ON A GENERALIZATION OF CQF-3' MODULES AND COHEREDITY TORSION THEORIES

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Throughout this paper we assume that \( R \) is a right perfect ring with identity and let \( \text{Mod-}R \) be the category of right \( R \)-modules. Let \( M \) be a right \( R \)-module. We denote by \( 0 \to K(M) \to P(M) \to M \to 0 \) the projective cover of \( M \). \( M \) is called a CQF-3' module, if \( P(M) \) is \( M \)-generated, that is, \( P(M) \) is isomorphic to a homomorphic image of a direct sum \( \oplus M \) of some copies of \( M \).

A subfunctor of the identity functor of \( \text{Mod-}R \) is called a preradical. For a preradical \( \sigma \), \( T_\sigma := \{ M \in \text{Mod-}R : \sigma(M) = M \} \) is called the class of \( \sigma \)-torsion right \( R \)-modules, and \( F_\sigma := \{ M \in \text{Mod-}R : \sigma(M) = 0 \} \) is called the class of \( \sigma \)-torsionfree right \( R \)-modules. A right \( R \)-module \( M \) is called \( \sigma \)-projective if the functor \( \text{Hom}_R(M, -) \) preserves the exactness for any exact sequence \( 0 \to A \to B \to C \to 0 \) with \( A \in F_\sigma \). We put \( P_\sigma(M) = P(M)/\sigma(K(M)) \) for a module \( M \). We call a right \( R \)-module \( M \) a \( \sigma \)-CQF-3' module if \( P_\sigma(M) \) is \( M \)-generated.

In this paper, we characterize \( \sigma \)-CQF-3' modules and give some related facts.

1. CQF-3' MODULES RELATIVE TO A COHEREDITY TORSION THEORIES

F. F. Mbuntum and K. Varadarajan defined a CQF-3' module as a dualization of a QF-3' module and characterized it in [10]. In this paper we generalize a CQF-3' module by using an idempotent radical. A preradical \( \sigma \) is idempotent [radical] if \( \sigma(\sigma(M)) = \sigma(M) \) [\( \sigma(M/\sigma(M)) = 0 \)] for a module \( M \), respectively. It is well known that if \( \sigma \) is idempotent preradical, then \( F_\sigma \) is closed under taking extensions. It is also well known that if \( \sigma \) is a radical, then \( T_\sigma \) is closed under taking extensions. A preradical \( t \) is called epi-preserving if \( t(M/N) = (t(M) + N)/N \) holds for any submodule \( N \) of a module \( M \). It holds that any epi-preserving preradical is a radical. For a preradical \( \sigma \) we say that \( t \) is \( \sigma \)-epi-preserving if \( t(M/N) = (t(M) + N)/N \) holds for any module \( M \) and any submodule \( N \) of \( M \) with \( N \in F_\sigma \). For modules \( M \) and \( N \), \( t_N(M) \) denote \( \sum_{f\in \text{Hom}_R(N,M)} \text{im} f \). It holds that \( t_N \) is an idempotent preradical for any module \( N \) and that \( F_{t_A} = \{ M \in \text{Mod-}R : \text{Hom}_R(A, M) = 0 \} \) and \( T_{t_A} = \{ M \in \text{Mod-}R : \oplus A \to M \to 0 \} \)

A short exact sequence \( 0 \to K(M) \to P(M) \xrightarrow{f} M \to 0 \) is called a projective cover of a module \( M \) if \( P(M) \) is projective and \( K(M) := \ker f \) is

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small in \( P(M) \). For \( X, Y \in \text{Mod}-R \) we call an epimorphism \( g \in \text{Hom}_R(X,Y) \) a minimal epimorphism if \( g(H) \subsetneq Y \) holds for any proper submodule \( H \) of \( X \). It is well known that a minimal epimorphism is an epimorphism having a small kernel. A short exact sequence \( 0 \rightarrow X \rightarrow Y \rightarrow M \rightarrow 0 \) is called \( \sigma \)-projective cover of a module \( M \) if \( Y \) is \( \sigma \)-projective, \( X \) is \( \sigma \)-torsionfree and \( X \) is small in \( Y \). If \( \sigma \) is an idempotent preradical, then \( P(M)/\sigma(K(M)) \) is \( \sigma \)-projective for any module \( M \) by Lemma 1.4 in [11]. If \( \sigma \) is a radical, \( K(M)/\sigma(K(M)) \in \mathcal{F}_\sigma \). We put \( K_\sigma(M) = K(M)/\sigma(K(M)) \) and \( \sigma(M) = P(M)/\sigma(K(M)) \) for a preradical \( \sigma \). Then \( K_\sigma(M) \) is small in \( \sigma(M) \). Thus if \( \sigma \) is an idempotent radical, then a module \( M \) has a \( \sigma \)-projective cover and it is given by \( 0 \rightarrow K_\sigma(M) \rightarrow \sigma(M) \rightarrow M \rightarrow 0 \).

Let \( \sigma \) be a preradical and \( C \) a class of \( R \)-modules. We say that \( C \) is closed under taking \( \mathcal{F}_\sigma \)-extensions if the following condition holds: if \( N, M/N \in C \) and \( N \in \mathcal{F}_\sigma \), then \( M \in C \). Next we say that \( C \) is closed under taking \( \mathcal{F}_\sigma \)-factor modules if : if \( M \in C \) and \( N \) is a \( \sigma \)-torsionfree submodule of \( M \) then \( M/N \in C \). For a preradical \( \sigma \) we say that \( M \) is a \( \sigma \)-coessential extension of \( X \) if there exists a minimal epimorphism \( h : M \rightarrow X \) with \( \ker h \in \mathcal{F}_\sigma \). We say that \( C \) is closed under taking \( \sigma \)-coessential extensions if : for any minimal epimorphism \( f : M \rightarrow X \) with \( \ker f \in \mathcal{F}_\sigma \) if \( X \in C \) then \( M \in C \).

