Combinatorial Boundary Tracking of a 3D Lattice Point Set

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Boundary tracking and surface generation are ones of main topological topics for three-dimensional digital image analysis. However, there is no adequate theory to make relations between these different topological properties in a completely discrete way. In this paper, we present a new boundary tracking algorithm which gives not only a set of border points but also the surface structures by using the concepts of combinatorial/algebraic topologies. We also show that our boundary becomes a triangulation of border points (in the sense of general topology), that is, we clarify relations between border points and their surface structures.

1. Introduction

Several algorithms have been presented for border/boundary tracking [4, 19, 20, 30] and surface generation [4, 6, 8, 17, 22, 24, 31] of 3-dimensional digital images in the purposes of visualization, calculation of geometric features such as surface areas, calculation of topological features such as Euler characteristics, numerical analysis for deformable objects, etc. Even if both two topological properties of borders and surfaces are required simultaneously (sometimes implicitly) for many applications as listed above, it is not easy to find a useful theory to allow us to discuss both topological properties for each point in a 3-dimensional lattice space, i.e. for each voxel in a 3-dimensional digital image. Note that there are many approximation techniques, but we are interested in completely discrete techniques because our input is digital images and our computation for image analysis are also digital. Even in the Euclidean space, it is not easy to make relations between borders in the sense of general topology and surfaces in the sense of combinatorial topology [23]; the more discussions on the historical backgrounds are found in Section 2.

In this paper, we tackle a problem for clarifying relations between border points and surface points in a 3-dimensional lattice space by using polyhedral complexes such that all vertices are lattice points and the adjacent vertices are neighboring each other in the sense of 3-dimensional digital topology [19]. By using such polyhedral complexes called discrete polyhedral complexes, we also present a new effective algorithm for tracking all border points which constitute a combinatorial surface with their surface structures simultaneously.

The definition of border points is based on general topology [9, 23] and it has been shown that we can obtain border points by a set operation using neighborhoods [19, 27, 30]. Because we need to carry out the set operation for each point in a 3-dimensional lattice space, i.e. each voxel in a 3-dimensional digital image, the computational time is linear to the size of a digital image.

In two dimensions, some effective border tracking algorithms have been proposed by using curve structures of borders [19, 25] such that a border is given as a sequence of points (or pixels) and each point (or pixel) has exactly two neighboring points (or pixels) of a border. Each border point is tracked by a "left-hand-on-wall" border following algorithm from the previous point in a sequence; therefore, we do not have to scan all points in a whole digital image; the computational time becomes linear to the number of border points.

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In three dimensions, the completely different approach from that of two dimensions is commonly used because of the difficulty for finding "surface structures" of border points. An algebraic-topology-based approach is taken so that unit cubes (or voxels) whose centroid are lattice points are first considered and then for border tracking the common faces between two voxels centered at the points \( p \) in a object region and \( q \) in the complement are considered \cite{4}. Such faces are represented by the ordered pair \( (p, q) \). Therefore, boundaries are represented by surfaces which are sets of square faces and whose topological structures are given as cellular complexes as shown in \cite{20}. We can also consider the set of all \( p \) (resp. \( q \)) of such pair \( (p, q) \) as the internal border \cite{30}. However, for such internal borders, we can not obtain any topological surface structures.

There are some axiomatic definitions of discrete surfaces such that all points of discrete surfaces are lattice points and not voxel faces \cite{6, 8, 17, 24}. However, the relations between border points and those surface points are not yet clarified; the connectedness of border points are shown in \cite{16, 18}, but the concept of connectedness is clearly not sufficient for providing surface structures.

To solve our problem, i.e. to give the relations between border points and surface points, we need some other approach to define surface points. In the sense of combinatorial topology \cite{3}, this is a special formulation of a triangulation problem for border points. In Section 3 we define 3-dimensional discrete polyhedral complexes and then give the combinatorial boundary which contains 2-dimensional surface structures. In Section 4 we present an algorithm to provide a combinatorial boundary from any given 3-dimensional lattice point set. Because our practical algorithm is like a marching cubes algorithm \cite{22, 31} by using a look-up table, our computational time is linear to the size of a 3-dimensional digital image. We then derive the relations between borders in the sense of general topology and our combinatorial boundaries. From the relations, we finally conclude that our combinatorial boundary tracking algorithm gives a triangulation for border points without applying set operations of border points with respect to a given 3-dimensional lattice point set.

