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Modeling uncertainty for unknown modal parameters in large flexible structures

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A procedure for control-oriented modeling is proposed for large flexible structures with unknown modal parameters. Techniques on quantification of errors in modal truncated nominal models are developed for the case where a finite number of upper and lower bounds of the unknown modal parameters are given. A feasible set of systems matching the conditions is introduced, and then error bounds covering the feasible set are established in the frequency domain. The bounds are easily checked using linear programming for any user-specified frequency. The feasibility of the proposed scheme is illustrated by numerical study on an ideal flexible beam example.

1 Introduction

In modeling for controller design, it is required to characterize a nominal model describing essential dynamics of the plant and also bounds of magnitudes of the uncertainty for the plant[1]. While many researchers have attempted to identify nominal models and uncertainty magnitudes[2], there proved to be some theoretical issues to quantify uncertainty bounds. It is hard to derive the bounds without any *a priori* information, and the estimated bounds tend to be overly sensible to the information[3].

Efforts, on the other hand, have been made on bounding uncertainty using physical knowledge or first principles. It has been shown that for elastic equations with a finite number of known eigenparameters, the minimum worst case error and the nominal model such that the minimum is attained, can be characterized explicitly in the frequency domain [4]. Efficient numerical bounding techniques are also developed for the case where a finite number of upper and lower bounds of the unknown parameters are available[5].

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While such a physical modeling can be valuable in having insight into essential plant dynamics, it is widely recognized that there are some fundamental limitations on precise modeling of complicated practical physical phenomena. And even if we obtain some estimates on nominal models and uncertainty bounds just based on first-principles, these may not necessarily be consistent with actual measurement data. Therefore, modeling procedures using both physical *a priori* knowledge of the plant and experimental data has been strongly desired for the sake of quantification of uncertainty bounds.

In this paper, according to the requirement and issues mentioned above, techniques on bounding uncertainty for elastic vibrating systems are presented. In the formulation adopted in [4, 5], we have no other way to estimate parameters needed, than doing modal analysis just theoretically, and it is not so easy to identify them from experimental data. Here we formulate our problem using modal parameters with which we can reflect results of both theoretical analysis on physical laws and experimental data. If we consider that dynamics of vibrating systems can be described by just superposition of simple second order vibrating modes, then analysis techniques for modal parameters are roughly summarized as in the following two categories[6]; one is theoretical analysis determining modal parameters from eigenvalue/eigenvector analysis of high dimensional matrices about inertia and stiffness of the equation of motion, and the other experimental one to identify modal parameters using least squares or curve fitting of frequency domain based on input-output data in time or frequency domain.

Finally, to illustrate feasibility of the proposed quantification formula, we theoretically characterize an interval for a finite number of modal parameters, and demonstrate controller design for an flexible beam example.

2 System and Problem Formulation

2.1 System Description

We consider elastic vibrating systems described by

$$G(s) = \sum_{i=1}^{\infty} \frac{k_i}{1 + 2\zeta_i(s/\omega_i) + (s/\omega_i)^2} \tag{1}$$

where $0 < \omega_1 \leq \omega_2 \leq \cdots \to \infty$ and (ω_i, k_i) satisfy

$$\sum_{i=1}^{\infty} |k_i| \leq \rho \tag{2}$$

for some given $\rho > 0$.

We assume that first ℓ triples of (k_i, ω_i, ζ_i) where $i = 1, \ldots, \ell$ are known but all the rest unknown, but each upper and lower bounds to the first $(p - \ell)$-th unknown pairs (ω_i, k_i) $i = \ell + 1, \ldots, p$ are given as $(\underline{\omega}_i,\ \overline{\omega}_i,\ \underline{k}_i,\ \overline{k}_i)$, respectively, where

$$\underline{\omega}_i \le \omega_i \le \overline{\omega}_i$$
$$\underline{k}_i \le k_i \le \overline{k_i}\ (i = \ell + 1, \ldots, p) \tag{3}$$

The ζ_i $(i > \ell)$ is assumed to satisfy

$$\zeta_i \ge \gamma \omega_i \tag{4}$$

for given $\gamma > 0$.

In a subsequent section we illustrate an example to determine ρ and γ from equations of motion for an elastic vibrating system.

2.2 Problem Statement

We choose the ℓ-th partial sum, the known part of (1)

$$G_\ell(s) = \sum_{i=1}^{\ell} \frac{k_i}{1 + 2\zeta_i(s/\omega_i) + (s/\omega_i)^2} \tag{5}$$

as a nominal model.

