X,X') \times \varinjlim_\lambda F(X'_\lambda, Y) \\ &\to \varinjlim_\mu F(X_\mu, Y) \end{aligned}$$

which maps (f,α) to $i_{f^\sharp\lambda}(F(f_\lambda, 1_Y)(\alpha))$, is continuous for every covering λ of X.

Let $R_\lambda\colon\ \mathbf{map}_0(X,X') \to \varinjlim_\mu \mathbf{map}_0(X_\mu, X'_\lambda)$ be the map which assigns to $f : X \to X'$ the image of $\mathbf{map}_0(X,X')$, $f_\lambda \in \mathbf{map}_0(X_{f^\sharp\lambda}, X'_\lambda)$ in $\varinjlim_\mu \mathbf{map}_0(X_\mu, X'_\lambda)$, and let Q_λ be the map

$$\varinjlim_\mu \mathbf{map}_0(X_\mu, X'_\lambda) \times F(X'_\lambda, Y) \to \varinjlim_\mu F(X_\mu, Y),$$

$$[f, \alpha] \mapsto i_{f^\sharp\lambda} f_\lambda \circ \alpha = i_{f^\sharp\lambda}(F(f_\lambda, 1_Y)(\alpha)).$$

Since we have $\check{\mathrm{F}}' \circ (1 \times i_\lambda) = Q_\lambda \circ (R_\lambda \times 1)$, we need only show the continuity of Q_λ and R_λ. Since Q_λ is induced by the maps $\mathbf{map}_0(X_\mu, X'_\lambda) \times F(X'_\lambda, Y) \to F(X_\mu, Y)$, we see Q_λ is continuous.

To see that R_λ is continuous, let $W_{K^f,U}$ be a neighborhood of f_λ in $\mathbf{map}_0(X_{f^\sharp\lambda}, X'_\lambda)$, where K^f is a compact subset of $X_{f^\sharp\lambda}$ and U is an open subset of X'_λ. Since K^f is compact, there is a finite subcomplex S^f of $X_{f^\sharp\lambda}$ such that $K^f \subset S^f$. Let τ_i^f, $0 \le i \le m$, be simplexes of S^f. By taking a suitable subdivision of $X_{f^\sharp\lambda}$, we may assume that there is a simplicial neighborhood $N_{\tau_i^f}$ of each τ_i^f, $1 \le i \le m$, such that $K^f \subset S^f \subset \cup_i N_{\tau_i^f} \subset f_\lambda^{-1}(U)$.

Let $\{x_k^i\}$ be the set of vertices of τ_i^f and let W be the intersection of all $W_{\{x_k^i\}, U_{(\tau_i^f)'}}$,where $U_{(\tau_i^f)'}$ is an open set of X'_λ containing the set $\{f(x_k^i)\}$. Then W is a neighborhood of f. We need only show that $R_\lambda(W) \subset i_{f\#\lambda}(W_{K^f,U})$. Suppose that g belongs to W. Since $\{x_k^i\}$ is contained in $g^{-1}(U_{(\tau_i^f)'})$ for any i, a simplex τ_i^g spanned by the vertices is contained in $X_{g^\sharp\lambda}$. Let S^g be the finite subcomplex of $X_{g^\sharp\lambda}$ consisting of simplexes τ_i^g. By the construction, S^f and S^g are isomorphic. Moreover there is a compact subset K^g of $X_{g^\sharp\lambda}$ such that K^g and K^f are homeomorphic. On the other hand, since $g(\{x_k^i\}) \subset U_{(\tau_i^f)'}$, there is a simplex of X'_λ having $g_\lambda(\tau_i^g)$ and $(\tau_i^f)'$ as its faces. This means that $g_\lambda(\tau_i^g) \subset f_\lambda(\cup_i N_{\tau_i^f})$. Thus we have $g_\lambda(K^g) = \cup_i g_\lambda(\tau_i^g) \subset f_\lambda(\cup_i N_{\tau_i^f})$.

