IRREDUCIBILITIES OF THE INDUCED CHARACTERS
OF CYCLIC $p$-GROUPS

KATSUSUKE SEKIGUCHI

Abstract. We denote by $C_n$ the cyclic group of order $p^n$, where
$p$ is an odd prime. Let $\phi$ be a faithful irreducible character of $C_n$.
In this paper, we study the $p$-group $G$ containing $C_n$ such that the
induced character $\phi^G$ is also irreducible. The purpose of this paper
is to determine such groups $G$ in the case when $G$ has a subgroup $H$
containing $C_n$ such that $C_n \triangleleft H$ and $[G : H] = p$.

1. Introduction

Let $G$ be a finite group. We denote by $\text{Irr}(G)$ the set of complex
irreducible characters of $G$ and by $\text{Firr}(G)$ ($\subset \text{Irr}(G)$) the set of faithful
irreducible characters of $G$.

For a prime $p$, we denote by $C_n$ the cyclic group of order $p^n$. A
finite group $G$ is called an M-group, if every $\phi \in \text{Irr}(G)$ is induced from
linear character of a subgroup of $G$.

It is well-known that every nilpotent group is an M-group. So, for
any $\chi \in \text{Irr}(G)$, where $G$ is a $p$-group, there exists a subgroup $H$ of $G$
and the linear character $\phi$ of $H$ such that $\phi^G = \chi$. If we set $N = \text{Ker}\phi$,
then $N \triangleleft H$ and $\phi$ is a faithful irreducible character of $H/N \cong C_n$, for
some non-negative integer $n$. In this paper, we will consider the case when
$N = 1$, that is, $\phi$ is a faithful linear character of $H \cong C_n$.

We consider the following:

Problem. Let $p$ be an odd prime, and $\phi$ be a faithful irreducible
character of $C_n$. Determine the $p$-group $G$ such that $C_n \subset G$ and the
induced character $\phi^G$ is also irreducible.

Since all faithful irreducible character of $C_n$ are algebraically conjugate
to each other, the irreducibility of $\phi^G$ ($\phi \in \text{Firr}(C_n)$) is independent
of the choice of $\phi$, but depends only on $n$.

Recently, Iida [2] solved this problem in the case when $C_n$ is a normal
subgroup of $G$. 

27
The purpose of this paper is to solve this problem in the case when $G$ has a subgroup $H$ containing $C_n$ such that $C_n \triangleleft H$ and $[G : H] = p$.

The problem of this type was considered by Yamada and Iida [3]. They studied the 2-groups $G$ such that $H \subset G$ and the similar properties of our problem hold, where $H = Q_n$ or $D_n$ or $SD_n$. Here, we denote by $Q_n$ and $D_n$ the generalized quaternion group and the dihedral group of order $2^{n+1}(n \geq 2)$, respectively, and by $SD_n$ the semidihedral group of order $2^{n+1}(n \geq 3)$.

Throughout this paper, $\mathbb{Z}$, and $\mathbb{N}$ denote the rational integers and the natural numbers, respectively.

2. Statements of the results

For the rest of this paper, we assume that $p$ is an odd prime.

First, we introduce the following groups:

(i): $G(n,m) = \langle a, u_m \rangle$ with
$$a^{p^n} = u_m^{p^m} = 1, \quad u_m a u_m^{-1} = a^{1+p^{n-m}}, \quad (m \leq n - 1).$$

(ii): $G(n,m,1) = \langle a, u_m, v \rangle$ ($\triangleright G(n,m) = \langle a, u_m \rangle$) with
$$a^{p^n} = u_m^{p^m} = 1, \quad u_m a u_m^{-1} = a^{1+p^{n-m}}, \quad v a v^{-1} = a^{1+p^{n-m-1}} u_m^{p^m-1},$$
$$v^p = u_m, \quad v u_m v^{-1} = u_m, \quad (2m \leq n - 1).$$

We can see that $G(n,m,1)$ is an extension group of $G(m,n)$ by using Proposition 1 below:

**Proposition 1.** Let $N$ be a finite group such that $G \triangleright N$ and $G/N = \langle uN \rangle$ is a cyclic group of order $m$. Then $u^m = c\in N$. If we put $\sigma(x) = uxu^{-1}$, $x \in N$, then $\sigma \in \text{Aut}(N)$ and (i) $\sigma^m(x) = cxc^{-1}$, $(x \in N)$ (ii) $\sigma(c) = c$.