Our problem is to find a least upper bound of errors between the nominal model and the system

$$r(\omega) := \sup_{G \in \mathcal{P}_\ell} |G(j\omega) - G_\ell(j\omega)| \tag{6}$$

for each user-specified frequency ω where $\mathcal{P}_\ell$ is the set of systems satisfying all the conditions shown above.

We can rewrite $r(\omega)$ as in

$$\begin{aligned} r(\omega) \ = \ \sup_{\substack{\omega_i, k_i, \\ i=\ell+1,\ldots}} \Bigg\{ & \sum_{i=\ell+1}^{\infty} \frac{k_i}{1 + 2\zeta_j(j\omega/\omega_i) - (\omega/\omega_i)^2} \\ & \Bigg| \sum_{j=\ell+1}^{\infty} |k_j| \le \overline{\rho}^{(\ell)}; \underline{k}_i \le k_i \le \overline{k}_i, \\ & \underline{\omega}_i \le \omega_i \le \overline{\omega}_i,\ (\ell + 1 \le i \le p) \Bigg\} \end{aligned} \tag{7}$$

where $\tilde{k}_i := \min\{|x| | \underline{k}_i \le x \le \overline{k}_i\}$ and

$$\overline{\rho}^{(p)} := \rho - \sum_{j=1}^{p} \tilde{k}_j.$$

Evaluating $r(\omega)$ represented as in (7) is apparently not an easy task since it contains infinite number of unknown parameters.

3 Quantification of the Error Bounds

From relation (2), it follows that $|k_i| \leq \acute{k}_i$, here $\acute{k}_i := \overline{\rho}^{(p)} + \widetilde{k}_i$. We suppose that this restriction is taken into account in (3) and we assume the relation $\max\{|\underline{k}_i|, |\overline{k}_i|\} \leq \acute{k}_i$.

3.1 Upper Bounds

An upper bound to $r(j\omega)$ is shown in
Theorem 1. For q where $\ell + 1 \leq q \leq p$, $r(\omega) \leq \overline{r}_q(\omega)$ holds, here

$$\overline{r}_q(\omega) := \max_{x_{\ell+1},\ldots,x_{q+1}} \Bigg\{ \sum_{i=\ell+1}^{q+1} x_i h_i(\omega) + x_{q+1}\overline{h}_{q+1}(\omega) \\ \Bigg| \sum_{i=\ell+1}^{q+1} x_i \leq \overline{\rho}^{(\ell)}, \quad x_{q+1} \geq 0 \\ \widetilde{k}_i \leq x_i \leq \max\{|\underline{k}_i|, |\overline{k}_i|\}, \ (\ell + 1 \leq i \leq \overline{\ell}) \Bigg\} \tag{8}$$

and

$$h_i(\omega) = \sup_{\underline{\omega}_i \leq \theta \leq \overline{\omega}_i} |H_\theta(j\omega)|, \ \overline{h}_i(\omega) = \sup_{\underline{\omega}_i \leq \theta} |H_\theta(j\omega)|$$

where

$$H_\theta(s) = \frac{1}{1 + 2\zeta(s/\theta) + (s/\theta)^2}, \ \zeta = \gamma\theta.$$

Proof. Let $X_i := k_i$, then

$$|G(j\omega) - G_\ell(j\omega)| = \left| \sum_{i=\ell+1}^{\infty} X_i H_{\omega_i}(j\omega) \right|.$$

For all q such that $\ell + 1 \leq q \leq p$, the following evaluations hold true.

$$\begin{aligned} \left| \sum_{i=\ell+1}^{\infty} X_i H_{\omega_i}(j\omega) \right| &\leq \sum_{i=\ell+1}^{\infty} |X_i||H_{\omega_i}(j\omega)| \\ &\leq \sum_{i=\ell+1}^{q} |X_i| \sup_{\underline{\omega}_i \leq \theta \leq \overline{\omega}_i} |H_\theta(j\omega)| \\ &\quad + \sum_{i=q+1}^{\infty} |X_i| \sup_{\underline{\omega}_{q+1} \leq \theta} |H_\theta(j\omega)| \\ &= \sum_{i=\ell+1}^{q} |X_i| \cdot h_i(\omega) + \left(\sum_{i=q+1}^{\infty} |X_i| \right) \cdot \overline{h}_{q+1}(\omega) \end{aligned} \tag{9}$$

Here, if we set $x_i := |X_i|$ $(i = \ell + 1, \ldots, q)$, $x_{q+1} := \sum_{i=q+1}^{\infty} |X_i|$ then the result follows.