For the sake of simplicity we say that \( M \) is a \( \sigma \)-coessential extension of \( M/N \) if \( N \) is a \( \sigma \)-torsionfree small submodule of \( M \). We say that \( C \) is closed under taking \( \sigma \)-coessential extensions if : if \( M/N \in C \) then \( M \in C \) for any \( \sigma \)-torsion free small submodule \( N \) of any module \( M \).

**Theorem 1.** Let \( \sigma \) be a preradical. We consider the following conditions.

1. \( A \) is a \( \sigma \)-CQF-3' module.
2. \( t_A(P_\sigma(A)) = P_\sigma(A) \)
3. \( t_A(M) = t_{P_\sigma(A)}(M) \) for any module \( M \).
4. \( t_A(-) \) is \( \sigma \)-epi-preserving.
5. (a) \( T_{t_A} \) is closed under taking \( \mathcal{F}_\sigma \)-extensions.
   (b) \( \mathcal{F}_{t_A} \) is closed under taking \( \mathcal{F}_\sigma \)-factor modules.
6. \( T_{t_A} \) is closed under taking \( \sigma \)-projective covers.
7. \( T_{t_A} \) is closed under taking \( \sigma \)-coessential extensions.
8. If \( \text{Hom}_R(A,f) = 0 \), then \( \text{Hom}_R(A,M/N) = 0 \) holds for any submodule \( N \in \mathcal{F}_\sigma \) of a module \( M \), where \( f \) is the canonical epimorphism \( f : M \rightarrow M/N \).

Then we have implications \( (1) \rightarrow (3) \rightarrow (2) \rightarrow (1) \) and \( (4) \rightarrow (5) \).

If \( \sigma \) is idempotent, then \( (3) \rightarrow (4) \), \( (1) \rightarrow (8) \) and \( (6) \rightarrow (5) \), \( (7) \) hold.

If \( \sigma \) is a radical, then \( (7) \rightarrow (6) \), \( (4) \rightarrow (2) \), \( (6) \) hold.

If \( \sigma \) is an epi-preserving radical and \( A \) is in \( \mathcal{F}_\sigma \), then \( (8) \rightarrow (5) \) holds, moreover if \( \sigma \) is idempotent then \( (5) \rightarrow (2) \) hold.
Thus if $\sigma$ is an epi-preserving idempotent radical and $A$ is in $\mathcal{F}_\sigma$, all conditions are equivalent.

**Proof.** (1)$\rightarrow$(3): Let $M$ be a module in Mod-$R$. By the assumption there exists an exact sequence $\oplus A \rightarrow P_\sigma(A) \rightarrow 0$, and hence $t_A(M)$ contains $t_{P_\sigma(A)}(M)$. Since $P_\sigma(A) \rightarrow A \rightarrow 0$ is exact, $t_A(M)$ is contained in $t_{P_\sigma(A)}(M)$. Thus it follows that $t_A(M) = t_{P_\sigma(A)}(M)$ for any module $M$.

(3)$\rightarrow$(2): It is clear, for $t_A(P_\sigma(A)) = t_{P_\sigma(A)}(P_\sigma(A)) = P_\sigma(A)$.

(2)$\rightarrow$(1): It is clear, for $P_\sigma(A) = t_A(P_\sigma(A))$ is a homomorphic image of direct sums of copies of $A$.

(3)$\rightarrow$(4): Suppose that $\sigma$ is an idempotent preradical. Then $P_\sigma(A)$ is $\sigma$-projective. Let $N \in \mathcal{F}_\sigma$ be a submodule of a module $M$. Consider the following diagram.

\[
\begin{array}{ccc}
P_\sigma(A) & \xrightarrow{h} & N \\
\downarrow{f} & & \downarrow{g} \\
0 & \rightarrow & M & \rightarrow & M/N & \rightarrow & 0,
\end{array}
\]

where $g$ is the canonical epimorphism, $f$ is any homomorphism from $P_\sigma(A)$ to $M/N$ and $h \in \text{Hom}_R(P_\sigma(A), M)$ is induced by the $\sigma$-projectivity of $P_\sigma(A)$ such that $f = gh$.

Thus $t_{P_\sigma(A)}(M/N) \subseteq (t_{P_\sigma(A)}(M) + N)/N$. By the assumption, it holds that $t_A(M/N) \subseteq (t_A(M) + N)/N$. Since $t_A(-)$ is a preradical, $t_A(M/N) \supseteq (t_A(M) + N)/N$ holds, and so $t_A(-)$ is a $\sigma$-epi-preserving preradical.

(4)$\rightarrow$(2): Here we assume that $\sigma$ is a radical.

Then it holds that $K_\sigma(A) = K(A)/\sigma(K(A)) \in \mathcal{F}_\sigma$. Thus it holds $(t_A(P_\sigma(A)) + K_\sigma(A))/K_\sigma(A) = t_A(P_\sigma(A)/K_\sigma(A))$. Since $t_A(A) = A$ and $A \cong P_\sigma(A)/K_\sigma(A)$, it follows that $t_A(P_\sigma(A)/K_\sigma(A)) = P_\sigma(A)/K_\sigma(A)$. Thus $t_A(P_\sigma(A)) + K_\sigma(A) = P_\sigma(A)$ holds. Consequently $P_\sigma(A) = t_A(P_\sigma(A))$ for $K_\sigma(A)$ is small in $P_\sigma(A)$.

(4)$\rightarrow$(5)(a): Let $N$ be a submodule of a module $M$ such that $N \in \mathcal{F}_\sigma \cap \mathcal{T}_{tA}$ and $M/N \in \mathcal{T}_{tA}$, then $N = t_A(N) \subseteq t_A(M)$ and $t_A(M/N) = M/N$. By the assumption $t_A(M/N) = (t_A(M) + N)/N$, and so $M = t_A(M) + N = t_A(M)$, as desired.