2. Historical Background

2.1 Borders, Frontiers and Combinatorial Boundaries

2.1.1 Borders, frontiers and combinatorial boundaries in \( \mathcal{R}^n \)

General topology studies topological spaces defined by open and closed sets \cite{9, 23}, allowing to introduce interior \( \text{Int}(A) \), border \( \text{Br}(A) \) and frontier \( \text{Fr}(A) \) of a point set \( A \subset \mathcal{R}^2 \) (b) the border of \( A \), (c) the border of \( \bar{A} \), and (d) the frontier.

![Figure 1: Examples of (a) a point set \( A \subset \mathcal{R}^2 \), (b) the border of \( A \), (c) the border of \( \bar{A} \), and (d) the frontier.](image)

Figure 1 shows examples of the border and frontier of a point set \( A \subset \mathcal{R}^3 \).

In this paper we also consider the combinatorial boundary of a \( n \)-dimensional polyhedral complex \( K \) \cite{3, 23} because we would like to treat boundaries as \( (n-1) \)-dimensional manifolds (more precisely, \( (n-1) \)-dimensional pure polyhedral complexes \cite{3}). If \( K \) is

\[
\begin{align*}
\text{Int}(A) & = \{x \in A : U_\varepsilon(x) \subseteq A\}, \\
\text{Br}(A) & = A \setminus \text{Int}(A), \\
\text{Fr}(A) & = \bar{A} \setminus U_\varepsilon(x) \subseteq A,
\end{align*}
\]

of radius \( \varepsilon > 0 \) define a basis of open sets for this Euclidean space.

If a point \( x \in A \subset \mathcal{R}^n \) is such that there exists a neighborhood \( U_\varepsilon(x) \subseteq A \), then it is called an interior point of \( A \). Otherwise, a point \( x \in A \) is called a border point of \( A \). Let \( \text{Int}(A) \) and \( \text{Br}(A) \) be the sets of all interior and border points such that

\[
\begin{align*}
\text{Int}(A) & = \{x \in A : U_\varepsilon(x) \subseteq A\}, \\
\text{Br}(A) & = A \setminus \text{Int}(A),
\end{align*}
\]

called the interior and border of \( A \), respectively. Then we have \( A = \text{Int}(A) \cup \text{Br}(A) \).

Let \( \bar{A} \) be the complement of \( A \) such that \( \mathcal{R}^n = A \cup \bar{A} \). Then, the interior points of \( \bar{A} \) are also the exterior points of \( A \). The union of the borders of \( A \) and \( \bar{A} \) yields the frontiers \( \text{Fr}(A) \) and \( \text{Fr}(\bar{A}) \) such that

\[
\text{Fr}(A) = \text{Br}(A) \cup \text{Br}(\bar{A}) = \text{Fr}(\bar{A}).
\]

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a 3-dimensional polyhedral complex, then the combinatorial boundary $\partial K$ is the set of all 2-polyhedra $\sigma$ of $K$ such that $\sigma$ lies only one 3-polyhedron of $K$ together with all faces of such simplices $\sigma$. The precise definitions of polyhedral complexes and the combinatorial boundaries will be given in Section 3. Let $K$ be a 3-dimensional polyhedral complex which is a triangulated 3-manifold with boundary and $|K|$ be the union of the elements of $K$, with the subspace topology induced by the topology of $\mathbb{R}^n$. Then, the relation between the frontier and the combinatorial boundary is derived such that

$$|\partial K| = Fr(|K|) \quad (5)$$

if $|K|$ is closed; see [23] for the proof.

2.1.1 Borders and Boundaries in $\mathbb{Z}^2$ and $\mathbb{Z}^3$

Let us consider the set $\mathbb{Z}^n$ of all lattice points in $\mathbb{R}^n$ such that their coordinates are all integers. For any point set $V \subset \mathbb{Z}^n$, which is given as an object component in an $n$-dimensional binary image, borders are also defined similarly to borders in $\mathbb{R}^n$. In this paper, we consider the cases $n = 2, 3$.