Let $f^\sharp\lambda \cap g^\sharp\lambda$ be an open covering

$$\{f^{-1}(U) \cap g^{-1}(V) \mid U, V \in \lambda\}$$

of X. We regard $X_{f^\sharp\lambda}$ and $X_{g^\sharp\lambda}$ as a subcomplex of $X_{f^\sharp\lambda\cap g^\sharp\lambda}$. Since $g_\lambda | X_{f^\sharp\lambda\cap g^\sharp\lambda}$ is contiguous to $f_\lambda | X_{f^\sharp\lambda\cap g^\sharp\lambda}$, we have a homotopy equivalence $g_\lambda | X_{f^\sharp\lambda\cap g^\sharp\lambda} \simeq$

$f_\lambda|X_{f^\sharp\lambda\cap g^\sharp\lambda}$. By the homotopy extension property of $g_\lambda|X_{f^\sharp\lambda\cap g^\sharp\lambda} : X_{f^\sharp\lambda\cap g^\sharp\lambda} \to X'_\lambda$ and $f_\lambda : X_{f^\sharp\lambda} \to X'_\lambda$, we see that $g_\lambda|X_{f^\sharp\lambda\cap g^\sharp\lambda}$ extends to a map $G : X_{f^\sharp\lambda} \to X'_\lambda$.

We have the relation $G \sim \pi^{f^\sharp\lambda\cap g^\sharp\lambda}_{f^\sharp\lambda} G = g_\lambda|X_{f^\sharp\lambda\cap g^\sharp\lambda} = \pi^{f^\sharp\lambda\cap g^\sharp\lambda}_{g^\sharp\lambda} g_\lambda \sim g_\lambda$, where $\sim$ is the relation of the direct limit. Moreover by $G(K^f) \subset f_\lambda(\cup_i N_{\tau_i^f}) \subset U$, we have $[g_\lambda] = [G] \in i_{f\#\lambda}(W_{K^f,U})$. Hence R_λ is continuous, and so is $\check{\mathrm{F}}'$.

3.3 Proofs of Theorems 2 and 3

To prove Theorems 2 and 3, we need several lemmas.

Lemma 3.3.1. *There exists a sequence $\lambda^n_1 < \lambda^n_2 < \cdots < \lambda^n_m < \cdots$ of open coverings of S^n such that:*

1. *For each open covering μ* of *S^n, there is an $m \in \mathbb{N}$ such that λ^n_m is a refinement of μ:*
2. *For any m, $S^n_{\lambda_m}$ is homotopy equivalent to S^n.*

Proof. We prove by induction on n. For $n = 1$, we define an open covering λ^1_m of S^1 as follows. For any i with $0 \le i < 4m$, we put

$$U(i,m) = \{(\cos\theta,\ \sin\theta) \mid \frac{i-1}{4m}\times 2\pi + \frac{1}{16m}\times 2\pi < \theta < \frac{i+1}{4m}\times 2\pi + \frac{1}{16m}\times 2\pi\}.$$

Let $\lambda^1_m = \{U(i,m)|\ 0 \le i < 4m\}$. Then the set λ^1_m is an open covering of S^1 and is a refinement of λ^1_{m-1}. Clearly $(S^1)^{\check{\mathrm{C}}}_{\lambda^1_m}$ is homeomorphic to S^1, and hence $S^1_{\lambda^1_m}$ is homotopy equivalent to S^1. Moreover for any open covering μ of

S^1, there exists an m such that λ_m^1 is a refinement of μ. Hence the lemma is true for $n = 1$. Assume now that the lemma is true for $1 \leq k \leq n-1$. Let λ'^n_m be the open covering $\lambda_m^{n-1} \times \lambda_m^1$ of $S^{n-1} \times S^1$ and let λ_m^n be the open covering of S^n induced by the natural map $p : S^{n-1} \times S^1 \to S^{n-1} \times S^1 / S^{n-1} \vee S^1$. Since $S^{n-1}_{\lambda_m^{n-1}}$ is a homotopy equivalence of S^{n-1}, we have a homotopy equivalence

$$S^n_{\lambda_m^n} \approx (S^{n-1} \times S^1 / S^{n-1} \vee S^1)_{\lambda_m^n} \approx (S^{n-1}_{\lambda_m^{n-1}} \times S^1_{\lambda_m}) / (S^{n-1}_{\lambda_m^{n-1}} \vee S^1_{\lambda_m}) \approx S^n.$$