(b): Let $N \in \mathcal{F}_\sigma$ be a submodule of a module $M \in \mathcal{F}_{tA}$, then we have the equation $t_A(M/N) = (t_A(M) + N)/N = N/N = 0$, as desired.

(1)$\rightarrow$(8): Suppose that $\sigma$ is idempotent. Then $P_\sigma(A)$ is $\sigma$-projective. Let $N$ be a submodule of a module $M$ such that $N \in \mathcal{F}_\sigma$. Since $A$ is $\sigma$-CQF-$3'$, there exists an epimorphism $\oplus A_i \xrightarrow{(\varphi_i)} P_\sigma(A)$, defined by $(\varphi_i)(a_i) = \sum_i \varphi_i(a_i)$ for $(a_i) \in \oplus A_i$, $\varphi_i \in \text{Hom}_R(A_i, P_\sigma(A))$, where $A_i \cong A$. 

We will show that if $\text{Hom}_R(A, f) = 0$ then $\text{Hom}_R(A, M/N) = 0$. Suppose that $\text{Hom}_R(A, M/N) \neq 0$. Then there exists a nonzero element $j$ in $\text{Hom}_R(A, M/N)$.

Let $f : M \rightarrow M/N$ be the canonical epimorphism, $g : P_\sigma(A) \rightarrow A$ a homomorphism associated with the $\sigma$-projectivity of $A$ and $h : P_\sigma(A) \rightarrow M$ a homomorphism induced by the $\sigma$-projectivity of $P_\sigma(A)$ such that $jg = fh$.

Consider the following commutative diagram with exact rows.

\[
\begin{array}{ccccccc}
P_\sigma(A) & \xrightarrow{g} & A & \rightarrow & 0 \\
\downarrow h & & \downarrow j & & \\
0 & \rightarrow & N & \rightarrow & M & \rightarrow & M/N \rightarrow 0
\end{array}
\]

There exists a nonzero element $x \in A$ such that $j(x) \neq 0$. Then there exists a nonzero element $y \in P_\sigma(A)$ such that $y = \sum \varphi_i(a_i)$ and $x = g(y) = g(\sum \varphi_i(a_i)) = \sum g(\varphi_i(a_i))$. Therefore it holds that $0 \neq j(x) = j(g(y)) = \sum j(g(\varphi_i(a_i)))$, and so there exists some $a_i$ in $A$ and some $\varphi_i$ in $\text{Hom}_R(A, P_\sigma(A))$ such that $j(g(\varphi_i(a_i))) \neq 0$. Then it holds that $0 \neq j(g(\varphi_i(a_i))) = f(h(\varphi_i(a_i)))$ for $jg = fh$. Since $h \varphi_i \in \text{Hom}_R(A, M)$, it holds that $0 \neq fh \varphi_i = \text{Hom}(A, f)(h \varphi_i)$. This is a contradiction, and so $\text{Hom}_R(A, M/N) = 0$, as desired.

(8)→(5): Here we assume that $\sigma$ is an epi-preserving preradical and $A \in \mathcal{F}_\sigma$.

(a): We show the stronger condition that $\mathcal{T}_{t_A}$ is closed under taking extensions. Let $N$ be a submodule of a module $M$ such that $M/N \in \mathcal{T}_{t_A}$ and $N \in \mathcal{T}_{t_A}$. Since $t_A(M)$ is a homomorphic image of a direct sum of copies of $A \in \mathcal{F}_\sigma$, it follows that $t_A(M) \in \mathcal{F}_\sigma$. Consider the following sequence. $\mathcal{F}_\sigma \ni t_A(M) \hookrightarrow M \rightarrow M/t_A(M)$. By the definition of $t_A(M)$ it follows that $\text{Hom}_R(A, f) = 0$. Consequently $\text{Hom}_R(A, M/t_A(M)) = 0$ by the assumption, and so $M/t_A(M) \in \mathcal{F}_{t_A}$.

Since $N \in \mathcal{T}_{t_A}$, $N = t_A(N) \subseteq t_A(M)$. Thus $M/t_A(M)$ is a factor module of $M/N \in \mathcal{T}_{t_A}$, and so $M/t_A(M) \in \mathcal{T}_{t_A}$.

Consequently it follows that $M/t_A(M) = 0$, as desired.

(b): Let $N \in \mathcal{F}_\sigma$ be a submodule of a module $M \in \mathcal{F}_{t_A}$. Consider the exact sequence $0 \rightarrow N \rightarrow M \xrightarrow{f} M/N \rightarrow 0$. Since $M \in \mathcal{F}_{t_A}$, $\text{Hom}_R(A, f) = 0$. Thus by the assumption $\text{Hom}_R(A, M/N) = 0$, and so $M/N \in \mathcal{F}_{t_A}$.

