ON THE SOLVABILITY OF CERTAIN (SSIE) WITH
OPERATORS OF THE FORM $B(r, s)$

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ABSTRACT. Given any sequence $z = (z_n)_{n \geq 1}$ of positive real numbers and any set $E$ of complex sequences, we write $E_z$ for the set of all sequences $y = (y_n)_{n \geq 1}$ such that $y/z = (y_n/z_n)_{n \geq 1} \in E$; in particular, $s^{(c)}$ denotes the set of all sequences $y$ such that $y/z$ converges. In this paper we deal with sequence spaces inclusion equations (SSIE), which are determined by an inclusion each term of which is a sum or a sum of products of sets of sequences of the form $\chi_\alpha(T)$ and $\chi_x(T)$ where $a$ is a given sequence, the sequence $x$ is the unknown, $T$ is a given triangle, and $\chi_\alpha(T)$ and $\chi_x(T)$ are the matrix domains of $T$ in the set $\chi$. Here we determine the set of all positive sequences $x$ for which the (SSIE) $s^{(c)}_x(B(r, s)) \subset s^{(c)}(B(r', s'))$ holds, where $r, r', s'$ and $s$ are real numbers, and $B(r, s)$ is the generalized operator of the first difference defined by $(B(r, s)y)_n = ry_n + sy_{n-1}$ for all $n \geq 2$ and $(B(r, s)y)_1 = ry_1$. We also determine the set of all positive sequences $x$ for which
\[
\frac{ry_n + sy_{n-1}}{x_n} \to l \text{ implies } \frac{r'y_n + s'y_{n-1}}{x_n} \to l \ (n \to \infty)
\]
and for some scalar $l$. Finally, for a given sequence $a$, we consider the $a$–Tauberian problem which consists of determining the set of all $x$ such that $s^{(c)}_x(B(r, s)) \subset s^{(c)}_a$.

1. Introduction

As usual we denote by $\omega$ the set of all complex sequences $x = (x_n)_{n \geq 1}$, and by $c_0$, $c$ and $\ell_\infty$ the subsets of all null, convergent and bounded sequences, respectively; we write $cs$ for the set of all convergent complex series. Also let $U^+$ denote the set of all sequences $u = (u_n)_{n \geq 1}$ with $u_n > 0$ for all $n$. Given a sequence $a \in \omega$ and a subset $E$ of $\omega$, Wilansky [15] introduced the notation $a^{-1} \ast E = \{y \in \omega : ay = (a_ny_n)_{n \geq 1} \in E\}$. The sets $s_\alpha$, $s^{(c)}_\alpha$ and $s^{(c)}_\alpha$ were introduced in [3] by $((1/a_n)_{n \geq 1})^{-1} \ast E$ for any sequence $a \in U^+$ and $E \in \{\ell_\infty, c_0, c\}$. In [4, 5] the sum $\chi_\alpha + \chi'_b$ and the product $\chi_\alpha \ast \chi'_b$ were defined, where $\chi$ and $\chi'$ are any of the symbols $s$, $s^{(c)}_a$, or $s^{(c)}_s$; also matrix transformations in the sets $s_\alpha + s^{(c)}_\alpha(\Delta q)$ and $s_\alpha + s^{(c)}_\alpha(\Delta q)$ were characterized, where $\Delta$ is the operator of the first difference. In [9] de Malafosse and

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Malkowsky gave the properties of the spectrum of the matrix of weighted means $\overline{N}_q$ considered as an operator in the set $s_\alpha$. In [10] characterizations can be found of the classes of matrix transformations from $s_\alpha(\Delta^q)$ into $\chi_b$, where $\chi$ is any of the symbols $s$, $s^0$, or $s^{(c)}$. Using the spectral properties of the operator of the first difference in the sets $s_\alpha^0$ and $s_\beta^{(c)}$, in [5] we were able to simply the set $s_\alpha^0((\Delta - \lambda I)^h) + s_\beta^{(c)}((\Delta - \mu I)^l)$, where $h$ and $l$ are complex numbers, and $\alpha$ and $\beta$ are given sequences; also matrix transformations in this set were characterized in [5]. In [11] de Malafosse and Rakočević gave applications of the measure of noncompactness to operators on the spaces $s_\alpha$, $s_\alpha^0$, $s_\alpha^{(c)}$ and $\ell_\alpha^p$ to determine compact operators between some of these spaces. Sequence spaces inclusion equations (SSIE) and sequence spaces equations (SSE) were introduced and studied in [2, 8, 7]. They are determined by an inclusion or identity each term of which is a sum or a sum of products of sets of the form $\chi_a(T)$ and $\chi_{f(x)}(T)$ where $\chi$ is any of the symbols $s$, $s^0$, or $s^{(c)}$, $a$ is a given sequence in $U^+$, $x$ is the unknown, $f$ maps $U^+$ to itself, and $T$ is a triangle. In this paper we use the operator represented by the triangle $B(r, s)$, called the generalized operator of the first difference and defined by $(B(r, s)y)_n = ry_n + sy_{n-1}$ for all $n \geq 2$ and $(B(r, s)y)_1 = ry_1$. Then we deal with the (SSIE) $s_x^{(c)}(B(r, s)) \subset s_x^{(c)}(B(r', s'))$, which is equivalent to

$$\frac{ry_n + sy_{n-1}}{x_n} \to l \implies \frac{r'y_n + s'y_{n-1}}{x_n} \to l' \quad (n \to \infty)$$

for all $x$. We then obtain extensions of results stated in [3, 2, 8, 7, 6]. The notion of an a–Tauberian theorem was introduced in [6] as follows. For a given sequence $a$, an a–Tauberian theorem is one in which the convergence of a sequence $y/a = (y_n/a_n)_{n \geq 1}$ is deduced from the convergence of some transform of the sequence together with some side conditions, the so–called a–Tauberian conditions. In [6], for given sequences $\lambda$ and $\mu$, we determined the set of all sequences $a$ such that

$$\frac{1}{\lambda_n} \sum_{k=1}^{n} \mu_k \left( \sum_{i=k}^{\infty} y_i \right) \to l \implies \frac{y_n}{a_n} \to l' \quad (n \to \infty)$$

for all $y \in cs$. In [6] a–Tauberian theorem is an extension of Hardy's Tauberian theorem. In Hardy's Tauberian theorem it is shown that under some condition for $y = (y_n)_{n \geq 1}$, we have $n^{-1} \sum_{k=1}^{n} y_k \to l$ implies $y_n \to l$ as $n$ tends to infinity. In a similar way, for a given sequence $a$, we will determine the set of all positive sequences $x$ for which

$$\frac{ry_n + sy_{n-1}}{x_n} \to l \implies \frac{y_n}{a_n} \to l \quad (n \to \infty)$$

for all $y$. Would you like to proceed with this text or do you have any other questions?
If \( a_n = 1 \) for all \( n \) we obtain the classical Tauberian problems. In [14] we considered the \((C, \lambda, \mu)\) summability that generalizes the \((C, 1)\) summability and established conditions for the equivalence between the convergence of \( x_n/\mu_n \) and the convergence of the sequence

\[
\mu'_n = 1/\lambda_n \sum_{m=1}^{n} \hat{\mu}_m(x),
\]

where \( \hat{\mu}_n(x) = (x_1 + \ldots + x_n)/\mu_n \), and also for the equivalence between the convergence of \( \hat{\mu}_n(x) \) and the convergence of \( \mu'_n \).

