A GENERALIZATION OF THE DADE'S THEOREM ON
LOCALIZATION OF INJECTIVE MODULES

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In general the localization does not preserve the injectivity. E. C. Dade
gave a necessary and sufficient condition which assures the preservation of
the injectivity for commutative rings in [1] Theorem 13. Recently one of
the authors tried to generalize it to non commutative rings in [3]. But it
does not seem to be enough sufficient. So we retried to solve this problem.
We refer to [2] on the terminologies and notations mainly.

Let $A$ be a ring not necessary commutative, $\mathcal{F}$ a class of right ideals
of $A$ which defines a hereditary torsion theory on the category of right
$A$-modules. First, we refer to [2] Chapter IX, Proposition 2.7, namely,

Let $M$ be a torsion free $A$-module. Then the localization $M_\mathcal{F}$ of $M$ is
$A_\mathcal{F}$-injective if and only if the following holds:

For any right ideal $I$ of $A$ and any homomorphism $g : I \to M$
there exists $B \in \mathcal{F}$ containing $I$ and a homomorphism $p : B \to$
$M$ such that $p|_I = g$.

We start with the following diagram:

\[
\begin{array}{c}
B \\
i \\
I \\
g \\
0 \longrightarrow t(E) \overset{\alpha}{\longrightarrow} E \overset{\beta}{\longrightarrow} \tilde{E} \longrightarrow 0
\end{array}
\]

where $E$ is an injective module, $t(E)$ the $\mathcal{F}$-torsion submodule of $E$, $\tilde{E} = E/t(E)$ and $I$ and $B$ are right ideals of $A$. From this we can construct the
following row exact commutative diagram:
where $F$ and $G$ are projective modules. Furthermore if there was given a homomorphism $p : B \to E$ then we obtain the following:

\begin{equation}
\begin{array}{ccccccccc}
0 & \rightarrow & H & \overset{\alpha_2}{\rightarrow} & G & \overset{\beta_2}{\rightarrow} & B & \overset{0}{\rightarrow} \\
& & k & & j & & i & & \\
0 & \rightarrow & K & \overset{\alpha_1}{\rightarrow} & F & \overset{\beta_1}{\rightarrow} & I & \overset{0}{\rightarrow} \\
& & e & & f & & g & & \\
0 & \rightarrow & t(E) & \overset{\alpha}{\rightarrow} & E & \overset{\beta}{\rightarrow} & E & \overset{0}{\rightarrow}
\end{array}
\end{equation}

in which the rectangular parts are commutative.

**Lemma 1.** In the above diagram if $g = pi$ holds then there exists a homomorphism $u : F \to t(E)$ such that $e = rk + u\alpha_1$.

As the proof is an easy exercise we shall omit it.

Next we start from the following row exact commutative diagram:

\begin{equation}
\begin{array}{ccccccccc}
0 & \rightarrow & H & \overset{\alpha_2}{\rightarrow} & G & \overset{\beta_2}{\rightarrow} & B & \overset{0}{\rightarrow} \\
& & k & & j & & i & & \\
0 & \rightarrow & K & \overset{\alpha_1}{\rightarrow} & F & \overset{\beta_1}{\rightarrow} & I & \overset{0}{\rightarrow} \\
& & e & & f & & g & & \\
0 & \rightarrow & t(E) & \overset{\alpha}{\rightarrow} & E & \overset{\beta}{\rightarrow} & E & \overset{0}{\rightarrow}
\end{array}
\end{equation}
where \( E \) is injective and \( i \) is a monomorphism. Furthermore if there was given a homomorphism \( r : H \to t(E) \), then we obtain the following diagram:

\[
\begin{array}{ccccccccc}
O & \to & H & \xrightarrow{\alpha_2} & G & \xrightarrow{\beta_2} & B & \to & 0 \\
\downarrow{k} & & \downarrow{j} & & \downarrow{i} & & & & \\
0 & \to & K & \xrightarrow{\alpha_1} & F & \xrightarrow{\beta_1} & I & \to & 0 \\
\downarrow{e} & \downarrow{r} & \downarrow{f} & \downarrow{g} & \downarrow{p} & & & & \\
0 & \to & t(E) & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & E & \to & 0
\end{array}
\]

(2)

**Lemma 2.** In the above diagram, if there exists a map \( u : F \to t(E) \) such that \( e = rk + u\alpha_1 \), then there exists a homomorphism \( p_0 : B \to \hat{E} \) such that \( g = p_0i \).

**Proof.** By the commutativity of the diagram there holds

\[
0 = \alpha(e - rk - u\alpha_1) = f\alpha_1 - q\alpha_2k - au\alpha_1 = (f - qj - \alpha u)\alpha_1.
\]

Therefore \( f - qj - \alpha u \) induces a homomorphism \( v : I \to E \) such that \( f - qj - \alpha u = v\beta_1 \) holds. Then there holds

\[
0 = \beta(f - qj - \alpha u - v\beta_1) = g\beta_1 - p\beta_2j - \beta v\beta_1 = (g - pi - \beta v)\beta_1.
\]

As \( \beta_1 \) is an epimorphism there holds \( g = pi + \beta v \). Last, since \( i \) is a monomorphism and \( \hat{E} \) is injective \( v \) is extended to a homomorphism \( v_1 : B \to \hat{E} \) such that \( v_1i = v \). Set \( p_0 = p + \beta v_1 \) then there holds \( g = p_0i \).  

Now we are in a position to treat the main theorem. We remark that if \( e = rk + u\alpha_1 \) holds then by setting \( M = \ker u \) and \( N = \ker r \) there holds \( k^{-1}(N) \cap \alpha_1^{-1}(M) \subseteq \ker e \).