Conversely, if $\sigma \in \text{Aut}(N)$ and $c \in N$ satisfy (i) and (ii), then there exists one and only one extension group $G$ of $N$ such that $G/N = \langle uN \rangle$ is a cyclic group of order $m$ and $\sigma(x) = vxv^{-1}$ $(x \in N)$ and $v^m = c$.

*Proof.* For instance, see [4, III, §7] \qed

We state the theorem of Iida ([2]).

**Theorem 0** (Iida [2]). Let $G$ be a $p$-group which contains $C_n$ as a normal subgroup of index $p^m$. Let $\phi \in \text{FIrr}(C_n)$. Suppose that $\phi^G \in \text{Irr}(G)$. Then $G \cong G(n,m)$.

In particular, when $C_n \subset G$ and $[G : C_n] = p$, $C_n$ is always a normal subgroup of $G$, so we have

**Corollary 0.** Let $\phi \in \text{FIrr}(C_n)$. Suppose that $C_n \subset G$ such that $[G : C_n] = p$ and $\phi^G \in \text{Irr}(G)$. Then $G \cong G(n,1)$. 

Our main theorem is the following:

**Theorem.** Let $G$ be a $p$-group which contains $C_n$ with $[G : C_n] = p^{m+1}$, where $p$ is an odd prime. Let $\phi \in \text{F Irr}(C_n)$. Suppose that $\phi^G \in \text{Irr}(G)$, and $n-3 \geq 2m$. Further, suppose that there exists a subgroup $H$ of $G$ such that $H \triangleright C_n$ and $[G : H] = p$. Then $G \cong G(n, m+1)$ or $G(n, m, 1)$.

**Corollary.** Let $G$ be a $p$-group which contains $C_n$ with $[G : C_n] = p^2$. Let $\phi \in \text{F Irr}(C_n)$. Suppose that $\phi^G \in \text{Irr}(G)$ and $n \geq 5$. Then $G \cong G(n, 2)$ or $G(n, 1, 1)$.

3. SOME PRELIMINARY RESULTS

In this section, we state some results concerning the criterion of the irreducibilities of induced characters and others, which we need in section 4.

We denote by $\zeta = \zeta_{p^n}$ a primitive $p^n$th root of unity. It is known that, for $C_n = \langle a \rangle$, there are $p^m$ irreducible characters $\phi_\nu$ ($1 \leq \nu \leq p^n$) of $C_n$:

$$\phi_\nu(a^i) = \zeta^{\nu i}, \quad (1 \leq i \leq p^n).$$

The irreducible character $\phi_\nu$ is faithful if and only if $(\nu, p) = 1$.

First, we state the following result of Shoda (cf [1, p.329]):

**Proposition 2.** Let $G$ be a group and $H$ be a subgroup of $G$. Let $\phi$ be a linear character of $H$. Then the induced character $\phi^G$ of $G$ is irreducible if and only if, for each $x \in G - H = \{ g \in G \mid g \notin H \}$, there exists $h \in xHx^{-1} \cap H$ such that $\phi(h) \neq \phi(x^{-1}hx)$. In particular, when $\phi$ is faithful, the condition $\phi(h) \neq \phi(x^{-1}hx)$ is equivalent to that of $h \neq x^{-1}hx$.