This paper is organized as follows. In Section 2 we recall some results on AK and BK spaces and on the set \( S_{a,b} \). In Section 3 we consider the operator \( C(\xi) \) and its inverse \( \Delta(\xi) \), and recall the definitions and properties of the sets \( \hat{\Gamma}, \hat{C}, \Gamma \) and \( \hat{C}_1 \). In Section 4 we solve the (SSIE) \( s_x^{(c)}(B(r, s)) \subset s_x^{(c)}(B(r', s')) \) where \( B(r, s) \) is the generalized operator of the first difference defined above. In Section 5 we determine the set of all sequences \( x \) of positive real numbers such that \( (r y_n + s y_{n-1})/x_n \to l \) implies \( (r' y_n + s' y_{n-1})/x_n \to l \) as \( n \) tends to infinity, for some scalar \( l \) and for given reals \( r, s, r' \) and \( s' \). Finally in Section 6 we consider some \( a\)–Tauberian theorems; this is achieved by determining the set of all \( x \) such that \( s_x^{(c)}(B(r, s)) \subset s_x^{(a)} \).

2. Notations and preliminary results

Let \( A = (a_{nk})_{n,k \geq 1} \) be an infinite matrix and \( y = (y_k)_{k \geq 1} \) be a sequence. Then we write

\[
A_n y = \sum_{k=1}^{\infty} a_{nk} y_k \text{ for any integer } n \geq 1
\]

and \( A y = (A_n y)_{n \geq 1} \) provided all the series in (2.1) converge.

Let \( E \) and \( F \) be any subsets of \( \omega \). Then we write \((E, F)\) for the class of all infinite matrices \( A \) for which the series in (2.1) converge for all \( y \in E \) and all \( n \), and \( A y \in F \) for all \( y \in E \). So if \( A \in (E, F) \) then we are led to the study of the operator \( \Lambda = \Lambda_A : E \to F \) defined by \( \Lambda y = A y \) and we identify the operator \( \Lambda \) with the matrix \( A \).

A Banach space \( E \) of complex sequences is said to be a BK space if each projection \( P_n : E \to \mathbb{C} \) defined by \( P_n(y) = y_n \) for all \( y = (y_n)_{n \geq 1} \in E \) is continuous. A BK space \( E \) is said to have \( AK \) if every sequence \( y = (y_k)_{k \geq 1} \in E \) has a unique representation \( y = \sum_{k=1}^{\infty} y_k e^{(k)} \) where \( e^{(k)} \) is the sequence with 1 in the \( k \)-th position and 0 otherwise.
If \( u \) and \( v \) are sequences and \( E \) and \( F \) are two subsets of \( \omega \), then we write \( uv = (u_nv_n)_{n \geq 1} \) and 
\[
M(E,F) = \{ u = (u_n)_{n \geq 1} : uv \in F \text{ for all } v \in E \},
\]
for the multiplier space of \( E \) and \( F \).

To simplify notations, we use the diagonal matrix \( D_a \) defined by \([D_a]_{nn} = a_n\) for all \( n \), write 
\[
D_a * E = (1/a)^{-1} * E = \{ (y_n)_{n \geq 1} \in \omega : (y_n/a_n)_{n \in E} \}
\]
for any \( a \in U^+ \) and any \( E \subset \omega \), and define \( s_a = D_a * \ell_\infty \), \( s_0^a = D_a * c_0 \) and 
\[
s_{a,c} = D_a * c \text{ (see, for instance, \([4, 3, 11]\))}.
\]

Each of the spaces \( D_a * \chi \), where \( \chi \in \{ \ell_\infty, c_0, c \} \), is a BK space normed by \( \| \xi \|_{s_a} = \sup_{n \geq 1} (|\xi_n|/a_n) \) and \( s_0^a \) has AK (see \([15, \text{Theorem 4.3.6}]\)).

Now let \( a = (a_n)_{n \geq 1}, b = (b_n)_{n \geq 1} \in U^+ \). By \( S_{a,b} \) we denote the set of all infinite matrices \( \Lambda = (\lambda_{nk})_{n,k \geq 1} \) such that 
\[
\| \Lambda \|_{S_{a,b}} = \sup_{n \geq 1} \left( \frac{1}{b_n} \sum_{k=1}^\infty |\lambda_{nk}|a_k \right) < \infty.
\]

It is well known that \( \Lambda \in (s_a, s_b) \) if and only if \( \Lambda \in S_{a,b} \). So we can write \( (s_a, s_b) = S_{a,b} \).

When \( s_a = s_b \) we obtain the Banach algebra with identity \( S_{a,b} = S_a \) (see \([3]\)), normed by \( \| \Lambda \|_{S_a} = \| \Lambda \|_{S_{a,a}} \). We also have \( \Lambda \in (s_a, s_a) \) if and only if \( \Lambda \in S_a \).

If \( a = (r^n)_{n \geq 1} \), the sets \( S_a, s_a, s_0^a \) and \( s_{a,c} \) are denoted by \( S_r, s_r, s_0^r \) and 
\( s_{r,c} \), respectively (see \([4]\)). When \( r = 1 \), we obtain \( s_1 = \ell_\infty \), \( s_1^0 = c_0 \) and 
\( s_{1,c} = c \), and writing \( e = (1,1,...) \) we have \( S_1 = S_e \). It is well known that 
\( (s_1, s_1) = (c_0, s_1) = (c, s_1) = S_1 \) (see, for instance, \([15, \text{Example 8.4.5A}]\)).

In the sequel we will frequently use the obvious fact that \( \Lambda \in (\chi_a, \chi'_b) \) if and only if \( D_{1/b} \Lambda D_a \in (\chi_e, \chi'_e) \) where \( \chi, \chi' \) are any of the symbols \( s_0^0, s_{(c)} \), or \( s \).

For any subset \( E \) of \( \omega \), we put \( \Lambda E = \{ \eta \in \omega : \eta = \Lambda y \text{ for some } y \in E \} \).