While an upper bound to $r(j\omega)$ for each q such that $\ell+1 \leq q \leq p$ is shown in theorem 1, we can expect from the above proof that the larger the number q, the smaller the corresponding upper bounds. This is shown in

Proposition 1. Suppose $\ell+1 \leq q_1 \leq q_2 \leq p$. Then

$$\overline{r}_{q_1}(\omega) \geq \overline{r}_{q_2}(\omega)$$

hold true.

That is, $\overline{r}_q(\omega)$ is always minimal if $q = p$, but minimizing q is not necessarily equal to p, as in

Proposition 2. Let $\overline{\ell}$ be the smallest n such that for all $i \geq n+1$

$$\acute{k}_i = \max\{|\underline{k}_i|, |\overline{k}_i|\}, \ \overline{\omega}_i \geq \underline{\omega}_{i+1} \tag{10}$$

hold. Such an $\overline{\ell}$ will exist uniquely in $\ell \leq \overline{\ell} \leq p-1$. Then

$$\overline{r}_q(\omega) = \overline{r}_{\overline{\ell}}(\omega) \quad (q \geq \overline{\ell})$$

and

$$\overline{r}_q(\omega) > \overline{r}_{\overline{\ell}}(\omega) \quad (q < \overline{\ell})$$

hold.

In other words, among upper bounds $\overline{r}_q(\omega)$'s of $r(\omega)$, the smallest is no other than $\overline{r}_{\overline{\ell}}(\omega)$, and we can see the problem is reduced to a linear programming problem with $\overline{\ell} - \ell + 1$ variables.

3.2 Lower Bounds to the Solution

A lower bound to the $r(j\omega)$ can be found by just choosing an element from the feasible set. It is desirable to find a lower bound as large as possible, in light of evaluating tightness of upper bounds. We develop a technique to obtain such a lower bound utilizing the result of upper bound as in the following

Corollary 1. For q such that $\overline{\ell}+1 \leq q \leq p$, $r(\omega) \geq \underline{r}_q(\omega)$ hold true here

$$\underline{r}_q(\omega) := \left| \sum_{i=\ell+1}^{q+1} \widehat{X}_i^{(\omega)} H_{\omega_{p,i(\omega)}(\omega)}(j\omega) \right|$$

and, $\omega_{p,i}(\omega) = \arg\max_{\underline{\omega}_i \leq \theta \leq \overline{\omega}_i} |H_\theta(j\omega)|$ where $\widehat{X}_i^{(\omega)}$ is defined such that

$$|\widehat{X}_i^{(\omega)}| = \widehat{x}_i^{(\omega)}, \quad \underline{k}_i \leq \widehat{X}_i^{(\omega)} \leq \overline{k_i}$$

and $(\widehat{x}_1^{(\omega)}, \ldots, \widehat{x}_{\overline{\ell}+1}^{(\omega)})$ is the $(x_1, \ldots, x_{\overline{\ell}+1})$ maximizing (9).

Proof. Suppose $q \geq \bar{\ell}+1$. If we define

$$G^{(\omega)}(s) := G_\ell(s) + \sum_{i=\ell+1}^{q} \widehat{X}_i^{(\omega)} H_{\omega_{p,i(\omega)}}(s)$$

for some fixed frequency ω, we can see that this belongs to the feasible set and conclude $\underline{r}(\omega) := |G^{(\omega)}(j\omega) - G_\ell(j\omega)|$ is a lower bound to $r(j\omega)$.

The upper bound $\bar{r}(\omega)$ can be rewritten as in

$$\bar{r}(\omega) := \sum_{i=\ell+1}^{\bar{\ell}+1} |\widehat{X}_i^{(\omega)} H_{\omega_{p,i(\omega)}(\omega)}(j\omega)|$$

and we can see highly explicit relationship between $\bar{r}(\omega)$ and $\underline{r}(\omega)$.

3.3 Evaluating Least Upper Bounds

Suppose $\bar{\ell} = \ell$ is verified, that is, consider a case where intervals for high order modal parameters are unknown or no information contributing to uncertainty quantification is available, then from the discussion in the previous section we can see

Corollary 2. The relation

$$\underline{r_\ell}(\omega) = \overline{r_\ell}(\omega) = r(\omega) = \bar{\rho}^{(\ell)} \bar{h}_\ell(\omega)$$

will hold true for $\bar{\ell} = \ell$.