Thus the sequence $\lambda_1^n < \lambda_2^n < \cdots < \lambda_m^n < \cdots$ satisfies the required conditions. $\square$

Lemma 3.3.2. *$h_n(X, \check{\mathrm{F}}) \cong \pi_n \operatorname{\underleftarrow{holim}}_\mu T(X_\mu^{\check{\mathrm{C}}})$ for $n \geq 0$.*

Proof. By Lemma 3.3.1, we have an isomorphism

$$\varinjlim_\lambda [S^n_\lambda, \operatorname{\underleftarrow{holim}}_\mu T(X_\mu^{\check{\mathrm{C}}})] \cong [S^n, \operatorname{\underleftarrow{holim}}_\mu T(X_\mu^{\check{\mathrm{C}}})].$$

Thus we have

$$\begin{aligned}
h_n(X, \check{\mathrm{F}}) &= \pi_0 \, \check{\mathrm{F}}(S^n, X) \\
&= \pi_0 \varinjlim_\lambda \mathbf{map}_0(S^n_\lambda, \operatorname{\underleftarrow{holim}}_\mu T(X_\mu^{\check{\mathrm{C}}})) \\
&\cong \varinjlim_\lambda [S^0, \mathbf{map}_0(S^n_\lambda, \operatorname{\underleftarrow{holim}}_\mu T(X_\mu^{\check{\mathrm{C}}})] \\
&\cong \varinjlim_\lambda [S^n_\lambda, \operatorname{\underleftarrow{holim}}_\mu T(X_\mu^{\check{\mathrm{C}}})] \\
&\cong [S^n, \operatorname{\underleftarrow{holim}}_\mu T(X_\mu^{\check{\mathrm{C}}})] \\
&\cong \pi_n \operatorname{\underleftarrow{holim}}_\mu T(X_\mu^{\check{\mathrm{C}}}).
\end{aligned}$$

$\square$

Now we are ready to prove Theorem 2. Let X be a compact metric space and let $\mathbb{S} = \{T(S^k) \mid k \geq 0\}$. Since X is a compact metric space, there is a sequence $\{\mu_i\}_{i\geq 0}$ of finite open covering of X with $\mu_0 = X$ and μ_i refining μ_{i-1} such that $X = \varprojlim_i X^{\check{C}}_{\mu_i}$ holds. Let us denote $X^{\check{C}}_{\mu_i} = X^{\check{C}}_i$ and $X_{\mu_i} = X_i$ if there is no possibility of confusion. According to [F], there is a short exact sequence

$$0 \longrightarrow \varprojlim^1_i H_{n+1}(X^{\check{C}}_i, \mathbb{S}) \longrightarrow H^{st}_n(X, \mathbb{S}) \longrightarrow \varprojlim_i H_n(X^{\check{C}}_i, \mathbb{S}) \longrightarrow 0$$

where $H_n(X, \mathbb{S})$ is the homology group of X with coefficients in the spectrum $\mathbb{S}$. (This is a special case of the Milnor exact sequence [Mi].) On the other hand, by [BK], we have the following.

Lemma 3.3.3. *([BK]) There is a natural short exact sequence*

$$0 \longrightarrow \varprojlim^1_i \pi_{n+1}T(X^{\check{C}}_i) \longrightarrow \pi_n \operatorname{\underleftarrow{holim}}_i T(X_i) \longrightarrow \varprojlim_i \pi_n T(X^{\check{C}}_i) \longrightarrow 0.$$