If \( F \) is a subset of \( \omega \), we write \( F(\Lambda) = F_{\Lambda} = \{ y \in \omega : \Lambda y \in F \} \) for the matrix domain of \( \Lambda \) in \( F \).

3. The operators \( C(\xi), \Delta(\xi) \) and the sets \( \tilde{\Gamma}, \tilde{C}, \Gamma \) and \( \widetilde{\Gamma} \)

An infinite matrix \( T = (t_{nk})_{n,k \geq 1} \) is said to be a triangle if \( t_{nk} = 0 \) for \( k > n \) and \( t_{nn} \neq 0 \) for all \( n \). Now let \( U \) be the set of all sequences \((u_n)_{n \geq 1} \in \omega \) with \( u_n \neq 0 \) for all \( n \). If \( \xi = (\xi_n)_{n \geq 1} \in U \), we write \( C(\xi) \) for the triangle
with
\[
[C(\xi)]_{nk} = \begin{cases} 
\frac{1}{\xi_n} & \text{if } k \leq n, \\
0 & \text{otherwise}, 
\end{cases}
\]
(see, for instance, [12]-[14]). It is easy to see that the triangle \(\Delta(\xi)\) defined by
\[
[\Delta(\xi)]_{nk} = \begin{cases} 
\xi_n & \text{if } k = n, \\
-\xi_{n-1} & \text{if } k = n-1 \text{ and } n \geq 2, \\
0 & \text{otherwise}, 
\end{cases}
\]
is the inverse of \(C(\xi)\), that is, \(C(\xi)(\Delta(\xi)y) = \Delta(\xi)(C(\xi)y) = y\) for all \(y \in \omega\). If \(\xi = e\) we get \(\Delta(e) = \Delta\), where \(\Delta\) is the well–known operator of the first difference defined by \(\Delta_ny = y_n - y_{n-1}\) for all \(y \in \omega\) and all \(n \geq 1\), with the convention \(y_0 = 0\). It is usual to write \(\Sigma = C(e)\). We note that \(\Delta\) and \(\Sigma\) are inverse to one another, and \(\Delta, \Sigma \in \mathcal{S}_R\) for any \(R > 1\).

To simplify notation, for \(t > 0\) and \(\xi \in U^+\), we write \(\xi'_n = t^{-n}\xi_n\) and
\[
c_n(t, \xi) = [C(\xi') \xi'_n]_n = \frac{t^n}{\xi_n} \sum_{k=1}^{n} \frac{\xi_k}{t^k} \quad \text{for all } n,
\]
and
\[
c_n(\xi) = c_n(1, \xi) = \frac{1}{\xi_n} \sum_{k=1}^{n} \xi_k \quad \text{for all } n.
\]
We also consider the sets
\[
\hat{C} = \{ \xi \in U^+ : c_n(\xi) \to l \ (n \to \infty) \text{ for some scalar } l \},
\]
\[
\hat{C}_1 = \left\{ \xi \in U^+ : \sup_n c_n(\xi) < \infty \right\},
\]
\[
\hat{\Gamma} = \left\{ \xi \in U^+ : \lim_{n \to \infty} \left( \frac{\xi_{n-1}}{\xi_n} \right) < 1 \right\},
\]
\[
\Gamma = \left\{ \xi \in U^+ : \limsup_{n \to \infty} \left( \frac{\xi_{n-1}}{\xi_n} \right) < 1 \right\}
\]
and
\[
G_1 = \{ \xi \in U^+ : \text{there are } C > 0 \text{ and } \gamma > 1 \text{ such that } \xi_n \geq C\gamma^n \text{ for all } n \}.
\]
We obtain the next lemma by [3, Proposition 2.1, p. 1786] and [9, Proposition 2.2, p. 88].

**Lemma 3.1.** We have \(\hat{C} = \hat{\Gamma} \subset \Gamma \subset \hat{C}_1 \subset G_1\).
4. On the (SSIE) $s_x^{(c)}(B(r, s)) \subset s_x^{(c)}(B(r', s'))$ for real numbers $r$, $s$, $r'$ and $s'$

In this subsection we determine, for given real numbers $r$, $s$, $r'$ and $s'$, the set of all $x \in U^+$ such that

$$\frac{ry_n + sy_{n-1}}{x_n} \to l \quad \text{implies} \quad \frac{r'y_n + s'y_{n-1}}{x_n} \to l' \quad (n \to \infty)$$

for all $y$ and for some scalars $l$ and $l'$. We will see that this is equivalent to determining the set of all $x \in U^+$ that satisfy the (SSIE)

$$(4.1) \quad s_x^{(c)}(B(r, s)) \subset s_x^{(c)}(B(r', s')),$$

where $B(r, s)$ and $B(r', s')$ are the generalized operators of the first difference.

We recall the next result which is a direct consequence of the famous Silverman-Toeplitz theorem.

**Lemma 4.1.** We have:

i) $\Lambda \in (c, c)$ if and only if

$$\Lambda \in S_1, \quad \lim_{n \to \infty} \sum_{k=1}^{\infty} \lambda_{nk} = l \quad \text{and} \quad \lim_{k \to \infty} \lambda_{nk} = l_k \quad \text{for all} \quad k \geq 1$$

for some scalars $l$ and $l_k$ (see, for instance, [15, Theorem 1.3.6]).

ii) Let $\Lambda \in (c, c)$ and $y \in c$. If $\lim_{k \to \infty} \lambda_{nk} = 0$ for all $k \geq 1$, then

$$\lim_{n \to \infty} y_n = L \quad \text{implies} \quad \lim_{n \to \infty} \Lambda_n y = lL$$

(see, for instance, [15, Theorem 1.3.8]).

To state the next theorem we need the following result.

**Proposition 4.2.** Let $x \in U^+$. Then

$$c_n(x) = \frac{1}{x_n} \sum_{k=1}^{n} x_k \to l \quad \text{if and only if} \quad \frac{x_{n-1}}{x_n} \to 1 - \frac{1}{l} \quad (n \to \infty)$$

for some scalar $l$.

**Proof.** We put $L = 1 - 1/l$ and $\Sigma_n = \sum_{k=1}^{n} x_k$ and note that $l \geq 1$, since $\Sigma_n/x_n = 1 + \Sigma_{n-1}/x_n \geq 1$ for all $n$.

It was shown in [3, Proposition 2.1, p. 1786] that $c_n(x) \to l \quad (n \to \infty)$ implies $x_{n-1}/x_n \to 1 - 1/l \quad (n \to \infty)$. To show the converse implication, we assume $x_{n-1}/x_n \to 1 - 1/l \quad (n \to \infty)$. 