In other words, in the case of $\bar{\ell} = \ell$, the upper and lower bounds shown in the above theorems coincide, and become equal to the least upper bounds $r(\omega)$ and explicit formula for the least upper bounds can be available.

Remark 1. It is easy to show the following fact from comparing with the result in [4]: for $\bar{\ell} = \ell$, the complex number $G_n(\omega)$ minimizing

$$\sup_{G \in \mathcal{P}_\ell} |G(j\omega) - G_n(\omega)| \tag{11}$$

for ω is no other than $G_\ell(j\omega)$, and the minimized value is $r(\omega)$. That is,

$$r(\omega) = \min_{G_n(\omega) \in \mathbf{C}} \sup_{G \in \mathcal{P}_\ell} |G(j\omega) - G_n(\omega)| \tag{12}$$

hold true.

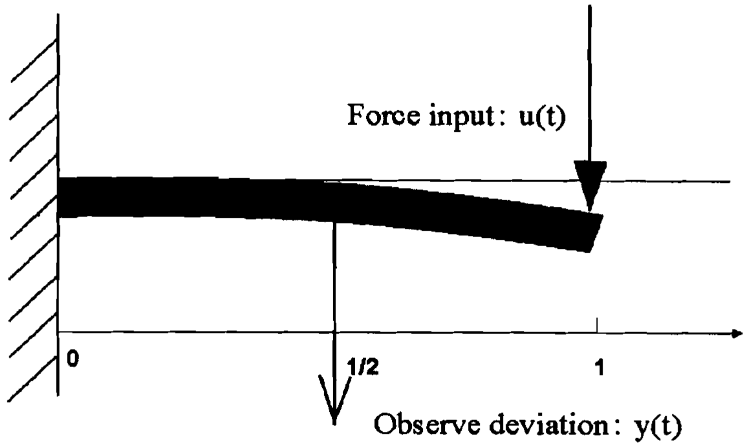

Figure 1: An ideal flexible beam

4 Example

4.1 Flexible Beam: Equation of Motion

We consider an example for controller design to suppress the bending vibration for an ideal flexible beam(Figure 1). Dynamics of bending motion of canti-levered beam where sensors and actuators may not be collocated is described as

$$\frac{\partial^2 v}{\partial t^2}(t,\xi) + 2\gamma\frac{\partial^5 v}{\partial \xi^4 \partial t}(t,\xi) + \frac{\partial^4 v}{\partial \xi^4}(t,\xi) = \delta(\xi - \xi_i)u(t) \quad (0 < \xi < 1) \tag{13.a}$$

$$v(t,0) = \frac{\partial v}{\partial \xi}(t,0) = \frac{\partial^2 v}{\partial \xi^2}(t,1) = \frac{\partial^3 v}{\partial \xi^3}(t,1) = 0 \tag{13.b}$$

$$y(t) = v(t,\xi_o) \tag{13.c}$$

where $\delta(\xi)$ is Dirac's delta function. Furthermore $\gamma = 1 \times 10^{-4}$, and $\xi_i = 1$ and $\xi_o = 0.5$ represent the location of point input and output.

4.2 Evaluating ρ

It is well-known that a countable infinite number of non-trivial solutions to the Eigenvalue/Eigenvector problem

$$\varphi''''(\xi) = \mu\varphi(\xi) \quad (0 < \xi < 1)$$

$$\varphi(0) = \frac{\partial \varphi}{\partial \xi}(0) = \frac{\partial^2 \varphi}{\partial \xi^2}(1) = \frac{\partial^3 \varphi}{\partial \xi^3}(1) = 0$$

Table 1. Bounding k_i's and ω_i's.

i	ω_i		k_i		$\overline{\omega}_i \geq \underline{\omega}_{i+1}$?	$k_i = \max\lvert k_i\rvert$?
1	3.51	$\pm 1.84\times10^{-8}$	1.35	$\pm 2.87\times10^{-4}$	No	No
2	2.20×10^{1}	$\pm 2.85\times10^{-4}$	-2.85	$\pm 1.38\times10^{-1}$	No	No
3	6.16×10^{1}	$\pm 1.68\times10^{-2}$	1.11×10^{1}	$\pm 8.33\times10^{-1}$	No	No
4	1.20×10^{2}	$\pm 2.62\times10^{-1}$	1.47	± 9.35	No	No
5	1.98×10^{2}	± 2.64	0	$\pm 2.96\times10^{1}$	No	Yes
6	2.80×10^{2}	$\pm 3.64\times10^{1}$	0	$\pm 6.60\times10^{1}$	Yes	Yes
7	3.79×10^{2}	$\pm 7.47\times10^{1}$	0	$\pm 1.28\times10^{2}$	Yes	Yes