(5)→(2): Let $\sigma$ be an epi-preserving idempotent radical and $A \in \mathcal{F}_\sigma$. Since $\mathcal{F}_\sigma$ is closed under taking extensions and $K_\sigma(A) \in \mathcal{F}_\sigma$, it follows that $P_\sigma(A) \in \mathcal{F}_\sigma$ and so $t_A(P_\sigma(A)) \in \mathcal{F}_\sigma$ since $\mathcal{F}_\sigma$ is closed under taking submodules. We put $K = t_A(P_\sigma(A))$. We will show that $K = P_\sigma(A)$.
Suppose $K \subseteq P_\sigma(A)$. Since $K_\sigma(A)$ is small in $P_\sigma(A)$, $K + K_\sigma(A) \subseteq P_\sigma(A)$. Since $A \cong P_\sigma(A)/K_\sigma(A) \to P_\sigma(A)/(K_\sigma(A) + K) \neq 0$, it follows that $\text{Hom}_R(A, P_\sigma(A)/(K_\sigma(A) + K)) \neq 0$, and so $P_\sigma(A)/(K_\sigma(A) + K) \notin \mathcal{F}_t_A$. As $(K_\sigma(A) + K)/K$ is an epimorphic image of $K_\sigma(A) \in \mathcal{F}_\sigma$, it follows that $(K_\sigma(A) + K)/K \in \mathcal{F}_\sigma$ since $\mathcal{F}_\sigma$ is closed under taking factor modules. Consider the exact sequence $0 \to (K_\sigma(A) + K)/K \to P_\sigma(A)/K \to P_\sigma(A)/(K_\sigma(A) + K) \to 0$. By the assumption (b), it follows that $(P_\sigma(A)/K) \notin \mathcal{F}_t_A$. We put $X/K = t_A(P_\sigma(A)/K)(\neq 0)$. Consider the exact sequence $0 \to K \to X \to X/K \to 0$. As $K = t_A(P_\sigma(A)) \in \mathcal{F}_\sigma$, $K \in \mathcal{F}_\sigma \cap \mathcal{T}_{t_A}$. Since $X/K \in \mathcal{T}_{t_A}$, it follows that $X \in \mathcal{T}_{t_A}$ by the assumption (a). As $X \subseteq P_\sigma(A)$, $X = t_A(X) \subseteq t_A(P_\sigma(A)) = K$. Thus it follows that $X = K$. But this is a contradiction, for $X/K = t_A(P_\sigma(A)/K) \neq 0$. It concludes that $t_A(P_\sigma(A)) = K = P_\sigma(A)$, as desired.

(4)$\to$(6): We assume that $\sigma$ is a radical. Then $K_\sigma(X) \in \mathcal{F}_\sigma$ for any module $X$. Let $M \in \mathcal{T}_{t_A}$. Consider the exact sequence $0 \to K_\sigma(M) \to P_\sigma(M) \to P_\sigma(M)/K_\sigma(M) \to 0$. Since $K_\sigma(M) \in \mathcal{F}_\sigma$ and $P_\sigma(M)/K_\sigma(M) \cong M \in \mathcal{T}_{t_A}$, it follows that $P_\sigma(M)/K_\sigma(M) = t_A(P_\sigma(M)/K_\sigma(M)) = (t_A(P_\sigma(M)) + K_\sigma(M))/K_\sigma(M)$. Thus it follows that $P_\sigma(M) = t_A(P_\sigma(M)) + K_\sigma(M)$. As $K_\sigma(M)$ is small in $P_\sigma(M)$, it follows that $P_\sigma(M) = t_A(P_\sigma(M)) \in \mathcal{T}_{t_A}$, as desired.

(6)$\to$(5): We assume that $\sigma$ is idempotent. Then $P_\sigma(X)$ is $\sigma$-projective for any module $X$.

(a): Let $N \in \mathcal{F}_\sigma \cap \mathcal{T}_{t_A}$ be a submodule of a module $M$ such that $M/N \in \mathcal{T}_{t_A}$. Consider the following diagram.

\[
\begin{array}{cccc}
P_\sigma(M/N) & \overset{f}{\to} & M & \overset{g}{\to} & M/N & \to 0, \\
0 & \to & N & \to & M & \to h, \\
\end{array}
\]

where $g$ is an epimorphism associated with the $\sigma$-projective cover of $M/N$, $h$ is the canonical epimorphism and $f$ is a homomorphism induced by the $\sigma$-projectivity of $P_\sigma(M/N)$. By the assumption it follows that $P_\sigma(M/N) \in \mathcal{T}_{t_A}$. Thus it follows that $f(P_\sigma(M/N)) = f(t_A(P_\sigma(M/N))) \subseteq t_A(M)$. Since $N \in \mathcal{T}_{t_A}$, $N = t_A(N) \subseteq t_A(M)$. Then the following equalities hold. $M/N = g(A(P_\sigma(M/N))) = h(f(P_\sigma(M/N))) = (f(P_\sigma(M/N)) + N)/N \subseteq (t_A(M) + N)/N = t_A(M)/N \subseteq t_A(M/N) = M/N$. Thus we conclude that $M = t_A(M)$, as desired.

(b): Let $N \in \mathcal{F}_\sigma$ be a submodule of a module $M \in \mathcal{F}_{t_A}$. Consider the following diagram.
\[ P_\sigma(t_A(M/N)) \xrightarrow{g} t_A(M/N) \rightarrow 0 \]

\[
\begin{array}{cccc}
0 & \rightarrow & N & \rightarrow & M & \rightarrow & M/N & \rightarrow & 0, \\
& & f & & i & & h & & \\
\end{array}
\]

where \( g \) is an epimorphism associated with the \( \sigma \)-projective cover of \( t_A(M/N) \), \( i \) is the canonical monomorphism and \( f \) is a homomorphism induced by \( \sigma \)-projectivity of \( P_\sigma(t_A(M/N)) \).

By the assumption \( P_\sigma(t_A(M/N)) \in T_{t_A} \). Since \( M \in F_{t_A} \), it follows that \( f = 0 \), and so \( ig = 0 \). Hence \( i = 0 \), and so we conclude that \( t_A(M/N) = 0 \), as desired.

(7) \( \rightarrow \) (6): We assume that \( \sigma \) is a radical, and then \( K_\sigma(M) \in F_\sigma \). Thus it is clear, for \( P_\sigma(M) \) is a \( \sigma \)-coessential extension of \( M \).