Since we have $\hat{C} = \hat{\Gamma}$ by Lemma 3.1, we can write $\Sigma_n/x_n \to l_1 (n \to \infty)$ for some scalar $l_1$, and must show $l_1 = l$. We have for every $n > 2$

$$\frac{x_{n-1}}{x_n} = \frac{\Sigma_{n-1} - \Sigma_{n-2}}{x_n} = \frac{\Sigma_{n-1}}{x_{n-1}} - \frac{\Sigma_{n-2}}{x_{n-2}} \frac{x_{n-1}}{x_n}$$

and

$$\frac{\Sigma_{n-1} - \Sigma_{n-2}}{x_n} \to l_1 L - l_1 L^2 = L \ (n \to \infty).$$

If $L \neq 0$ then we have $l_1 = 1/(1 - L)$ and since $L = 1 - 1/l$, we conclude

$$l_1 = \frac{1}{1 - \left(1 - \frac{1}{l}\right)} = l.$$

If $L = 0$ then we have $l = 1$ and

$$\frac{\Sigma_n}{x_n} = \frac{\Sigma_{n-1}}{x_{n-1}} \frac{x_{n-1}}{x_n} + 1 \to 1 \ (n \to \infty).$$

\[\square\]

We recall that $B(r, s)$, where $r$ and $s$ are real numbers, is the lower triangular matrix

$$B(r, s) = \begin{pmatrix} r & s & 0 \\ s & r & 0 \\ 0 & s & r \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

For $r, s \neq 0$, the matrix $B(r, s)$ was introduced by Altay and Basar [1] and was called the generalized operator of the first difference.

In the next theorem we confine our studies to the case when $\alpha = -s/r > 0$ if $\delta = rs' - r's \neq 0$.

**Theorem 4.3.** Let $r, s, r'$ and $s'$ be real numbers with $r, s \neq 0$, and $\delta = rs' - r's \neq 0$.

i) If $\delta = 0$, then (SSIE) (4.1) holds for all $x$.

ii) If $\delta \neq 0$ and $\alpha = -s/r > 0$, then (4.1) holds if and only if

$$\lim_{n \to \infty} \frac{x_{n-1}}{x_n} < \frac{1}{\alpha}.$$

**Proof.** Inclusion (4.1) is equivalent to $I \in (s_x^c(B(r, s)), s_x^c(B(r', s')))$, that is, to

$$\tilde{B} = B(r', s')B^{-1}(r, s) \in \left(s_x^c, s_x^c\right).$$

This means

$$D_{1/x} \tilde{B} D_x \in (c, c).$$

(4.2)
Since \( r \neq 0 \), the matrix \( B(r, s) \) is invertible, its inverse is a triangle and elementary calculations give

\[
[B^{-1}(r, s)]_{nk} = \frac{1}{r} \alpha^{n-k} \quad \text{for} \quad 1 \leq k \leq n.
\]

Then we obtain \( \tilde{B}_{nn} = r'/r \), and have for \( k \leq n - 1 \)

\[
\tilde{B}_{nk} = s' \left[B^{-1}(r, s)\right]_{n-1,k} + r' \left[B^{-1}(r, s)\right]_{nk}
\]

\[
= s' \frac{1}{r} \alpha^{n-k-1} + r' \frac{1}{r} \alpha^{n-k}
\]

\[
= \alpha^{n-k-1} \left( \frac{s'}{r} + \frac{r'}{r} \frac{1}{\alpha} \right) = \alpha^{n-k-1} \frac{\delta}{r^2}.
\]

It follows that

\[
\left[D_{1/x} \tilde{B}D_x\right]_{nk} = \begin{cases} 
\frac{1}{x_n} \alpha^{n-k-1} \frac{\delta}{r^2} x_k & \text{for } k \leq n - 1, \\
\frac{r'}{r} & \text{for } k = n.
\end{cases}
\]

We deduce from the characterization of \((c, c)\) in Lemma 4.1 (i) that (4.2) holds if and only if

\[
\sum_{k=1}^{n} \left[D_{1/x} \tilde{B}D_x\right]_{nk} = \frac{r'}{r} - \frac{\delta}{rs} \bar{C}_n(\alpha, x) \to l \quad (n \to \infty)
\]

for some scalar \( l \), where

\[
\bar{c}_n(\alpha, x) = c_n(\alpha, x) - 1 = \frac{1}{x_n} \sum_{k=1}^{n-1} \frac{x_k}{\alpha^k}.
\]

Indeed this condition implies \( D_{1/x} \tilde{B}D_x \in S_1 \) and \((x_n/\alpha^n)_n \in \tilde{C}\). Since we have \( \tilde{C} \subset G_1 \) by Lemma 3.1, we deduce \( x_n/\alpha^n \to \infty \) \((n \to \infty)\) and have for each \( k \) and for \( n > k \)

\[
\left[D_{1/x} \tilde{B}D_x\right]_{nk} = \frac{1}{x_n} \alpha^{n-k-1} \frac{\delta}{r^2} x_k = \frac{1}{x_n} \alpha^{n-k} \left( \alpha^{k-1} \frac{\delta}{r^2} x_k \right) = o(1) \quad (n \to \infty).
\]

i) If \( \delta = 0 \) then the sum in (4.3) reduces to \( r'/r \) and inclusion (4.1) holds for all \( x \).

ii) If \( \delta \neq 0 \) then inclusion (4.1) means that (4.3) is convergent and

\[
\bar{c}_n(\alpha, x) \to -\frac{l - \frac{r'}{r}}{1 - \frac{r}{rs} \delta} \quad (n \to \infty),
\]
so we have \((x_n/\alpha^n)_n \in \widehat{C}\). By Lemma 3.1 we have \(\widehat{C} = \widehat{\Gamma}\), and so (4.2) is equivalent to
\[
\lim_{n \to \infty} \frac{x_{n-1} \alpha^n}{\alpha^{n-1} x_n} = \alpha \lim_{n \to \infty} \frac{x_{n-1}}{x_n} < 1.
\]
This shows ii).

□

The following result can easily be shown when \(r = 0\) or \(s = 0\).

Theorem 4.4. Let \(r, s, r'\) and \(s'\) be real numbers.

i) Let \(r \neq 0\) and \(s = 0\).
   a) If \(s' \neq 0\), then (4.1) holds if and only if
      \[
      \frac{x_{n-1}}{x_n} \to l \ (n \to \infty) \text{ for some scalar } l.
      \]
   b) If \(s' = 0\), then (4.1) holds for all \(x\).

ii) Let \(r = 0\) and \(s \neq 0\).
   a) If \(r' \neq 0\), then (4.1) holds if and only if
      \[
      \frac{x_n}{x_{n-1}} \to l' \ (n \to \infty) \text{ for some scalar } l'.
      \]
   b) If \(r' = 0\), then (4.1) holds for all \(x\).

iii) Let \(r = s = 0\).
    a) If \(r' \neq 0\), or \(s' \neq 0\), then (4.1) has no solution.
    b) If \(r' = s' = 0\), then (4.1) holds for all \(x\).