exist, and let $0 < \mu_1 \leq \mu_2 \leq \cdots$ and corresponding $\varphi_i(\xi)$ the transfer function can be written as

$$G(s) = \sum_{i=1}^{\infty} \frac{c_i b_i / \mu_i}{1 + 2\gamma s + s^2/\mu_i} \tag{14}$$

where $c_i = \varphi_i(\xi_o)$ and $b_i = \varphi_i(\xi_i)$. If we apply this to (1), we see $\omega_i^2 = \mu_i$, $\zeta_i = \gamma\omega_i$ $(\forall i)$, $k_i = c_i b_i/\mu_i$.

Furthermore,

$$\begin{aligned} \sum_{i=1}^{\infty} |k_i| &= \sum_{i=1}^{\infty} |c_i\omega_i^{\alpha'}||b_i\omega_i^{-2-\alpha'}| \\ &\leq \left(\sum_{i=1}^{\infty} |c_i\omega_i^{\alpha'}|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} |b_i\omega_i^{-2-\alpha'}|^2\right)^{\frac{1}{2}} \end{aligned} \tag{15}$$

while

$$\sum_{i=1}^{\infty} |c_i|^2 = \int_0^1 \delta^2(\xi - \xi_o)dx,\ \sum_{i=1}^{\infty} |b_i|^2 = \int_0^1 \delta^2(\xi - \xi_i)dx$$

should be bounded. So $\alpha' < 0$, $-2 - \alpha' < 0$. On the other hand $\sum_{i=1}^{\infty} |c_i/\omega_i|^2 = \eta_{\xi_o}(\xi_o)$ where $\eta_{\xi_o}(\xi)$ is a solution to the following boundary value problem:

$$\eta_{\xi_o}''''(\xi) = \delta(\xi - \xi_o),\ 0 < \xi < 1$$

$$\eta_{\xi_o}(0) = \eta_{\xi_o}'(0) = \eta_{\xi_o}''(1) = \eta_{\xi_o}'''(1) = 0$$

$\eta_{\xi_i}(\xi)$ can be defined similarly, and $\sum_{i=1}^{\infty} |b_i/\omega_i|^2 = \eta_{\xi_i}(\xi_i)$. We obtain that if $\alpha' = -1$, then Evaluation like $\rho = \sqrt{\eta_{\xi_o}(\xi_o) \cdot \eta_{\xi_i}(\xi_i)}$ proves to be true.

4.3 Quantification of Error Bounds

Quintic B-splines defined on equidistant 21 spatial nodes on $[0, 1]$, adjusted to the boundary conditions, are used as a set of coordinate functions. And we evaluate as $\eta_{\xi_i}(\xi_i) = 0.333$, $\eta_{\xi_o}(\xi_o) = 0.0417$ by Galerkin approximation using the coordinate functions. A finite dimensional system is derived, and the eigenvalues of the approximating system are guaranteed from principles of variations to be upper bounds of the eigenvalues of the original system. The computed results of bounding parameters ω_i, k_i are shown in Table 1. Refer to [5] for detail of the parameter bounding.

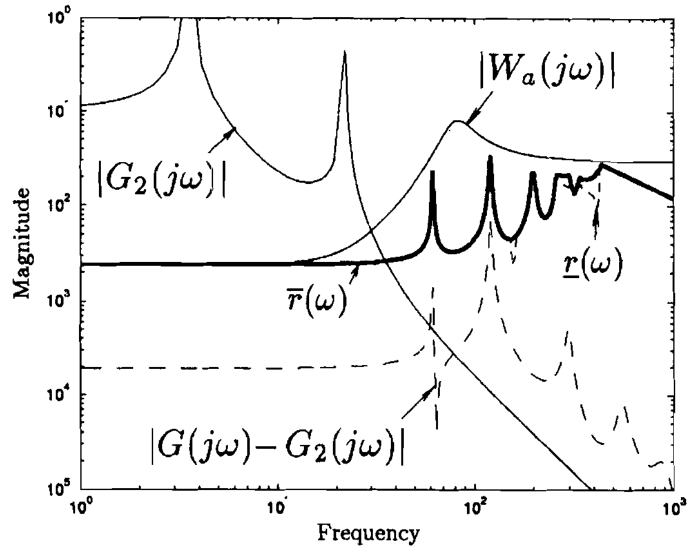

Figure 2: Evaluation of upper and lower bounds to the least upper bounds.