Traditionally, the following $m$-neighborhoods

$$N_m(x) = \{y \in \mathbb{Z}^n : \|x - y\| \leq t\}$$

with $t = 1, \sqrt{2}$ (resp. $t = 1, \sqrt{2}, \sqrt{3}$) in common use for $x \in \mathbb{Z}^2$ (resp. $x \in \mathbb{Z}^3$), and $m = 4, 8$ (resp. $m = 6, 18, 26$) stands for the cardinality of these neighborhood systems [19]. In distinction to $e$-neighborhoods of (1), the radius $t$ is only one of the three numbers $1, \sqrt{2}$ or $\sqrt{3}$. It follows that these $m$-neighborhoods do not establish a basis of open sets of a topology on $\mathbb{Z}^n$, and that image analysis normally only assumes adjacency graphs in $\mathbb{Z}^n$ for defining concepts of connectivity [19].

Let $m \in \{4, 8\}$ for $n = 2$ and $m \in \{6, 18, 26\}$ for $n = 3$. If a point $x$ in $V \subset \mathbb{Z}^n$ is such that $N_m(x) \subseteq V$, then $x$ is called an interior point$^2$ (with respect to $m$-neighborhoods) [19, 30]. The set of interior points of $V$ is called the interior of $V$ and denoted by

$$Int_m(V) = \{x \in V : N_m(x) \subseteq V\} \quad (6)$$

similarly to (2) for $A \subset \mathbb{R}^n$. If a point $x \in V$ is not an interior point of $V$, then $x$ is called a border point of $V$ with respect to $m$-neighborhoods [19, 30]. The set of all border points of $V$ is called the $m$-boundary of $V$, denoted by

$$Br_m(V) = V \setminus Int_m(V). \quad (7)$$

Equation (7) corresponds to (3).

In terms of mathematical morphology [27] it follows that an interior set $Int_m(V)$ of (6) coincides with

$$\text{the erosion of } V \text{ with the structure element } N_m(o) \text{ where } o \text{ is the origin of } \mathbb{Z}^n \text{ [30]. We see that (7) also defines } Br_m(V) \text{ via a set operation such as}$$

$$Br_m(V) = \{x \in V : N_m(x) \cap V \neq \emptyset\}. \quad (8)$$

This is because the radii $t$ of $m$-neighborhoods are constant. Therefore, no set operation corresponding to (8) exists for $Br(A)$ in $\mathbb{R}^n$ of (3). Figure 2 shows examples of the 4-borders of $V \subset \mathbb{Z}^2$ and of the complement $V$.

Let us consider the boundary points of $V \subset \mathbb{Z}^n$, corresponding to the frontier points of $A \subset \mathbb{R}^n$, in the sense of general topology. From (4), a point set $A \subset \mathbb{R}^n$ and the complement $A$ has the frontier which is the “common boundary” as shown in Figure 1 (d). Similarly, we can define the $m$-boundary of $V$ as the union of the $m$-borders of $V$ and of $V$. Such $m$-boundaries are used for the composition of boundaries by contributions from both participating sets [22, 26]. In digital image analysis, however, $Br_m(V)$ and $Br_m(\overline{V})$ are considered separately [19, 25, 30] not only for boundary tracking but also for thinning$^3$.

They are called internal and external $m$-boundaries, respectively [27].

Such concept of “different boundaries” in a discrete space has been pointed out by W. K. Clifford. In [5], he explained the “different boundaries” using an example of a heap of white marbles on the top of which black marbles are put. The boundary of the white part would be a layer of white marbles and the boundary of the black part would be a layer of the black marbles, that is, the two adjacent parts have different boundaries when they are divided into two parts. He also referred to the Aristotelian definitions of continuous and discontinuous: the continuous as that of which two adjacent parts have the same boundary; the discontinuous as that of which two adjacent parts have different boundaries.

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$^2$We follow the terminology in [19], even if the term “inner point” is used instead of “interior point” in [30], to make a correspondence between interior points in $\mathbb{R}^n$ and $\mathbb{Z}^n$.