Proof. We only prove Part i), the proofs of the other parts are left to the reader.

i) Let \(r \neq 0\) and \(s = 0\).

Since \(B(r, s) = rI\) we have \(s_x^{(c)}(B(r, s)) = s_x^{(c)}\). So inclusion (4.1) is equivalent to \(D_{1/\delta}B(r', s')D_x \in (c, c)\). This means that there are \(K \geq 0\) and \(L\) such that
\[
(*) \quad \begin{cases}
|r'| + |s'| \frac{x_{n-1}}{x_n} \leq K \text{ for all } n, \\
r' + s' \frac{x_{n-1}}{x_n} \to L \ (n \to \infty).
\end{cases}
\]

a) If \(s' \neq 0\) then we have
   \[
   \frac{x_{n-1}}{x_n} \to \frac{L - r'}{s'} \ (n \to \infty).
   \]
   b) If \(s' = 0\) then the system (*) is satisfied for all \(x\).

□

In the general case when \(r, s, \delta, \alpha \neq 0\) we can state the following remark.
Remark. Condition (4.1) holds if and only if

\[(i) \quad \frac{\alpha_n}{x_n} \sum_{k=1}^{n-1} x_k \alpha_k \to l \quad (n \to \infty),\]

\[(ii) \quad \frac{|\alpha_n|}{x_n} \sum_{k=1}^{n-1} \frac{x_k}{|\alpha_k|^k} \leq K \text{ for all } n\]

and

\[(iii) \quad \frac{\alpha_n}{x_n} \to l' \quad (n \to \infty)\]

for some scalars \(l\) and \(l'\), and a constant \(K > 0\). This result is a direct consequence of condition (4.2) in the proof of Theorem 4.3.

5. The case of regularity

5.1. The set of all \(x \in U^+\) such that \(x_n^{-1}B(r,s)y_n \to l\) implies \(x_n^{-1}B(r',s')y_n \to l \quad (n \to \infty)\) for all \(y\) and for some \(l\). A matrix \(A \in (c,c)\) and the corresponding operator \(\Lambda\) are said to be regular if \(y_n \to l\) implies \(A_n y \to l \quad (n \to \infty)\) for all \(y \in \omega\) and for some scalar \(l\). We then write \(A \in (c,c)_{reg}\). As a direct consequence of Lemma 4.1, we have the known result (see, for instance, [15, Theorem 1.3.9])

**Lemma 5.1.** We have \(\Lambda \in (c,c)_{reg}\) if and only if the next statements hold,

a) \(\Lambda \in S_1\),

b) \(\sum_{k=1}^{\infty} \lambda_{nk} \to 1 \quad (n \to \infty)\),

c) \(\lambda_{nk} \to 0 \quad (n \to \infty)\) for \(k = 1, 2, \ldots\).

Now we consider the next question, where \(r, s, r'\) and \(s'\) are real numbers. What is the set of all \(x \in U^+\) such that

\[(5.1) \quad \frac{ry_n + sy_n-1}{x_n} \to l \text{ implies } \frac{r'y_n + s'y_n-1}{x_n} \to l \quad (n \to \infty)\]

for some scalar \(l'\)? The answer to this question is given by the following theorem where we confine our studies to the case \(-s/r > 0\) when \(\delta \neq 0\).

**Theorem 5.2.** Let \(r, s, r'\) and \(s'\) be real numbers.

i) Let \(\delta \neq 0\) and \(\alpha = -s/r > 0\).

a) If \(\tau = (r - r')/(s - s') \leq 0\), then (5.1) holds if and only if

\[\lim_{n \to \infty} \frac{x_{n-1}}{x_n} = -\tau.\]

b) If \(\tau > 0\), then (5.1) has no solutions.

ii) Let \(\delta = 0\) and \(r \neq 0\).

a) If \(r = r'\), then (5.1) holds for all \(x\).
b) If $r \neq r'$, then (5.1) has no solution.

Proof. First we note that statement (5.1) obviously means that

(5.2)

$$z_n = \left[ D_{1/x} B(r,s)y \right]_n \to l$$

for all $y$ and for some scalar $l$. Since $y = B^{-1}(r,s)D_xz$, for $r \neq 0$ statement (5.2) is equivalent to

$$z_n \to l \implies \left[ D_{1/x} \tilde{B}D_xz \right]_n \to l \quad (n \to \infty)$$

where $\tilde{B} = B(r',s')B^{-1}(r,s)$. Then (5.1) is equivalent to

(5.3)

$$D_{1/x} \tilde{B}D_x \in (c,c)_{\text{reg}},$$

which, by Lemma 5.1, is equivalent to

$$D_{1/x} \tilde{B}D_x \in S_1,$$

and

$$\sum_{k=1}^{n} \left[ D_{1/x} \tilde{B}D_x \right]_{nk} \to 1 \quad (n \to \infty),$$

Using this characterization of $(c,c)_{\text{reg}}$ and reasoning as in Theorem 4.3, we deduce that (5.3) holds if and only if

(5.4)

$$\sum_{k=1}^{n} \left[ D_{1/x} \tilde{B}D_x \right]_{nk} = \frac{r'}{r} - \frac{\delta}{rs} \tilde{c}_n(\alpha, x) \to 1 \quad (n \to \infty).$$

i) Now we can show a) and b).

Putting $z_n = x_n\alpha^{-n}$, we have

$$\tilde{c}_n(z) = \frac{1}{z_n} \sum_{k=1}^{n-1} z_k \to L \quad (n \to \infty),$$

where

(5.5)

$$L = \frac{1 - \frac{r'}{r}}{\frac{r'}{\delta} - \frac{r'}{rs}} \geq 0.$$
Using (5.5) we immediately obtain \( L/(L + 1) = -\alpha \tau \). We conclude

\[
\frac{x_{n-1}}{x_n} = \frac{z_{n-1}}{z_n} \frac{1}{\alpha} \rightarrow -\tau \geq 0 \ (n \rightarrow \infty).
\]

ii) If \( \delta = 0 \) the sum defined in (5.4) reduces to \( r'/r = 1 \), that is, \( r = r' \). We then have \( s = s' \) and (5.1) holds for all \( x \).

Now give a remark in which we consider a Tauberian problem using the operator of the generalized difference sequence.