Applying the formula of the main result, both sufficient and necessary magnitudes are plotted in Figure 1. We see that their gap is so small that they do not affect controller design. Two major vibrating modes are considered in the nominal model ($\ell = 2$). We obtain the 8th order controller that satisfies the specs for $\sigma_0 = 1$ and $\omega_d = 1$ using the MATLAB linear matrix inequalities (LMI) toolbox. The design results of sensitivity function $|S(j\omega)|$, performance weight $|W_d(j\omega)|$, and an error bound $|W_a(j\omega)|$ are plotted in Figure 2.

4.4 Discussion

The feasible set in this example is broader than that in [5] since we are not using internal eigenstructures but just modal parameters of input-output relations. Thus uncertainty bounds become larger by +10dB, but we found little effects on achievable control performance.

5 Conclusion

In this paper we proposed a modeling of uncertainty in elastic vibrating systems. Here we formulate the system as input-output relation, and presented a methods to bound of uncertainty in the frequency domain. The class of systems considered

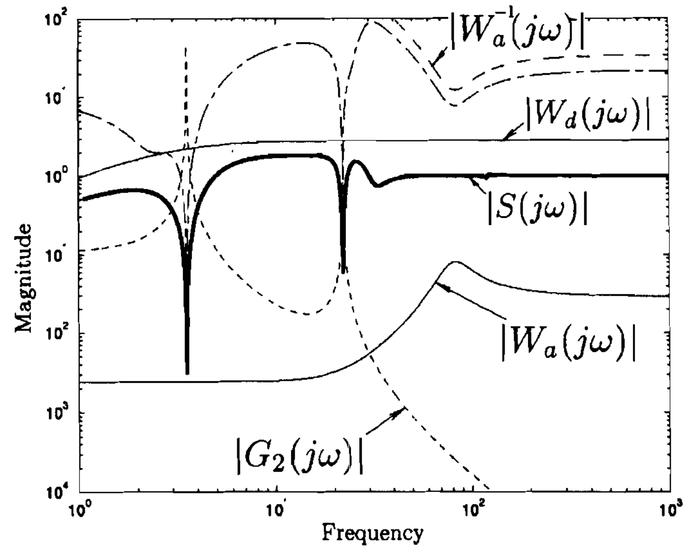

Figure 3: Design result of controller ($|S(j\omega)|$,$|G_2(j\omega)|$, $|W_d(j\omega)|$, $|W_a(j\omega)|$)

here include ones in [5] and the results are its generalization in some sense. Our approach is based on feasible set of systems, and it is to find upper and lower bounds of uncertainty magnitudes by solving linear programming. The minimum size of linear programming problem to be solved is given for bounding the errors allowing for reasonable effiencient computation. We pointed out that the least magnitude of uncertainties is given in explicit form for the case where intervals of high order modal parameters are unknown; we clarified a meaning of the result in [4] in another view. We illustrate the feasibility of the proposed scheme by numerical study on an ideal flexible beam example.

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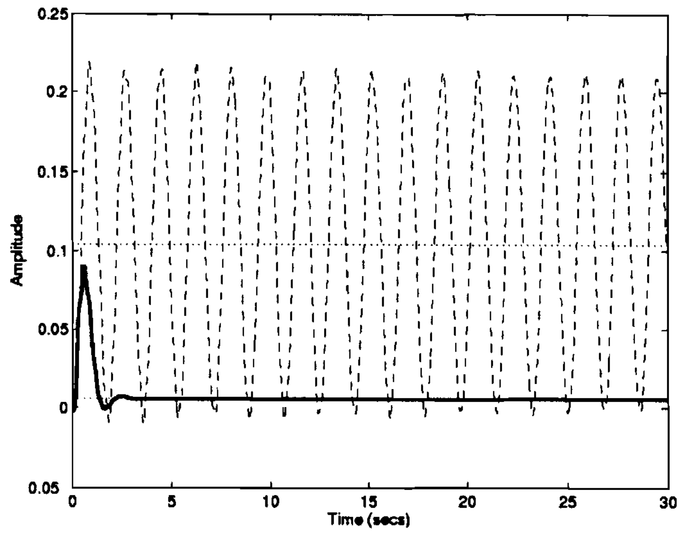

Figure 4: Step response of the plant in open and closed loop

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