$^3$For thinning of $V \subset \mathbb{Z}^n$, we consider simple points which can be removed without collapsing the criteria of digital topology [19]. Obviously, simple points are related to border/boundary points of $V \subset \mathbb{Z}^n$. 

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Figure 2: Examples of (a) a point set $V \subset \mathbb{Z}^2$, (b) the 4-border of $V$ and (c) the 4-border of $V$. 

and the discontinuous or discrete as that of which two adjacent parts have different boundaries. Figure 2 (b) and (c) show examples of the internal 4-boundary $Br_4(V)$ and the external 4-boundary $Br_4(V)$, respectively.

This paper adopts the boundary approach as follows: the internal boundary $Br_m(V)$ (resp. the external boundary $Br_m(V)$) defines the initial point set, we consider the combinatorial boundary of an n-dimensional polyhedral complex $K$ such that all vertices are in $Br_m(V)$ (resp. $Br_m(V)$). We call such n-dimensional polyhedral complexes *discrete polyhedral complexes* to distinguish them with other (general) polyhedral complexes. This is a special formulation of a triangulation problem for $Br_m(V)$ or $Br_m(V)$ in the sense of combinatorial topology [3]. In the following sections, we give definitions which are necessary for construction of discrete polyhedral complexes and present a solution for the triangulation problem. We afterwards derive the relations between $Br_m(V)$ and the combinatorial boundary of a discrete polyhedral complex with respect to the relation of (5) for $R^3$. Note that we succeed to derive these relations because we take the the different-boundary approach.

2.2 Border Tracking and Surface Representation in $Z^n$

2.2.1. Connectedness of Border Points and Border Tracking

A set $B \subset Z^n$ is said to be connected or m-connected if any pair of $x, y \in B$ has a point sequence $x_1 = x, x_2, \ldots, x_k = y$ such that all $x_i \in B$ and $x_{i+1} \in N_m(x_i)$ [19].

In $Z^2$, it is known that the m-border $Br_m(V)$ of $V \subset Z^2$ is m'-connected if $V$ is m'-connected without hole where $(m, m') = (4, 8), (8, 4)$ [19, 25]. All m-border points are then tracked as a sequence of points such as $x_1, x_2, x_3, \ldots$ and every point $x_i$ in the sequence is found as an element of m'-neighborhood of the previous point $x_{i-1}$ [19]. We see in this approach that the definition of a "curve" is implicitly given as a sequence of points. In other words, border points are tracked by using the "curve structures" in $Z^2$.

In $Z^3$, it has been shown in [16, 18] that the m-border $Br_m(V)$ of $V \subset Z^3$ is m'-connected if $V$ and $V$ are m'- and m-connected respectively for any pair $(m, m') \in \{6, 18, 26\} \times \{6, 18, 26\} \setminus \{6, 6\}$.

4 Similarly to the case of two dimensions, for border tracking of $V$ in $Z^3$, we need a definition of a "surface" instead of that of a "curve" in $Z^2$. Clearly, the connectivity is not enough for representing the structures of surfaces such as triangulated surfaces.

2.2.2 Surface Representation in $Z^3$

The definition of "surfaces" in $Z^3$ is more complicated than that of "curves" in $Z^2$. There exist various definitions of two-dimensional surfaces in $Z^3$. The approaches are mainly classified into the following four types:

1. the graph-theory-based approach: a surface is defined as a set of lattice points which satisfies some conditions based on the neighborhood relations or the connectedness [6, 24, 30]. Every point on surfaces is considered to have a characteristic of spatial separation according to the Jordan surface theorem in a local sense.

2. the algebraic-topology-based approach: surfaces are defined as the combinatorial boundaries of 3-dimensional cellular complexes. In [4, 10, 20], cells are considered to be unit cubes (or voxels) whose centroids correspond to lattice points and surfaces are represented by sets of faces of unit cubes. In [11], simplicial complexes are used instead of cellular complexes so that the vertices of simplexes are all lattice points.

3. the combinatorial-manifold-based approach: surfaces are triangulated and any point on a surface is topologically equivalent to the central point of an open disc [6, 8, 17].