By Proposition 2.6.3, we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & \varprojlim^1_i H_{n+1}(X^{\check{C}}_i, \mathbb{S}) \arrow[r] \arrow[d, "\cong"] & H^{st}_n(X, \mathbb{S}) \arrow[r] & \varprojlim_i H_n(X^{\check{C}}_i, \mathbb{S}) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \varprojlim^1_i \pi_{n+1}(T(X^{\check{C}}_i)) \arrow[r] & \pi_n(\operatorname{\underleftarrow{holim}}_i T(X^{\check{C}}_i)) \arrow[r] & \varprojlim_i \pi_n(T(X^{\check{C}}_i)) \arrow[r] & 0.
\end{tikzcd} \tag{3.1}$$

Hence it suffices to construct a natural homomorphism

$$H^{st}_n(X, \mathbb{S}) \to \pi_n(\operatorname{\underleftarrow{holim}}_i T(X^{\check{C}}_i))$$

making the diagram (3.1) commutative.

Since T is continuous, the identity map $X \wedge S^k \to X \wedge S^k$ induces a continuous map $i' : X \wedge T(S^k) \to T(X \wedge S^k)$. Hence we have a composite homomorphism

$$
\begin{aligned}
H_n^{st}(X,\mathbb{S}) &= \pi_n \underleftarrow{\text{holim}}_i (X_i^{\check{\text{C}}} \wedge \mathbb{S}) \\
&\cong \varinjlim_k \pi_{n+k}(\underleftarrow{\text{holim}}_i (X_i^{\check{\text{C}}} \wedge T(S^k)) \\
&\xrightarrow{I} \varinjlim_k \pi_{n+k}(\underleftarrow{\text{holim}}_i T(X_i^{\check{\text{C}}} \wedge S^k)) \\
&\cong \pi_n(\underleftarrow{\text{holim}}_i T(X_i^{\check{\text{C}}}))
\end{aligned}
$$

where $I = \varinjlim_k i'^k_*$ is induced by the homomorphisms

$$
i'^k_* : \pi_{n+k}(\underleftarrow{\text{holim}}_i (X_i^{\check{\text{C}}} \wedge T(S^k)) \to \pi_{n+k}(\underleftarrow{\text{holim}}_i T(X_i^{\check{\text{C}}} \wedge S^k)).
$$

Clearly the resulting homomorphism $H_n^{st}(X,\mathbb{S}) \to \pi_n(\underleftarrow{\text{holim}}_i T(X_i^{\check{\text{C}}}))$ makes the diagram (3.1) commutative. Thus $h_n(X,\check{\text{F}})$ is isomorphic to the Steenrod homology group coefficient in the spectrum $\mathbb{S}$.

Finally, to prove Theorem 3 it suffices to show that $h^n(X,\check{\text{C}})$ is isomorphic to the Čech cohomology group of X.

By Lemma 3.3.1, we have a homotopy commutative diagram

$$
\begin{tikzcd}
\cdots \arrow[r, "="] & AG(S^n) \arrow[r, "="] \arrow[d, "\simeq"] & AG(S^n) \arrow[r, "="] \arrow[d, "\simeq"] & \cdots \\
\cdots \arrow[r] & AG(S^n_{\lambda^n_{m-1}}) \arrow[r, "\simeq"] & AG(S^n_{\lambda^n_m}) \arrow[r] & \cdots .
\end{tikzcd}
$$

Hence we have $AG(S^n) \simeq \underleftarrow{\text{holim}}_i AG(S^n_{\lambda^n_i})$.

Thus we have

$$
\begin{aligned}
h^n(X,\check{C}) &= \pi_0 \check{C}(X,S^n) \\
&= \pi_0 \varinjlim_\lambda \mathbf{map}_0(X_\lambda, \underleftarrow{\text{holim}}_\mu AG((S^n)_\mu^{\check{\text{C}}})) \\
&\cong [S^0, \varinjlim_\lambda \mathbf{map}_0(X_\lambda, AG(S^n)] \\
&\cong \varinjlim_\lambda [S^0, \mathbf{map}_0(X_\lambda, AG(S^n)] \\
&\cong \varinjlim_\lambda [S^0 \wedge X_\lambda, AG(S^n)] \\
&\cong \varinjlim_\lambda [X_\lambda, AG(S^n)].
\end{aligned}
$$

Hence $h^n(X, \check{C})$ is isomorphic to the Čech cohomology group of X.

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