(6) \( \rightarrow \) (7): We assume that \( \sigma \) is idempotent, and then \( P_\sigma(X) \) is \( \sigma \)-projective for any module \( X \). Let \( N \) be a small submodule of a module \( M \) such that \( M/N \in T_{t_A} \) and \( N \in F_\sigma \). Consider the following diagram.

\[
\begin{array}{cccc}
P_\sigma(M/N) & & f & & h & \check{\leftarrow} \\
0 & \rightarrow & N & \rightarrow & M & \rightarrow & M/N & \rightarrow & 0, \\
& & g & & \\
\end{array}
\]

where \( f \) is an epimorphism associated with the \( \sigma \)-projective cover of \( M/N \), \( g \) is the canonical epimorphism and \( h \) is a homomorphism induced by the \( \sigma \)-projectivity of \( P_\sigma(M/N) \). Since \( g \) is a minimal epimorphism and \( f \) is an epimorphism, it follows that \( h \) is also an epimorphism. By the assumption, \( M/N \in T_{t_A} \) implies \( P_\sigma(M/N) \in T_{t_A} \). Since \( h \) is an epimorphism, it follows that \( M \in T_{t_A} \). \( \square \)

If \( \sigma \) is zero functor, then \( \sigma \) is an epi-preserving idempotent radical and \( A \) is \( \sigma \)-torsionfree. Thus then \( \sigma \)-CQF-3′ modules are CQF-3′ modules.

**Proposition 2.** Let \( \sigma \) be an epi-preserving idempotent radical. Then the following conditions on a module \( A \) are equivalent.

1. \( F_{t_A} \) is closed under taking \( F_\sigma \)-factor modules.
2. \( F_{t_A} = F_{t_{P_\sigma(A)}} \)

**Proof.** (1) \( \rightarrow \) (2): Since \( P_\sigma(A) \rightarrow A \) is surjective, it follows that \( F_{t_{P_\sigma(A)}} \subseteq F_{t_A} \). Next we will show that \( F_{t_{P_\sigma(A)}} \supseteq F_{t_A} \). Let \( M \) be in \( F_{t_A} \). We will show that \( M \in F_{t_{P_\sigma(A)}} \). Suppose that \( M \notin F_{t_{P_\sigma(A)}} \). Then it holds that \( \text{Hom}_R(P_\sigma(A), M) \neq 0 \), and there exists \( 0 \neq f \in \text{Hom}_R(P_\sigma(A), M) \). Since \( \ker f \nsubseteq P_\sigma(A) \) and \( K_\sigma(A) \) is small in \( P_\sigma(A) \), it follows that \( \ker f + K_\sigma(A) \nsubseteq P_\sigma(A) \). Since \( \sigma \) is an epi-preserving preradical, \( F_\sigma \) is closed under taking factor modules. Thus it follows that \( (K_\sigma(A) + \ker f)/\ker f \in F_\sigma \). Since
Consider the exact sequence 
\[ 0 \to (K_\sigma(A) + \ker f)/\ker f \to P_\sigma(A)/\ker f \to P_\sigma(A)/(K_\sigma(A) + \ker f) \to 0. \]
By the assumption it follows that \( P_\sigma(A)/(K_\sigma(A) + \ker f) \in \mathcal{T}_{tA}. \) Since \( A \in \mathcal{T}_{tA} \) and \( A \simeq P_\sigma(A)/K_\sigma(A) \to P_\sigma(A)/(K_\sigma(A) + \ker f), \) \( P_\sigma(A)/(K_\sigma(A) + \ker f) \in \mathcal{T}_{tA} \cap \mathcal{F}_{tA}. \) Thus \( P_\sigma(A)/(K_\sigma(A) + \ker f) = 0, \) this is a contradiction. Hence it follows that \( M \in \mathcal{F}_{tP_\sigma(A)}. \)

(2)→(1): By the assumption, it is sufficient to prove that \( \mathcal{F}_{tP_\sigma(A)} \) is closed under taking \( \mathcal{F}_\sigma \)-factor modules. Let \( N \in \mathcal{F}_\sigma \) be a submodule of a module \( M \in \mathcal{F}_{tP_\sigma(A)}. \) Suppose that \( M/N \not\in \mathcal{F}_{tP_\sigma(A)} \), then there exists \( 0 \neq f \in \text{Hom}_R(P_\sigma(A), M/N). \) Since \( P_\sigma(A) \) is \( \sigma \)-projective, there exists an \( h \in \text{Hom}_R(P_\sigma(A), M) \) such that \( gh = f. \) Since \( M \in \mathcal{F}_{tP_\sigma(A)} \), it follows that \( h = 0, \) and then \( f = 0. \) This is a contradiction, and so \( M/N \in \mathcal{F}_{tP_\sigma(A)}, \) as desired.

**Lemma 3.** Let \( \sigma \) be an idempotent radical. For a module \( M \) and its submodule \( N, \) consider the following diagram with exact rows:

\[
\begin{array}{ccccccc}
0 & \to & K_\sigma(M) & \to & P_\sigma(M) & \xrightarrow{f} & M & \to & 0 \\
0 & \to & K_\sigma(M/N) & \to & P_\sigma(M/N) & \xrightarrow{g} & M/N & \to & 0,
\end{array}
\]
where \( f \) and \( g \) are epimorphisms associated with the \( \sigma \)-projective covers and \( j \) is the canonical epimorphism. Then there exists a homomorphism \( h : P_\sigma(M) \to P_\sigma(M/N) \) induced by the \( \sigma \)-projectivity of \( P_\sigma(M) \) such that \( jf = gh. \)

Then the following conditions hold.

(1) If \( M \) is a \( \sigma \)-coessential extension of \( M/N, \) then \( h : P_\sigma(M) \to P_\sigma(M/N) \) is an isomorphism.