4. the analytical approach: geometric surfaces such as planes and spheres are defined by using inequations in $Z^3$ instead of using equations in $R^3$ [1, 2, 7].

The analytical approach can be applied if only geometric objects such as planes and spheres are considered. In this paper, we would like to treat any free-form objects. Thus, we cannot take the analytical approach.

The graph-theory-based approach is the most classical, but is also axiomatic. Since it contains only neighborhood relations and not topological structures, the combinatorial-manifold-based approach has been taken in [6, 17] for making comparison between the graph-theory-based approach and the combinatorial-manifold-based approach. Clearly, the combinatorial-manifold-based approach has the strong power for investigating topological structures, but it is not evident that a set of border points can become a combinatorial manifold. For example, a set of border points may not construct a manifold as shown in Figure 3: the left one is a pseudomanifold [28].

For border tracking in $Z^3$, therefore, the algebraic-topology-based approach based on voxels is commonly used [4] such as tracking the common faces between two voxels centered at the points $p$ in $V \subset Z^3$ and $q$ in $V = Z^3 \setminus V$. Such faces are represented by the ordered pair $(p, q)$. Since $q \in N_b(p) \cap V$, the set
of all \( p \) of such pairs \((p, q)\) becomes equal to \( Br_6(V)\) of (8). In this approach, the “surface” is represented by a set of square faces of voxels and the topological structures of cellular complexes, i.e., voxels, voxel faces, etc., are shown in [20].

Because we would like to consider triangulated surfaces on the points of \( Br_m(V) \), we need another notion based on algebraic topology. In this paper, we extend our notions of discrete simplexes in [11] to discrete convex polyhedra and give the definition of discrete polyhedral complexes instead of discrete simplicial complexes in [11]. The following sections are devoted for presenting triangulation of \( Br_m(V) \).

3. Discrete Polyhedral Complexes and Combinatorial Boundaries

In this section, we give definitions of a polyhedral complex which consists of a finite set of convex polyhedra such that the vertices are all points in \( \mathbb{Z}^3 \) and any adjacent vertices are \( m \)-neighboring. Such polyhedral complexes are introduced here for a giving complicial representation of a finite subset of \( V \subset \mathbb{Z}^3 \). An algorithm for obtaining a polyhedral complex from \( V \) will be presented in the next section. Similar complicial representations for \( V \) are also found, for examples, in [14, 18, 29]. The differences between our complicial representation and them will be discussed in Section 6.

3.1 Convex Polyhedra and Polyhedral Complexes in \( \mathbb{R}^n \)

For the definitions of convex polyhedra and polyhedral complexes in \( \mathbb{R}^n \), we follow the notions in [32].

**Definition 1** A convex polyhedron \( \sigma \) is the convex hull of a finite set of points in some \( \mathbb{R}^d \).

The dimension of a convex polyhedron \( \sigma \) is the dimension of its affine hull. An \( n \)-dimensional convex polyhedron \( \sigma \) is abbreviated to an \( n \)-polyhedron. For instance, a point is a \( 0 \)-polyhedron, a line segment is a \( 1 \)-polyhedron, a triangle is a \( 2 \)-polyhedron, and a tetrahedron is a \( 3 \)-polyhedron. A linear inequality \( a \cdot x \leq z \) is valid for \( \sigma \) if it is satisfied for all points \( x \in \sigma \). A face of \( \sigma \) is then defined by any set of the form

\[
\delta = \sigma \cap \{x \in \mathbb{R}^d : a \cdot x = z\}
\]

where \( a \cdot x \leq z \) is valid for \( \sigma \). For instance, a \( 3 \)-polyhedron which is a tetrahedron has four \( 0 \)-polyhedra, six \( 1 \)-polyhedra and four \( 2 \)-polyhedra for its faces. The point of a \( 0 \)-polyhedron, the endpoints of a \( 1 \)-polyhedron and the vertices of \( 2 \) and \( 3 \)-polyhedra are called the vertices of each convex polyhedron.