Using this result, we have the following:

**Proposition 3.** Let $\langle a \rangle = C_n \subset G$, and $\phi$ be a faithful irreducible character of $C_n$. Then the following conditions are equivalent

1. $\phi^G$ is irreducible.
2. For each $x \in G - C_n$, there exists $y \in \langle a \rangle \cap x\langle a \rangle x^{-1}$ such that $yx^{-1} \neq y$.

**Definition.** When the condition (2) of Proposition 3 holds, we say that $G$ satisfies $(EX, C)$.

Finally, we state the following:

**Lemma 1.** Let $p$ be an odd prime and $n, m, k, j$ be integers satisfying $0 \leq m \leq n$. Then, if we put $s = 1 + kp^{n-m}$, we have the following equality:
\[
\frac{s^{jp^m} - 1}{s^j - 1} \equiv p^m \pmod{p^n}.
\]

4. Proof of Theorem

Let \( G \supseteq H \) be a \( p \)-group as is stated in Theorem, and let \( \phi \in \text{FIrr}(C_n) \). Since \( \phi^G = (\phi^H)^G \in \text{Irr}(G) \), we must have \( \phi^H \in \text{Irr}(H) \). Therefore, by Theorem 0, we can take an element \( u_m \) in \( H \) such that \( H = \langle a, u_m \rangle \cong G(n, m) \). For the sake of simplicity, we write \( u \) instead of \( u_m \). Since \([G : H] = p\), we may write as

\[
G = \langle H, y \rangle (> H),
\]

where \( y \in G - H = \{ g \in G \mid g \notin H \} \) and \( y^p \in H \).

Note that any element in \( H = \langle a, u \rangle \) is represented as \( a^i u^j \) for some \( i, j \in \mathbb{Z}, \ 0 \leq i \leq p^n - 1, \ 0 \leq j \leq p^m - 1 \).

Further, if we put \( s = 1 + p^{n-m} \), we have

\[
(a^i u^j)^p = a^{i (\frac{p^m j - 1}{s^j - 1})} u^{p^m j} = a^{p^m i};
\]

by Lemma 1.

First, we consider the elements \( yay^{-1} \) and \( yuy^{-1} \).

We will show the following

**Claim I.** We can write as

\[
yay^{-1} = a^{i+k p^{n-m} - 1} u^{p^n - 1} j,
\]

\[
yuy^{-1} = a^{p^n - m} d u,
\]

for some \( k, j, d \in \mathbb{Z} \).

**Proof of Claim I.** Write \( yay^{-1} = a^{i_0} u^{j_0} \) and \( yuy^{-1} = a^{d_0} u^{t_0} \). Since

\[
yap^{m} y^{-1} = a^{p^m t_0},
\]

we must have \( (p, i_0) = 1 \).

On the other hand, since

\[
1 = yu^{-1} y^{-1} = a^{d_0 p^n},
\]

we have

\[
d_0 \equiv 0 \pmod{p^{n-m}}.
\]

Therefore, we may write \( d_0 = p^{n-m} d \) and

\[
yuy^{-1} = a^{p^n - m} d u^{t_0},
\]

for some \( d \in \mathbb{Z} \). Since \( n - m \geq m \), by our assumption, we have

(1)

\[
yap^{n-m} y^{-1} = a^{p^n - m} t_0.
\]
Taking the conjugate of both sides of the equality, \( uau^{-1} = a^{1+p^{n-m}} \) by \( y \), we get
\[
(a^{p^{n-m}d}u^0)(a^{i_0}u^{j_0})(a^{p^{n-m}d}u^0)^{-1} = a^{i_0}u^{j_0}a^{p^{n-m}i_0}.
\]
Hence, we have
\[
a^{i_0(1+p^{n-m})}u^{j_0} = a^{i_0(1+p^{n-m})}u^{j_0}.
\]
Therefore,
\[
i_0(1 + t_0 \cdot p^{n-m}) \equiv i_0(1 + p^{n-m}) \pmod{p^n}.
\]
But \((i_0, p) = 1\), so we get \( t_0 \equiv 1 \pmod{p^m} \), and hence
\[
yuy^{-1} = a^{p^{n-m}d}u.
\]
For a normal subgroup \( N \) of \( G \), and any \( g, h \in G \), we write
\[
g \equiv h \pmod{N}
\]
when \( gh^{-1} \in N \).