**Remark.** If \( r > 1 \) or \( r < 0 \), then \( ry_n + (1 - r)y_{n-1} \rightarrow l \) implies \( y_n \rightarrow l \) \( (n \rightarrow \infty) \) for all \( y \) and for some scalar \( l \). Indeed, it is enough to take \( r' = 1 \), \( s' = 0 \) and \( x = e \) in Theorem 4.3. Then we have \( 1 = -(r - 1)/s \) with \(-s/r > 0\).

Now we consider the equivalence

\[
(5.6) \quad \frac{ry_n + sy_{n-1}}{x_n} \rightarrow l \quad \text{if and only if} \quad \frac{r'y_n + s'y_{n-1}}{x_n} \rightarrow l \quad (n \rightarrow \infty)
\]

and for some scalar \( l \). Note that in [3] we determined the set of all \( x \in U^+ \) such that \( s_{x}^{(c)}(\Delta) = s_{x}^{(c)} \). In [7] we gave a necessary and sufficient condition under which \( a, b \in U^+ \) satisfy \( s_{a}^{(c)}(\Delta) = s_{b}^{(c)} \). Since we have \( B(-1,1) = \Delta \) and \( B(1,0) = I \), then \( s_{x}^{(c)}(B(-1,1)) = s_{x}^{(c)}(\Delta) \) and \( s_{x}^{(c)}(B(1,0)) = s_{x}^{(c)} \). Thus we see that condition (5.6) is an extension of [3, 7].

We obtain the next result as a direct consequence of Theorem 5.2.

**Theorem 5.3.** Let \( r, s, r' \) and \( s' \) be real numbers, all different from zero.

i) Let \( \delta \neq 0 \) and \( r/s, r'/s' \leq 0 \).
   a) If \( \tau = (r - r')/(s - s') \leq 0 \), then the solutions of (5.6) are defined by

\[
\lim_{n \rightarrow \infty} \frac{x_{n-1}}{x_n} = -\tau.
\]

b) If \( \tau > 0 \), then (5.6) has no solutions.

ii) Let \( \delta = 0 \).
   a) If \( r = r' \), then (5.6) holds for all \( x \).
   b) If \( r \neq r' \), then (5.6) has no solution.

Now we deal with the case when \( r = 0 \) or \( s = 0 \).

**Theorem 5.4.** i) We assume \( r \neq 0 \) and \( s = 0 \).
   a) Let \( s' \neq 0 \).
      \( \alpha) \) If \( \tau_1 = (r - r')/s' \geq 0 \), then (5.1) holds if and only if

\[
(5.7) \quad \lim_{n \rightarrow \infty} \frac{x_{n-1}}{x_n} = \tau_1.
\]
β) If $\tau_1 < 0$, then (5.1) has no solution.

b) Let $s' = 0$.

α) If $r = r'$, then (5.1) holds for all $x$.

β) If $r \neq r'$, then (5.1) has no solution.

(ii) We assume $r = 0$ and $s \neq 0$.

a) Let $r' \neq 0$.

α) If $l = 0$, then (5.1) is equivalent to $(x_n/x_{n-1})_n \in \ell_\infty$.

β) If $l \neq 0$, then condition (5.1) holds if and only if

$$\lim_{n \to \infty} \frac{x_n}{x_{n-1}} = \frac{s - s'}{r'} \geq 0.$$  

b) Let $r' = 0$.

α) If $s' = s$, then (5.1) holds for all $x$.

β) If $s' \neq s$, then (5.1) has no solution.

(iii) Let $r = s = 0$.

a) If $r' \neq 0$, or $s' \neq 0$, then (5.1) has no solution.

b) If $r' = s' = 0$, then (5.1) holds for all $x$.

Proof. i) We assume $r \neq 0$ and $s = 0$. Since $B(r, s) = rI$, statement (5.1) is equivalent to $D_{1/x} B(r'/r, s'/r) D_x \in (c, c)_{\text{reg}}$, that is,

$$\left| \frac{r'}{r} \right| + \left| \frac{s'}{r} \right| \frac{x_{n-1}}{x_n} \leq K \text{ for all } n,$$

(5.8)

$$\frac{r'}{r} + \frac{s'}{r} \frac{x_{n-1}}{x_n} \to 1 \text{ } (n \to \infty).$$

(5.9)

a) Let $s' \neq 0$. Since condition (5.9) implies (5.8), statement (5.1) is equivalent to (5.7).

b) Let $s' = 0$.

α) If $r = r'$, then the previous system holds for all $x$.

β) If $r \neq r'$, then the system has no solution.

ii) We assume $r = 0$ and $s \neq 0$.

a) Let $r' \neq 0$. Then statement (5.1) reduces to

$$s \frac{y_{n-1}}{x_n} \to l \text{ implies } t_n = \frac{r'y_n + s'y_{n-1}}{x_n} \to l \text{ } (n \to \infty).$$

(5.10)

α) If $l = 0$, then we have

$$s^0_x (B(0, s)) = \left\{ y \in \omega : \frac{y_n}{x_{n+1}} = o(1) \text{ } (n \to \infty) \right\} = s_{x+},$$

where $x^+ = (x_{n+1})_n$. Then statement (5.1) with $l = 0$ is equivalent to $s^0_{x+} \subset s^0_x (B(r', s'))$, $B(r', s') \in (s^0_{x+}, s^0_x)$,
that is, to

\[(5.11)\quad |r'| \frac{x_{n+1}}{x_n} + |s'| \leq K \text{ for all } n.\]

Obviously the condition in (5.11) is equivalent to

\[(x_n/x_{n-1}) \in \ell_\infty.\]

\(\beta) \) If \( l \neq 0 \), we put \( z_n = s y_{n-1}/x_n \). Then (5.1) is equivalent to

\[z_n \to l \text{ implies } t_n = \frac{r'}{s} z_{n+1} + \frac{s'}{s} z_n \to l (n \to \infty),\]

that is, to

\[\frac{x_{n+1}}{x_n} = \frac{t_n - \frac{s'}{s} z_n}{r'} \to \frac{s - s'}{r'} (n \to \infty).\]

b) Let \( r' = 0 \). Then \( z_n = s y_{n-1}/x_n \to l \) implies \( s' y_{n-1}/x_n \to l = l s'/s \ (n \to \infty) \).

\(\alpha) \) If \( s' = s \), then statement (5.1) holds for all \( x \in U^+ \).

\(\beta) \) If \( s' \neq s \), then (5.1) has no solution.

iii) We assume \( r = s = 0 \). Then we must have \( B(r', s') \in (\omega, s_0^+) \) which implies \( r' = s' = 0 \). Indeed we assume either \( r' \neq 0 \) or \( s' \neq 0 \).

Let \( r' \neq 0 \). We consider the cases \( s'/r' \geq 0 \) and \( s'/r' < 0 \).