**Definition 2** A polyhedral complex \( K \) is a finite collection of convex polyhedra such that

1. the empty polyhedron is in \( K \),
2. if \( \sigma \in K \), then all faces of \( \sigma \) are also in \( K \),
3. the intersection \( \sigma \cap \tau \) of two convex polyhedra \( \sigma, \tau \in K \) is a face both of \( \sigma \) and of \( \tau \).

The dimension of \( K \) is the largest dimension of a convex polyhedron in \( K \).

3.2 Discrete Convex Polyhedra and Discrete Polyhedral Complexes

Now we consider polyhedral complexes such that the vertices of convex polyhedra are all lattice points in \( \mathbb{Z}^3 \) and the adjacent vertices are \( m \)-neighboring for \( m = 6, 18, 26 \). For constructing such polyhedral complexes, we first consider all possible convex polyhedra such that the vertices are all lattice points and any adjacent vertices of a convex polyhedron are \( m \)-neighboring each other for \( m = 6, 18, 26 \). Such convex polyhedra and polyhedral complexes are called discrete convex polyhedra and discrete polyhedral complexes hereafter. The constraints allow us to look for a discrete convex polyhedron which is not larger than the unit cubic region as follows.

Let us consider all possible convex polyhedra in a unit cubic region such that the vertices of each convex polyhedron are the vertices of a unit cube. A unit cube has eight lattice points for the vertices. For each lattice point we assign the value of either 1 or 0 and call the point a 1- or 0-point, respectively. There are 256 configurations of 1- and 0-points for the eight lattice points in a unit cubic region. In fact, we can reduce the number of the configurations from 256 to 23 with considering the congruent configurations by rotations as shown in Table 1.

For each configuration, we obtain a convex polyhedron such that the vertices of the polyhedron are 1-points. We then classify each convex polyhedron into a set of discrete convex polyhedra with the dimension of \( n = 0, 1, 2, 3 \) and with the \( m \)-neighborhood relations between the adjacent vertices for \( m = 6, 18, 26 \) as shown in Table 2. From Table 2, we see that there are a finite number of discrete convex polyhedra for

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6For the proof that the 23 configurations are complete, see the appendix B of [16]. Note that there are 22 configurations in [16] because symmetry is also considered.
Table 1: All possible 23 configurations of 1- and 0-points for the eight lattice points in a unit cubic region. With considering the congruent configurations by rotations, we obtain all 256 configurations from them.

For any neighborhood system, an isolated point of configuration P1 in Table 2 is regarded as a discrete 0-polyhedron. Similarly, the line segment for configuration P2a is regarded as a discrete 1-polyhedron for any neighborhood system because the adjacency between two points are m-neighboring for any \( m = 6,18,26 \). However, the line segment of configuration P2b is not considered to be a discrete 1-polyhedron for the 6-neighborhood system, but considered to be a discrete 1-polyhedron for the 18- and 26-neighborhood systems. The line segment of configuration P2c is considered to be a discrete 1-polyhedron only for the 26-neighborhood system. The discrete 2-polyhedra for each neighborhood system are illustrated in the last line of Table 2.

In Table 2, we see that every \( n' \)-dimensional face of any discrete \( n \)-polyhedron for \( n' < n \) is also a discrete \( n' \)-polyhedron for each \( m \)-neighborhood system, \( m = 6,18,26 \). This is important because it enables us to construct a discrete polyhedral complex which is a finite collection of discrete convex polyhedra satisfying the three conditions in Definition 2 for each \( m \)-neighborhood system.

If we cannot decompose a discrete \( n \)-polyhedron into other discrete \( n \)-polyhedra in one of the neighborhood systems, such a discrete \( n \)-polyhedron also called discrete \( n \)-simplex [11]. In \( \mathbb{R}^3 \), any \( n \)-dimensional simplex has \( n+1 \) vertices [3] while there exist, in \( \mathbb{Z}^3 \), discrete \( n \)-simplexes which has more than \( n+1 \) vertices such as the discrete simplexes of P4a and P8 for the 6-neighborhood system in Table 2. In mathematics such as combinatorial topology, simplexes are sometimes more focused on than cells or convex polyhedra. It is because polygonal 2-polyhedra are too general compared with triangular 2-simplexes. In our case, however, if we only use discrete simplexes for triangulation of a subset \( V \) of \( \mathbb{Z}^3 \), the simplicial decomposition of \( V \) may not be...
Definition 3 Let \( \mathcal{O} \) be a pure discrete 3-complex and \( \mathcal{H} \) be the set of all discrete 2-polyhedra in \( \mathcal{O} \) each of which is a face of exactly one discrete 3-polyhedron in \( \mathcal{O} \). The boundary of \( \mathcal{O} \) is defined as

\[
\partial \mathcal{O} = \text{Cl}(\mathcal{H}).
\]

From Definition 3, we obtain the following proposition.