Note that \( \langle a^{p^{n-m}} \rangle \) is a normal subgroup of \( G \), by (1).

It is easy to see that
\[
yuy^{-1} \equiv u \pmod{\langle a^{p^{n-m}} \rangle}.
\]

Further, we have
\[
y^{a}u^{-1} = (a^{i_0}u^{j_0})^l \equiv a^{iol}u^{jol} \pmod{\langle a^{p^{n-m}} \rangle}.
\]
for any \( l \in \mathbb{N} \).

Using these relations repeatedly, we get
\[
y^{s}ay^{-s} \equiv a^{i_0}u^{j_0(i_0^{-1}+\cdots+i_0+1)} \pmod{\langle a^{p^{n-m}} \rangle},
\]
for any \( s \in \mathbb{N} \).

In particular,
\[
y^{p}ay^{-p} \equiv a^{i_0}u^{j_0(i_0^{-1}+\cdots+i_0+1)} \pmod{\langle a^{p^{n-m}} \rangle}.
\]

Hence we may write as
\[
y^{p}ay^{-p} = a^{i_0+p^{n-m}}u^{j_0(i_0^{-1}+\cdots+i_0+1)},
\]
for some integer \( r \).

Since \( y^p \in H = \langle a, u \rangle \), we must have
\[
i_0^p \equiv 1 \pmod{p^{n-m}},
\]
and
\[
j_0(i_0^{-1} + \cdots + i_0 + 1) \equiv 0 \pmod{p^m}.
\]
By (2), we can write as \( i_0 = 1 + kp^{n-m-1} \), for some integer \( k \). So, \( j_0(i_0^{p-1} + \cdots + i_0 + 1) = j_0(i_0^{\frac{p-1}{i_0}} - 1) \equiv j_0p \quad (\text{mod } p^{n-m}). \)

Since \( n - m \geq m \), by our assumption, we have \( j_0p \equiv 0 \quad (\text{mod } p^m) \), by (3). Therefore we can write \( j_0 = p^{m-1}j \), for some \( j \in \mathbb{Z} \). Thus the proof of Claim I is completed. \( \square \)

Hence, in order to prove the theorem, we have only to consider the following two cases:

**Case I.** \( yay^{-1} = a^{1+kp^{n-m-1}}, \) and \( yuy^{-1} = a^{p^{n-m}d_u} \),

**Case II.** \( yay^{-1} = a^{1+kp^{n-m-1}}wp^{m-1}j, (j, p) = 1 \), and \( yuy^{-1} = a^{p^{n-m}d_u}. \)

First, we consider Case I. But in this case we can see that \( G \triangleright C_n \). Hence, by Iida's result, we have \( G \cong G(n, m+1) \).

Next, we consider Case II.

In this case, we have \( y(a)y^{-1} \cap \langle a \rangle = \langle a^p \rangle \), because \( ya^p y^{-1} = (a^{1+kp^{n-m-1}}wp^{m-1}j)^p = a^{(1+kp^{n-m-1})p} \in \langle a^p \rangle \).

Suppose that \( k \equiv 0 \pmod{p} \), then there exists \( s_0 \in \mathbb{Z}, 0 \leq s_0 \leq p^m - 1 \), such that \( (u^{s_0}y)a^p(u^{s_0}y)^{-1} = a^p \).

This contradicts the hypothesis that the condition (EX,C) holds. So, we must have \( (k, p) = 1 \).

Next, we consider the element \( y^p \in H = \langle a, u \rangle \). Write \( y^p = a^{l_0}u^{k_0} \).