(2) Moreover if \( \sigma \) is epi-preserving and \( h : P_\sigma(M) \to P_\sigma(M/N) \) is an isomorphism, then \( M \) is a \( \sigma \)-coessential extension of \( M/N. \)

**Proof.** (1): Let \( N \in \mathcal{F}_\sigma \) be a small submodule of a module \( M. \) Since \( jf \) is an epimorphism and \( g \) is a minimal epimorphism, \( h \) is also an epimorphism. Since \( j(f(\ker h)) = g(h(\ker h)) = g(0) = 0, \) it follows that \( f(\ker h) \subseteq \ker j = N \in \mathcal{F}_\sigma, \) and so \( f(\ker h) \in \mathcal{F}_\sigma. \) Let \( f|_{\ker h} \) be the restriction of \( f \) to \( \ker h. \) Then it follows that \( \ker(f|_{\ker h}) = \ker h \cap \ker f = \ker h \cap K_\sigma(M) \subseteq K_\sigma(M) \in \mathcal{F}_\sigma. \) Consider the exact sequence \( 0 \to \ker f|_{\ker h} \to \ker h \to f(\ker h) \to 0. \) Since \( \mathcal{F}_\sigma \) is closed under taking extensions, it follows that \( \ker h \in \mathcal{F}_\sigma. \) As \( P_\sigma(M/N) \) is \( \sigma \)-projective, the exact sequence \( 0 \to \ker h \to P_\sigma(M) \to P_\sigma(M/N) \to 0 \) splits, and so there exists a submodule \( L \) of \( P_\sigma(M) \) such that \( P_\sigma(M) = L \oplus \ker h. \) So it follows that \( f(P_\sigma(M)) = f(L) + f(\ker h). \) As \( f(\ker h) \subseteq N \) and \( f(P_\sigma(M)) = M, \) \( M = f(L) + N. \) Since \( N \) is small in \( M, \)
it follows that \( M = f(L) \). As \( f \) is a minimal epimorphism, it follows that \( P_\sigma(M) = L \) and \( \ker h = 0 \), and so \( h : P_\sigma(M) \simeq P_\sigma(M/N) \), as desired.

(2): Suppose that \( h : P_\sigma(M) \simeq P_\sigma(M/N) \). By the commutativity of the above diagram and \( h \), it follows that \( h(f^{-1}(N)) \subseteq K_\sigma(M/N) \in \mathcal{F}_\sigma \). Since \( h \) is an isomorphism, \( f^{-1}(N) \in \mathcal{F}_\sigma \). As \( f|_{f^{-1}(N)} : f^{-1}(N) \to N \to 0 \) and \( \sigma \) is an epi-preserving preradical, it follows that \( N \in \mathcal{F}_\sigma \).

Next we will show that \( N \) is small in \( M \). Let \( K \) be a submodule of \( M \) such that \( M = N + K \). If \( f^{-1}(K) \not\subseteq P_\sigma(M) \), then \( h(f^{-1}(K)) \not\subseteq P_\sigma(M/N) \) as \( h \) is an isomorphism. Since \( g(h(f^{-1}(K))) = j(f(f^{-1}(K))) = j(K) = (K + N)/N = M/N \) and \( g \) is a minimal epimorphism, this is a contradiction. Thus it holds that \( f^{-1}(K) = P_\sigma(M) \), and so \( K = f(f^{-1}(K)) = f(P_\sigma(M)) = M \). Thus it follows that \( N \) is small in \( M \). \( \square \)

**Proposition 4.** Let \( \sigma \) be an idempotent radical. The class of \( \sigma \)-CQF-3' modules is closed under taking \( \sigma \)-coessential extensions.

**Proof.** Let \( N \in \mathcal{F}_\sigma \) be a submodule of a module \( M \) such that \( P_\sigma(M/N) \) is \((M/N)\)-generated. Then by Lemma 3 it follows that \( \oplus M \to \oplus(M/N) \to P_\sigma(M/N) \simeq P_\sigma(M) \). Thus it follows that \( M \) is a \( \sigma \)-CQF-3' module. \( \square \)

2. **\( \sigma \)-epi-preserving preradical and \( \sigma \)-cohereditary torsion theories**

In this section we generalize epi-preserving preradicals by using torsion theories. If a module \( A \) is \( \sigma \)-CQF-3' and \( t = t_A \), then \( t \) is a \( \sigma \)-epi-preserving idempotent preradical by Theorem 1.

**Theorem 5.** Let \( \sigma \) be an idempotent radical. Consider the following conditions on a preradical \( t \).

1. \( t \) is a \( \sigma \)-epi-preserving preradical.
2. \( \mathcal{T}_t \) is closed under taking \( \sigma \)-coessential extensions.
3. \( \mathcal{T}_t \) is closed under taking \( \sigma \)-projective covers.
4. \( \mathcal{F}_t \) is closed under taking \( \mathcal{F}_\sigma \)-factor modules.
   1. \( \mathcal{F}_t \) is closed under taking \( \mathcal{F}_\sigma \)-extensions.

Then we have the implications (4)\( \iff \) (1)\( \iff \) (2)\( \iff \) (3).

If \( t \) is an idempotent preradical, then we have the implication (3)\( \implies \) (1).

If \( \sigma \) is an epi-preserving preradical and \( t \) is a radical, then (4)\( \implies \) (1) holds.

Thus if \( \sigma \) is an epi-preserving idempotent radical and \( t \) is an idempotent radical, then all conditions are equivalent.

**Proof.** By the assumption every module has its \( \sigma \)-projective cover.

(1)\( \implies \) (2): Let \( N \in \mathcal{F}_\sigma \) be a small submodule of a module \( M \) such that \( M/N \in \mathcal{T}_t \). By the assumption \( M/N = t(M/N) = (t(M) + N)/N \). Thus it follows that \( M = t(M) + N \), and so \( M = t(M) \), for \( N \) is small in \( M \).
(2)→(3): This is clear.
(3)→(2): Let $N \in \mathcal{F}_\sigma$ be a small submodule of a module $M$ such that $M/N \in \mathcal{T}_t$. Consider the following commutative diagram.

$$
\begin{array}{c}
P_\sigma(M/N) \\
\begin{array}{c}
h \\
\downarrow f
\end{array}
\end{array}
\xymatrix{
0 \ar[r] & N \ar[r] & M \ar[r]_g & M/N \ar[r] & 0,
}
$$

where $f$ is an epimorphism associated with the $\sigma$-projective cover of $M/N$, $g$ is the canonical epimorphism and $h$ is a homomorphism induced by the $\sigma$-projectivity of $P_\sigma(M/N)$.