For a right ideal \( I \) we fix a presentation of \( I \):

\[
0 \to K \to F \to I \to 0
\]

(3)

where \( F \) is a projective module. Consider the following condition for (3).

(4) Let \( L \) be a submodule of \( K \) such that \( K/L \) is a \( \mathcal{F} \)-torsion module. Then there exists \( B \) in \( \mathcal{F} \) containing \( I \) and if we construct the row exact
commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & H & \overset{\alpha_2}{\longrightarrow} & G & \overset{\beta_2}{\longrightarrow} & B & \longrightarrow & 0 \\
\downarrow k & & \downarrow j & & \downarrow i & & & & \\
0 & \longrightarrow & K & \overset{\alpha_1}{\longrightarrow} & F & \overset{\beta_1}{\longrightarrow} & I & \longrightarrow & 0
\end{array}
\]

where \( G \) is projective and \( i \) is the inclusion, there exist submodules \( M \) and \( N \) of \( F \) and \( H \) respectively such that

\[
\begin{align*}
(4a) & \quad F/M \text{ and } H/N \text{ are } \mathcal{F}\text{-torsion modules}, \\
(4b) & \quad k^{-1}(N) \cap \alpha_1^{-1}(M) \subseteq L.
\end{align*}
\]

**Theorem.** Let \( A \) be a ring, \( \mathcal{F} \) a class of right ideals of \( A \) which defines a hereditary torsion theory on the category of right \( A \)-modules. Then the localization \( E_{\mathcal{F}} \) of any injective module \( E \) is an injective \( A_{\mathcal{F}} \)-module if and only if each right ideal \( I \) of \( A \) has a presentation (3) satisfying (4). In that case any presentation (3) of any such right ideal satisfies (4).

**Proof.** Assume that for any injective module \( E \) the localization \( E_{\mathcal{F}} \) is \( A_{\mathcal{F}} \)-injective. In the presentation (3) of \( I \), let \( L \) be a submodule of \( K \) such that \( K/L \) is a torsion module. Now take an injective module \( E \) containing \( K/L \), then \( K/L \) is in \( t(E) \) and letting \( e \) be the composition map \( K \to K/L \hookrightarrow t(E) \), we can construct the following row exact commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & K & \overset{\alpha_1}{\longrightarrow} & F & \overset{\beta_1}{\longrightarrow} & I & \longrightarrow & 0 \\
\downarrow e & & \downarrow f & & \downarrow g & & & & \\
0 & \longrightarrow & t(E) & \overset{\alpha}{\longrightarrow} & E & \overset{\beta}{\longrightarrow} & \bar{E} & \longrightarrow & 0.
\end{array}
\]

By [2] Chapter IX, Proposition 2.7 there exists \( B \in \mathcal{F} \) containing \( I \) and a homomorphism \( p : B \to \bar{E} \) such that \( g = pi \) where \( i \) is the inclusion \( I \subseteq B \). From these we obtain the diagram (1). By Lemma 1, there exists a homomorphism \( u : F \to t(E) \) and there holds \( e = rk + u\alpha_1 \). Let \( M = \ker u \) and \( N = \ker r \) then \( M \) and \( N \) satisfy (4a) and (4b).

Conversely, suppose that for any right ideal \( I \) of \( A \) the presentation (3) of \( I \) satisfies (4). Let \( E \) be an injective module and a homomorphism \( g : I \to \bar{E} \) was given. Then we have the following row exact commutative
diagram:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & K & \overset{\alpha_1}{\longrightarrow} & F & \overset{\beta_1}{\longrightarrow} & I & \longrightarrow & 0 \\
\downarrow{e} & & \downarrow{f} & & \downarrow{g} & & \\
0 & \longrightarrow & t(E) & \overset{\alpha}{\longrightarrow} & E & \overset{\beta}{\longrightarrow} & \overline{E} & \longrightarrow & 0.
\end{array}
\]

Let \( L = \ker e \), then \( K/L \) is a torsion module. By (4) there exists \( B \in \mathcal{F} \) containing \( I \) and there are submodules \( M \) and \( N \) of \( F \) and \( H \) respectively satisfying (4a) and (4b). From these we have the following diagram:

\[
\begin{array}{ccccccc}
K/k^{-1}(N) \cap \alpha_1^{-1}(M) & \longrightarrow & K/k^{-1}(N) \oplus K/\alpha_1^{-1}(M) & \longrightarrow & H/N \oplus F/M \\
\downarrow{e} & & & & \\
K/L \subseteq t(E) & \overset{\alpha}{\longrightarrow} & E
\end{array}
\]

where the composition of the upper row is a monomorphism and \( \bar{e} \) is the natural map obtained from the condition (4b). As \( E \) is injective \( \bar{e} \) is extended to \( e_0 : H/N \oplus F/M \to E \) and its image is really in \( t(E) \) since \( H/N \oplus F/M \) is a \( \mathcal{F} \)-torsion module. The natural maps \( H \to H/N \) and \( F \to F/M \) composed with \( e_0 \) induce homomorphism \( r : H \to t(E) \) and \( u : F \to t(E) \). It is easily seen that there holds \( e = r \bar{k} + u \alpha_1 \) and from these we can construct the diagram (2). By Lemma 2 there exists \( p_0 : B \to \overline{E} \) such that \( g = p_0i \). Therefore \( E_{\mathcal{F}} \) is an \( A_{\mathcal{F}} \)-injective module by [2] Chapter IX, Proposition 2.7. This completes the proof. \( \square \)

REFERENCES


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