**Proposition 1** The boundary \( \partial \mathcal{O} \) of a pure discrete 3-complex \( \mathcal{O} \) is a pure discrete 2-subcomplex of \( \mathcal{O} \).
the following. If a discrete \( n \)-polyhedron \( \sigma \) for an \( m \)-neighborhood system exists with respect to an configuration of 1-points in \( D(x) \) in Table 2, we set \( C_m(x) \) to be a collection of \( \sigma \) and its faces where \( n = 0, 1, 2, 3 \). Otherwise, we consider discrete \( n \)-polyhedra \( \sigma \) such that \( n \) is as large as possible where \( n \leq 3 \) and the vertices of \( \sigma \) are all 1-points in \( D(x) \) and set \( C_m(x) \) to be a collection of such \( \sigma \) and their faces. For each 1-point configuration of \( D(x) \), we then obtain a discrete polyhedral complex \( C_m(x) \) for each \( m = 6, 18, 26 \) as shown in Table 3.

Now let

\[
C_m = \bigcup_{x \in \mathbb{Z}^3} C_m(x), \tag{9}
\]

and we verify that \( C_m \) is mostly a discrete polyhedral complex satisfying the conditions in Definition 2; there is an exceptional case that we need to replace \( C_m(x) \) to obtain a discrete polyhedral complex \( C_m \) by (9) for \( m = 18 \).

Say that \( C_m(x) \) and \( C_m(y) \) are adjacent if \( D(x) \cap D(y) \neq \emptyset \). Their adjacency types are classified into the following three:

\[
\#(D(x) \cap D(y)) = 1, 2 \text{ or } 4 \text{ (and never } 3)\]

where \( \#(A) \) represents the number of elements of the set \( A \). The adjacency types and the conceivable polyhedral decomposition at the joint are illustrated in Table 4. For each adjacent pair of \( C_m(x) \) and \( C_m(y) \), let

\[
C_m(x, y) = C_m(x) \cup C_m(y). \tag{10}
\]

We then verify, from Table 3, that \( C_m(x, y) \) is mostly a discrete polyhedral complex satisfying the conditions of Definition 2; there is an exceptional case that we need to replace \( C_m(x) \) and \( C_m(y) \) to obtain a discrete polyhedral complex \( C_m(x, y) \) by (10) for \( m = 18 \).

First, let us consider the case of \( \#(D(x) \cap D(y)) = 1 \). As shown in the first line of Table 4, the common point \( z \) is either 1- or 0-point. If \( z \) is a 0-point (Case 1), \( C_m(x) \) and \( C_m(y) \) include no common discrete convex polyhedron. Thus, we simply obtain \( C_m(x, y) \) by (10) as empty. If \( z \) is a 1-point (Case 2), both \( C_m(x) \) and \( C_m(y) \) include a common discrete 0-polyhedron \( \sigma_0 \). Let us introduce the notion of the skeleton \( Sk(\sigma) \) of a discrete convex polyhedron \( \sigma \) such as the set of all vertices of \( \sigma \) [3]. Then, we have \( Sk(\sigma_0) = \{ z \} \). Thus, we obtain a discrete 0-complex \( C_m(x, y) = \{ \sigma_0 \} \) by (10).