Since $f$ is an epimorphism and $g$ is a minimal epimorphism, it follows that $h$ is an epimorphism. By the assumption it holds that $P_\sigma(M/N) \in \mathcal{T}_t$, and so $M \in \mathcal{T}_t$, as desired.

(1)→(4): This is almost the same as (4)→(5) in Theorem 1.

(3)→(1): Let $N \in \mathcal{F}_{\sigma}$ be a submodule of a module $M$ and $t$ an idempotent preradical. Consider the following diagram.

$$
\begin{array}{c}
P_\sigma(t(M/N)) \\
\begin{array}{c}
h \downarrow f
\end{array}
\end{array}
\xymatrix{
t(M) \ar[r] & t(M/N) \\
0 \ar[r]_u & N \ar[r]_j & M \ar[r]_g & M/N \ar[r] & 0,
}
$$

where $i, j$ and $u$ are the inclusions, $f$ is an epimorphism associated with the $\sigma$-projective cover of $t(M/N)$ and $g$ is the canonical epimorphism from $M$ to $M/N$. By the assumption $P_\sigma(t(M/N)) \in \mathcal{T}_t$. Since $N \in \mathcal{F}_\sigma$, there exists an $h \in \text{Hom}_R(P_\sigma(t(M/N)), M)$ such that $if = gh$ by the $\sigma$-projectivity of $P_\sigma(t(M/N))$. Since $h(P_\sigma(t(M/N))) = h(t(P_\sigma(t(M/N)))) \subseteq t(M)$, $h \in \text{Hom}_R(P_\sigma(t(M/N)), t(M))$. Since $g(t(M)) \subseteq t(M/N)$, $g$ induces $g' \in \text{Hom}_R(t(M), t(M/N))$ such that $f = g'h$. As $f$ is an epimorphism, $g'$ is also an epimorphism. Thus $(t(M) + N)/N = g'(t(M)) = t(M/N)$, as desired.

(4)→(1): Let $N \in \mathcal{F}_\sigma$ be a submodule of a module $M$, $t$ a radical and $\sigma$ an epi-preserving preradical. Then $(N + t(M))/t(M) \simeq N/(N \cap t(M)) \leftarrow N \in \mathcal{F}_\sigma$. Consider the exact sequence $0 \rightarrow (N + t(M))/t(M) \rightarrow M/t(M) \rightarrow M/(N + t(M)) \rightarrow 0$. Since $M/t(M) \in \mathcal{F}_t$, it follows that $M/(N + t(M)) \in \mathcal{F}_t$ by the assumption (i). Hence $(M/N)/(N + t(M))/N) \in \mathcal{F}_t$, and so $t(M/N) \subseteq (N + t(M))/N$. Since $t$ is a preradical, it follows that $t(M/N) \supseteq (N + t(M))/N$, and so $t(M/N) = (N + t(M))/N$ holds.

\begin{proposition}
Let $\sigma$ be an epi-preserving radical and $t$ a preradical. Then the following conditions are equivalent.
\end{proposition}
(1) Let \( N \) be a submodule of a module \( M \) such that \( M \supseteq N \supseteq t(M) \). If
\[ N/t(M) \in \mathcal{F}_\sigma, \text{ then } M/N \in \mathcal{F}_t. \]

(2) \( t \) is both a radical and a \( \sigma \)-epi-preserving preradical.

**Proof.** (1)\( \rightarrow \)(2): We use \( t(M) \) instead of \( N \). Then it follows that \( M/t(M) \in \mathcal{F}_t \), and so \( t \) is a radical.

Next we will show that if \( N \in \mathcal{F}_\sigma \), then \( t(M/N) = (t(M) + N)/N \). We use \( N + t(M) \) instead of \( N \). Consider the sequence \( 0 \rightarrow (N + t(M))/t(M) \rightarrow M/t(M) \rightarrow M/(N + t(M)) \rightarrow 0 \). Since \( (N + t(M))/t(M) \simeq N/(N \cap t(M)) \subseteq N \in \mathcal{F}_\sigma \), \( (N + t(M))/t(M) \in \mathcal{F}_t \). It holds that \( (M/(N + t(M)) \in \mathcal{F}_t \), and so \( (M/N)/(N + t(M))/N) \in \mathcal{F}_t \). Thus \( t(M/N) \subseteq (N + t(M))/N \).

Since \( t \) is a preradical, \( (M/N) \supseteq (N + t(M))/N \), and so it follows that
\[ t(M/N) = (N + t(M))/N. \]

(2)\( \rightarrow \)(1): Let \( N \) be a submodule of a module \( M \) such that \( M \supseteq N \supseteq t(M) \) and \( N/t(M) \in \mathcal{F}_\sigma \). Consider the sequence \( 0 \rightarrow N/t(M) \rightarrow M/t(M) \rightarrow M/N \rightarrow 0 \). Since \( t \) is an \( \sigma \)-epi-preserving preradical and a radical,
\[ \{t(M/t(M)) + N/t(M))/N(t(M)) \simeq t(M/N), \text{ and so } 0 \simeq t(M/N), \] as desired. \( \Box \)

A torsion theory for \( \text{Mod-}R \) is a pair \((\mathcal{T}, \mathcal{F})\) of classes of objects of \( \text{Mod-}R \) such that
(i) \( \text{Hom}_R(T, F) = 0 \) for all \( T \in \mathcal{T}, F \in \mathcal{F} \)
(ii) If \( \text{Hom}_R(M, F) = 0 \) for all \( F \in \mathcal{F}, \text{ then } M \in \mathcal{T} \)
(iii) If \( \text{Hom}_R(T, N) = 0 \) for all \( T \in \mathcal{T}, \text{ then } N \in \mathcal{F} \)

We put \( t(M) = \sum_{\mathcal{T} \ni N \subseteq M} N (= \cap_{M \in \mathcal{N} \subseteq \mathcal{F}} N) \), then \( \mathcal{T} = \mathcal{T}_t \) and \( \mathcal{F} = \mathcal{F}_t \) hold.