If \( s'/r' \geq 0 \), then we take \( y = (R^n x_n) \in \omega \) with \( R > 1 \), and obtain

\[\left| \frac{B(r', s') y_n}{x_n} \right| = \left| \frac{r'}{x_n} \right| y_n + \frac{s'}{r'} y_{n-1} \geq |r'| R^n \text{ for all } n.\]

Then we have \( |B(r', s') y_n/x_n| \to \infty \ (n \to \infty) \) and \( \omega \subset s_x(B(r', s')) \) is impossible.

If \( s'/r' < 0 \), then we take \( y_n = (-R)^n x_n \) with \( R > 1 \), and obtain

\[\left| \frac{B(r', s') y_n}{x_n} \right| = \left| \frac{r'}{x_n} \right| \left( y_n + \frac{s'}{r'} y_{n-1} \right) \geq |r'| R^n \left( 1 - \frac{s' x_{n-1}}{r' R x_n} \right) \]

\[\geq |r'| R^n \text{ for all } n,\]

and we conclude as above.

The case \( s' \neq 0 \) can be treated similarly.
5.2. Applications. Let $r < 0$ and $s > -1$, and different from 0 and consider the sets
\[
S_1(r) = \left\{ x \in U^+ : \frac{ry_n + y_{n-1}}{x_n} \to l \text{ implies } \frac{\Delta y_n}{x_n} \to l \ (n \to \infty) \right\}
\]
for all $y \in \omega$ and for some scalar $l$}
and
\[
S_2(s) = \left\{ x \in U^+ : \frac{\Delta y_n}{x_n} \to l \text{ implies } \frac{sy_n}{x_n} \to l \ (n \to \infty) \right\}
\]
for all $y \in \omega$ and for some scalar $l$.
We can determine the set $S_1(r) \cap S_2(s)$. Since $\delta = -r + 1 \neq 0$, we have by Theorem 5.2
\[
S_1(r) = \left\{ x \in U^+ : \frac{x_{n-1}}{x_n} \to \frac{1 - r}{2} \ (n \to \infty) \right\},
\]
and similarly
\[
S_2(s) = \left\{ x \in U^+ : \frac{x_{n-1}}{x_n} \to \frac{1}{1 + s} \ (n \to \infty) \right\}.
\]
We conclude
\[
S_1(r) \cap S_2(s) = \begin{cases} S_2(s) & \text{if } s = (1 + r)/(1 - r), \\ \emptyset & \text{otherwise.} \end{cases}
\]
Note that if $r < 0$, then $S_1(r) \cap S_2(s) \neq \emptyset$ implies $|s| < 1$ and $s \neq 0$.

6. The $\alpha$-Tauberian (SSIE) $s^{(c)}_x(B(r, s)) \subset s^{(c)}_a$

6.1. $\alpha$-Tauberian (SSIE) with operators of the form $B(r, s)$. Here we consider the $\alpha$-Tauberian (SSIE) problem for given $a \in U^+$, (see [6]), stated as follows. Let $r$, $s$, $r'$ and $s'$ be real numbers, and let $a$ be a given sequence; what is the set $S_a$ of all $x \in U^+$ such that
\[
\frac{ry_n + sy_{n-1}}{x_n} \to l \text{ implies } \frac{y_n}{a_n} \to l' \ (n \to \infty) \text{ for all } y,
\]
and for some scalars $l$ and $l'$? This statement is equivalent to the solvability of the (SSIE)
(6.1) $s^{(c)}_x(B(r, s)) \subset s^{(c)}_a$.
As we will see in Proposition 6.1, since the condition on the sequence $a$ is less restrictive for (6.1) than for the (SSIE) $s^{(c)}_a(B(r, s)) \subset s^{(c)}_x$ it is natural to begin with the study of the set $S_a$. To state the next result, we use the
set $cs_b$ of all $x \in U^+$ such that $\sum_{k=1}^{\infty} x_k/b_k < \infty$, where $b \in U^+$. For $b = e$ we obtain $cs_e = cs \cap U^+$. Throughout this section we assume $\alpha = -s/r > 0$.

**Proposition 6.1.** We assume $(\alpha^n/a_n)_n \in c$. Then $x \in S_a$ if and only if

\[(6.2) \quad \left(\frac{\alpha^n}{a_n} \sum_{k=1}^{n} \frac{x_k}{\alpha^k}\right)_n \in c.\]

Moreover if $a_n \sim \lambda \alpha^n$ ($n \to \infty$) for $\lambda > 0$, that is, $a_n/\lambda \alpha^n \to 1$ ($n \to \infty$), then we have

$S_a = cs(\alpha^n)_n$.

**Proof.** We have $x \in S_a$ if and only if (6.1) holds, which is equivalent to

\[(6.3) \quad B^{-1}(r,s) \in \left(s_{x}^{(c)}, s_{a}^{(c)}\right),\]

that is, to $D_{1/a}B^{-1}(r,s)D_x \in (c,c)$. From the expression of $B^{-1}(r,s)$ in the proof of Theorem 5.2, and the characterization of $(c,c)$, condition (6.3) is equivalent to (6.2) and $(\alpha^n/a_n)_n \in c$. Now we assume $a_n/\alpha^n \to \lambda > 0$ ($n \to \infty$). Then we have $x \in S_a$ if and only if

$$u_n = \frac{\alpha^n}{a_n} \sum_{k=1}^{n} \frac{x_k}{\alpha^k} \to L \ (n \to \infty)$$

for some scalar $L$, that is,

$$\sum_{k=1}^{n} \frac{x_k}{\alpha^k} = \frac{u_n}{\alpha^n} \to \frac{L}{\lambda} \ (n \to \infty),$$

and $x \in cs(\alpha^n)_n$. \hfill $\Box$

When $a = e$, we obtain the next Tauberian result.

**Corollary 6.2.** i) If $0 < \alpha \leq 1$, then $x \in S_e$ if and only if

$$\left(\frac{\alpha^n}{\alpha^n} \sum_{k=1}^{n} \frac{x_k}{\alpha^k}\right)_n \in c.$$

ii) If $\alpha = 1$, then $S_e = cs \cap U^+$.

As a direct application we also have the next result,

**Corollary 6.3.** We assume $0 < \alpha < 1$. Then $(x^n)_n \in S_e$ if and only if $0 < x \leq 1$. 

Proof. First we assume $x \neq \alpha$. Since $x_k = x^k$ for all $k$, we have $(x^n)_n \in S_c$ if and only if

$$\alpha^n \sum_{k=1}^{n} \frac{x_k}{\alpha^k} = \alpha^n \frac{x}{\alpha} \frac{1}{1 - \frac{x}{\alpha}} - \alpha^n \left( \frac{x}{\alpha} \right)^{n+1} \frac{1}{1 - \frac{x}{\alpha}}$$

is convergent as $n$ tends to infinity, that is, for $0 < x \leq 1$ and $x \neq \alpha$. If $x = \alpha < 1$, we have $\alpha^n \sum_{k=1}^{n} (x/\alpha)^k = n\alpha^n = o(1) \ (n \to \infty)$.