Since \( ya^{p^{n-m+1}}y^{-1} = (a^{1+kp^{n-m-1}}wp^{m-1}j)^{p^{n-m+1}} = a^{p^{n-m+1}} \), we have \( ya^{p^t}y^{-1} = a^{p^t} \), for any \( t \geq m + 1 \). In particular, since \( n - m - 1 \geq m + 1 \), by our assumption, we have \( ya^{p^{n-m-1}}y^{-1} = a^{p^{n-m-1}}. \)
By a direct calculation, we have
\[ y^p a y^{-p} = a^{1+kp^{n-m}+p^{n-1}d(1+2+\cdots+(p-1))} u^{p^m} j = a^{1+kp^{n-m}}. \]
On the other hand, we have
\[ y^p a y^{-p} = (a^{l_0} u^{k_0}) a (a^{l_0} u^{k_0})^{-1} = a^{1+k_0 p^{n-m}}. \]
Hence, we have
\[ k \equiv k_0 \pmod{p^m}, \]
so, we may write
\[ y^p = a^{l_0} u^k. \]
We show the following

**Claim II.** There exists an integer \( e \), such that \( (a^e y)^p = u^k \).

**Proof of Claim II.** Since
\[ y a^{l_0} y^{-1} = a^{l_0} u^{p^{n-1}j l_0} \pmod{(a^{p^{n-1}})}, \]
we have
\[ a^{l_0} u^k = y^p = y y^p y^{-1} = y (a^{l_0} u^k) y^{-1} \equiv a^{l_0} u^{p^{n-1}j l_0 + k} \pmod{(a^{p^{n-1}})}. \]
Therefore we have
\[ j l_0 \equiv 0 \pmod{p}. \]
But, \((j, p) = 1\), by our assumption, so
\[ l_0 \equiv 0 \pmod{p}. \]
Hence we may write as \( l_0 = p l \) and
\[ y^p = a^{p l} u^k \]
for some \( l \in \mathbb{Z} \).

By a direct calculation, we get
\[ (a^s y)^p \equiv a^{p s} u^{p^{n-1} j s (1+2+\cdots+(p-1))} y^p = a^{p s} u^{p^{n-1} j s \frac{p(p-1)}{2}} y^p = a^{p(s+1)} u^k \pmod{(a^{p^{n-1}})}, \]
for any \( s \in \mathbb{N} \).
Therefore we may write as
\[ (a^s y)^p = a^{p(s+l+p^{n-2} \beta_{p,s})} u^k \]
for some integer $\beta_{p,s}$. Note that $\beta_{p,s}$ is not independent of the choice of $s$. If we set $y_1 = a^{-l}y$, we can write as

$$y_1^p = a^{\beta_{p,n-1}}u^k,$$

for some integer $\beta$. Further, set $e = -\beta_{p,n-2} - l$, and

$$y_2 = a^{e}y = a^{-\beta_{p,n-2} - l}y = a^{-\beta_{p,n-2}}y_1.$$

Since $n - m - 2 \geq m + 1$, by our assumption, we have

$$y_1 a^{p^{n-m-2}}y_1^{-1} = a^{p^{n-m-2}}.$$

So,

$$(a^{e}y)^p = y_2^p = (a^{-\beta_{p,n-2} - l}y_1)^p = a^{-\beta_{p,n-2}}y_1^p = u^k.$$

Thus the proof of Claim II is completed. \qed

Since $(k, p) = 1$, there exists $k' \in \mathbb{Z}$, such that $kk' \equiv 1 \pmod{p^m}$. Hence