We call a torsion theory \((\mathcal{T}, \mathcal{F})\) \( \sigma \)-cohereditary if \( \mathcal{F} \) is closed under taking \( \mathcal{F}_\sigma \)-factor modules for an idempotent radical \( \sigma \).

**Proposition 7.** Let \( t \) be a radical and \( \sigma \) an idempotent preradical such that \( \mathcal{T}_\sigma \subseteq \mathcal{T}_t \). If \( \mathcal{T}_t \) is closed under taking \( \sigma \)-projective covers, then \( \mathcal{T}_t \) is closed under taking projective covers.

**Proof.** For \( M \in \mathcal{T}_t \) it holds that \( P_\sigma(M) \in \mathcal{T}_t \) by the assumption. It holds that \( \sigma(K(M)) \subseteq \mathcal{T}_\sigma \) since \( \sigma \) is idempotent. As \( \mathcal{T}_\sigma \subseteq \mathcal{T}_t \), it follows that \( \sigma(K(M)) \in \mathcal{T}_t \). Consider the exact sequence \( 0 \rightarrow \sigma(K(M)) \rightarrow P(M) \rightarrow P_\sigma(M) \rightarrow 0 \). Since \( t \) is a radical, \( \mathcal{T}_t \) is closed under taking extensions. Therefore it follows that \( P(M) \in \mathcal{T}_t \). \( \Box \)

**Theorem 8.** Let \( \sigma \) be an epi-preserving idempotent radical. Let \((\mathcal{T}, \mathcal{F})\) be a torsion theory. Suppose that there exists \( Q \in \mathcal{T} \) such that \( \mathcal{F} = \{M_R : \text{Hom}_R(Q, M) = 0\} \). Then \((\mathcal{T}, \mathcal{F})\) is \( \sigma \)-cohereditary if and only if \( \mathcal{F} = \{M_R : \text{Hom}_R(P_\sigma(Q), M) = 0\} \).

**Proof.** Let \( \mathcal{F} = \{M_R : \text{Hom}_R(P_\sigma(Q), M) = 0\} \). Since it is easily verified that \( \mathcal{F} \) is closed under taking submodules, direct sums, and extensions by
Let \( M \) be a module in \( \mathcal{F} \) and \( N \) a \( \sigma \)-torsion free submodule of \( M \). Suppose that \( \text{Hom}_R(P_\sigma(Q), M/N) \neq 0 \). Consider the following diagram.

\[
P_\sigma(Q) \quad \downarrow^f \\
0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0,
\]

where \( f \) is a nonzero homomorphism from \( P_\sigma(Q) \) to \( M/N \) and \( h \) is the canonical epimorphism from \( M \) to \( M/N \).

Then there exists a homomorphism \( i \) from \( P_\sigma(Q) \) to \( M \) induced by the \( \sigma \)-projectivity of \( P_\sigma(Q) \) such that \( f = hi \). Since \( hi \neq 0 \), \( i \neq 0 \) for \( h \) is an epimorphism. Since \( P_\sigma(Q) \) is \( Q \)-generated by the assumption, there exists a homomorphism \( k : Q \rightarrow P_\sigma(Q) \) such that \( 0 \neq ik \in \text{Hom}_R(Q, M) \). Thus it follows that \( \text{Hom}_R(Q, M) \neq 0 \). This is a contradiction to the fact that \( M \in \mathcal{F} \). Thus \( \text{Hom}_R(Q, M/N) = 0 \), and so \( M/N \in \mathcal{F} \).

Conversely suppose that \( \mathcal{F} \) is closed under taking \( \mathcal{F}_\sigma \)-factor modules. Let \( t \) be a \( \sigma \)-epi-preserving idempotent radical associated with \( (\mathcal{T}, \mathcal{F}) \) such that \( \mathcal{T} = \mathcal{T}_t \) and \( \mathcal{F} = \mathcal{F}_t \). By Theorem 5, \( \mathcal{F} \) is closed under taking \( \mathcal{F}_\sigma \)-factor modules if and only if \( \mathcal{T} \) is closed under taking \( \sigma \)-projective covers. Since \( \mathcal{T} \) is closed under taking \( \sigma \)-projective covers, it follows that \( P_\sigma(Q) \in \mathcal{T} \).

Next we show that \( \mathcal{F} = \{ M : \text{Hom}_R(P_\sigma(Q), M) = 0 \} \).

If \( M \in \mathcal{F} \), then \( \text{Hom}_R(P_\sigma(Q), M) = 0 \) since \( P_\sigma(Q) \in \mathcal{T} \). Thus it follows that \( \mathcal{F} \subseteq \{ M : \text{Hom}_R(P_\sigma(Q), M) = 0 \} \).

Conversely suppose that \( \text{Hom}_R(P_\sigma(Q), M) = 0 \). Since \( P_\sigma(Q) \rightarrow Q \rightarrow 0 \), it follows that \( 0 \rightarrow \text{Hom}_R(Q, M) \rightarrow \text{Hom}_R(P_\sigma(Q), M) \), and so \( \text{Hom}_R(Q, M) = 0 \). Thus \( \mathcal{F} \supseteq \{ M : \text{Hom}_R(Q, M) = 0 \} \). Therefore it follows that \( \mathcal{F} = \{ M : \text{Hom}_R(P_\sigma(Q), M) = 0 \} \).

\( \square \)

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