We immediately deduce the next examples.

Example. Let $u, v > 0$. Then $x \in U^+$ satisfies the condition

$$\frac{uy_n - vy_{n-1}}{x_n} \to l \text{ implies } \left( \frac{u}{v} \right)^n y_n \to l' \ (n \to \infty)$$

for all $y$ and for some scalars $l$ and $l'$, if and only if $\sum_{k=1}^{\infty} (u/v)^k x_k < \infty$. This result can be obtained writing $\alpha = v/u$ and $a_n = \alpha^n$ in Proposition 6.1. In particular, if $u = v = 1$, then the set of all $x \in U^+$ such that

$$\frac{\Delta y_n}{x_n} \to l \text{ implies } y_n \to l' \ (n \to \infty)$$

is equal to $cs \cap U^+$.

Remark. We obtain a similar result when $a$ and $x$ are interchanged in (SSIE) (6.1). Indeed, let $a \in cs(\alpha_n)_n$ and let $\overline{S}_a$ be the set of all $x \in U^+$ such that the (SSIE) $\overline{S}_a^{(c)}(B(r, s)) \subset \overline{S}_x^{(c)}$ holds. Then $x \in \overline{S}_a$ if and only if

$$\left( \frac{\alpha^n}{x_n} \right)_n \in c.$$  \hfill (6.4)

This result follows from the fact that here the condition $D_{1/x}B^{-1}(r, s)D_a \in (c, c)$ is equivalent to (6.4) and

$$\left( \frac{\alpha^n \sum_{k=1}^{n} a_k}{x_n \alpha^k} \right)_n \in c,$$  \hfill (6.5)

and we conclude since (6.4) implies (6.5).

We immediately deduce the following Tauberian result.
Remark. If $a \in cs(\alpha_n)_n$, then
\[
\frac{B(r, s)y_n}{a_n} \to l \text{ implies } y_n \to l' \quad (n \to \infty)
\]
for all $y$ and for some scalars $l$ and $l'$, if and only if
\[
0 < -s/r \leq 1.
\]
This result comes from the fact that $e \in \overline{S}_a$ if and only if (6.6) holds.

6.2. The case of the operator of the first difference.

6.2.1. The general case. If $r = -s = 1$, then we obtain $B(r, s) = \Delta$. We confine our studies to the case when $a_n \to \infty$ ($n \to \infty$). We denote by $\tilde{S}_a$ the set of all $x \in U^+$ such that
\[
\frac{\Delta y_n}{x_n} \to l \text{ implies } \frac{y_n}{a_n} \to l' \quad (n \to \infty)
\]
for all $y$ and for some scalars $l$ and $l'$.

We state the next elementary result.

**Proposition 6.4.** We assume $a_n \to \infty$ ($n \to \infty$). Then the set $\tilde{S}_a$ is equal to the set of all $x \in U^+$ such that
\[
\frac{1}{a_n} \sum_{k=1}^{n} x_k \to L \quad (n \to \infty)
\]
for some scalar $L$; moreover we have $l' = lL$ in (6.7).

**Proof.** It is enough to apply Proposition 6.1 with $\alpha = 1$, and $a^n/a_n = 1/a_n \to 0$ ($n \to \infty$). By Lemma 4.1, we have $l' = lL$. \hfill \Box

6.2.2. Applications to the case when $a_n = n^{\beta + 1}$ with $\beta > -1$, or $a_n = \ln n$.

It is well known that if $\xi > -1$, then
\[
\sum_{k=1}^{n} k^\xi \sim \frac{n^{\xi+1}}{\xi+1} \quad (n \to \infty).
\]

The next result is a direct consequence of Proposition 6.4 and (6.9).

**Corollary 6.5.** Let $\beta$ be a real number.

i) If $\beta > -1$, then
\[
\frac{\Delta y_n}{n^{\beta}} \to l \text{ implies } \frac{y_n}{n^{\beta+1}} \to \frac{l}{\beta+1} \quad (n \to \infty)
\]
for all $y$ and for some scalar $l$. 
ii) If \( \beta = -1 \), then

\[
\frac{\Delta y_n}{n^\beta} = n \Delta y_n \to l \text{ implies } \frac{y_n}{\ln n} \to l \quad (n \to \infty)
\]

for all \( y \) and for some scalar \( l \).

**Proof.**

i) Part i) is a direct consequence of Proposition 6.4 and (6.9), since

\[
v_n = \frac{1}{n^{\beta+1}} \sum_{k=1}^{n} k^\beta \to \frac{1}{\beta+1} \quad (n \to \infty).
\]

ii) Trivially we have

\[
1 + \ln \left(\frac{n+1}{2}\right) = 1 + \int_{2}^{n+1} \frac{dx}{x} \leq s_n = \sum_{k=1}^{n} \frac{1}{k} \leq 1 + \int_{1}^{n} \frac{dx}{x}
\]

\[
= 1 + \ln n \quad \text{for all } n.
\]

We immediately deduce that \( s_n/\ln n \to 1 \quad (n \to \infty) \) and \( n \Delta y_n \to l \)

imply

\[
\frac{y_n}{\ln n} \to l \lim_{n \to \infty} s_n / \ln n = l \quad (n \to \infty)
\]

for all \( y \).

□

As a direct consequence of the preceding result we obtain,

**Corollary 6.6.**

i) If \( \beta > -1 \), then

\[
y_n - \left(1 - \frac{1}{n}\right)^\beta y_{n-1} \to L \text{ implies } \frac{y_n}{n} \to \frac{L}{\beta+1} \quad (n \to \infty)
\]

for all \( y \).

ii) If \( \beta = -1 \), then

\[
y_n - \left(1 - \frac{1}{n}\right)^\beta y_{n-1} = y_n - \frac{n}{n-1} y_{n-1} \to L
\]

implies

\[
\frac{y_n}{n \ln n} \to L \quad (n \to \infty)
\]

for all \( y \).
References


[5] de Malafosse, B., *Sum of sequence spaces and matrix transformations mapping in $s_0^\alpha \left((\Delta - \lambda I)^h + s_\beta^{(c)} \left((\Delta - \mu I)^l \right)\right)$, Acta Math. Hung. 122 (2008) 217-230


[9] de Malafosse, B., Malkowsky, E., *Matrix transformations in the sets $\chi (N^p N_q)$ where $\chi$ is in the form $s_c$, or $s_0^\alpha$, or $s_\beta^{(c)}$, Filomat 17 (2003) 85-106.


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