$$y_2^{k'} = u^{kk'} = u.$$

Therefore

$$y_2u^{k'-1} = u.$$

Further, we have

$$y_2a^{y_2^{-1}} = a^{-\beta_{p,n-2} - l}a^{y_2^{-1}a^l + \beta_{p,n-2}}$$

$$= a^{-l}a^{a^{1+kp^{n-1}u^{p^{m-1}j}}}a^{l}$$

$$= a^{1+(k+ljp^m)p^{n-1}u^{p^{m-1}j}}.$$

If we set $k_1 = k + lj p^m$, then

$$y_2a^{y_2^{-1}} = a^{1+k_1p^{n-1}u^{p^{m-1}j}},$$

and

$$y_2^p = u^k = u^{k_1}.$$

Summarizing the results, we have

$$y_2a^{y_2^{-1}} = a^{1+k_1p^{n-1}u^{p^{m-1}j}},$$

$$y_2^p = u^{k_1},$$

$$y_2u^{y_2^{-1}} = u.$$

There exists an integer $l_1$, such that

$$l_1k_1 \equiv 1 \pmod{p^{m+1}}.$$

Set $y_3 = y_2^{l_1}$. Since $n - m - 1 \geq m + 1$, by our assumption, we have

$$y_2a^{p^{n-m-1}}y_2^{-1} = a^{p^{n-m-1}}$$
Hence,
\[
y_3y_3y_3^{-1} = y_2y_2^{-1} = a^{1+p^{n-m-1}k_1}u^{p^{m-1}l_1j} = a^{1+p^{n-m-1}}u^{p^{m-1}l_1j}.
\]
and
\[
y_3^p = y_2^{pl_1} = u^{k_1l_1} = u, \quad y_3uy_3^{-1} = u.
\]
Take an integer \( s_1 \), such that
\[
l_1js_1 \equiv 1 \pmod{p}.
\]
Then
\[
\begin{align*}
y_3a^{s_1}y_3^{-1} &= (a^{1+p^{n-m-1}}u^{p^{m-1}l_1j})^{s_1} \\
&= a^{(1+p^{n-m-1})(s_1 + l_1jp^{n-1} + 1)}u^{l_1js_1p^{m-1}} \\
&= a^{(1+p^{n-m-1})(s_1 + l_1jp^{n-1} + 1)}u^{l_1js_1p^{m-1}} \\
&= a^{s_1(1+p^{n-m-1}+k_2p^{n-1})u^{p^{m-1}},}
\end{align*}
\]
where, \( k_2 = \frac{s_1(s_1-1)}{2}l_1j^2 \).
Set \( a_1 = a^{s_1} \), then
\[
\langle a \rangle = \langle a^{s_1} \rangle
\]
and
\[
y_3a_1y_3^{-1} = a_1^{1+p^{n-m-1}+k_2p^{n-1}}u^{p^{m-1}}.
\]
Further,
\[
a_1p^n = 1, \quad ua_1u^{-1} = a_1^{1+p^{n-m}}, \quad u^{p^m} = 1,
\]
Finally, we set
\[
y_4 = u^{-k_2p^{m-1}}y_3.
\]
Then,
\[
\begin{align*}
y_4a_1y_4^{-1} &= u^{-k_2p^{m-1}}y_3a_1y_3^{-1}u^{k_2p^{m-1}} \\
&= u^{-k_2p^{m-1}}(a_1^{1+p^{n-m-1}+k_2p^{n-1}}u^{p^{m-1}})u^{k_2p^{m-1}} \\
&= a_1^{1+p^{n-m-1}}u^{p^{m-1}},
\end{align*}
\]
and
\[
y_4^p = y_4^p = u, \quad y_4uy_4^{-1} = u.
\]
Therefore, \( G \) is generated by \( a_1, u \) and \( y_4 \) with relations (4), (5) and (6). These relations are the same as that of \( G(n, m, 1) \). Hence
\[
G = \langle a_1, u, y_4 \rangle \cong G(n, m, 1),
\]
as desired. This completes the proof of Theorem.
ACKNOWLEDGEMENT. The author would like to express his gratitude to the referee for his careful reading and pertinent suggestion.

REFERENCES


KATSUSUKE SEKIGUCHI
DEPARTMENT OF CIVIL ENGINEERING
FACULTY OF ENGINEERING
KOKUSHIKAN UNIVERSITY
4-28-1 SETAGAYA SETAGAYA-KU
TOKYO154-8515 JAPAN

(Received January 17, 2000)