In the case of \( \#(D(x) \cap D(y)) = 2 \), there are two common points \( z_1 \) and \( z_2 \) as shown in the second line of Table 4. Since each of \( z_1 \) and \( z_2 \) is either 1- or 0-point, there are three possible configurations of 1- and 0-points for the pair of \( z_1 \) and \( z_2 \). If both \( z_1 \) and \( z_2 \) are 0-points (Case 1), both \( C_m(x) \) and \( C_m(y) \) include no common discrete polyhedron. Thus, we obtain \( C_m(x, y) \) by (10) as empty. If either of \( z_1 \) and

---

**Table 3: Discrete convex polyhedral decomposition**

\( C_m(x) \) with respect to every 1-point configuration of a unit cubic region \( D(x) \) for (a) \( m = 6 \), (b) \( m = 18 \) and (c) \( m = 26 \).
Table 4: Three adjacency types of two unit cubic regions $D(x)$ and $D(y)$ such that $\#(D(x) \cap D(y)) = 1, 2$ and 4. For each adjacency type, all possible configurations of 1- and 0-points and a discrete convex polyhedral decomposition are shown.

<table>
<thead>
<tr>
<th>$#(D(x) \cap D(y))$</th>
<th>cellular decomposition of $D(x) \cap D(y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td><img src="image1" alt="Cellular Decomposition" /></td>
</tr>
<tr>
<td>2</td>
<td><img src="image2" alt="Cellular Decomposition" /></td>
</tr>
<tr>
<td>4</td>
<td><img src="image3" alt="Cellular Decomposition" /></td>
</tr>
</tbody>
</table>

$z_2$ is 1-point (Case 2), the 1-point becomes the common 0-polyhedron $\sigma$ in $C_m(x)$ and $C_m(y)$. Thus, we obtain a discrete 0-complex $C_m(x, y) = \{ \sigma \}$ by (10). If both $z_1$ and $z_2$ are 1-points (Case 3), $C_m(x)$ and $C_m(y)$ have a discrete 1-polyhedron $\sigma$ and its 0-dimensional faces as the common discrete polyhedron such that $Sk(\sigma) = \{z_1, z_2\}$. Thus, we obtain a discrete 1-complex $C_m(x, y) = Cl(\{\sigma\})$ by (10).

In the case of $\#(D(x) \cap D(y)) = 4$, let $z_i$ for $i = 1, 2, 3, 4$ be the four common points. Since each point is either 1- or 0-point, there are six configurations of 1- and 0-points for the four points as shown in the last line of Table 4. Similarly to the above approach, we see that a discrete complex $C_m(x, y)$ is obtained by (10) for almost every configuration; there is one exceptional case for the 18-neighborhood system. The last line of Table 4 shows possible common discrete polyhedra of $C_m(x)$ and $C_m(y)$: the empty set in Case 1; a discrete 0-polyhedron $\sigma_0$ such that $Sk(\sigma_0) = \{z_2\}$ in Case 2; a discrete 1-polyhedron $\sigma_1$ and its 0-dimensional faces such that $Sk(\sigma_1) = \{z_2, z_3\}$ in Case 3; two discrete 0-polyhedra $\sigma_0$ and $\sigma_1$ such that $Sk(\sigma_0) = \{z_2\}$ and $Sk(\sigma_1) = \{z_2\}$ for the 6-neighborhood system and a discrete 1-polyhedron $\sigma_1$ with its 0-dimensional faces such that $Sk(\sigma_1) = \{z_2, z_4\}$ for the 18- and 26-neighborhood systems in Case 4; two discrete 1-polyhedra $\sigma_{10}$ and $\sigma_{11}$ with their faces such that $Sk(\sigma_{10}) = \{z_2, z_3\}$ and $Sk(\sigma_{11}) = \{z_3, z_4\}$ for the 6-neighborhood system, and a discrete 2-polyhedron $\sigma_2$ with its 0- and 1-dimensional faces such that $Sk(\sigma_2) = \{z_2, z_3, z_4\}$ for the 18- and 26-neighborhood systems in Case 5.

In Case 6 of the last line of Table 4, $C_m(x)$ and $C_m(y)$ have common discrete polyhedra such as a discrete 2-polyhedron $\sigma$ and its 0- and 1-dimensional faces including such that $Sk(\sigma) = \{z_1, z_2, z_3, z_4\}$ for any neighborhood system except for the case such that adjacent unit cubes $D(x)$ and $D(y)$ have both the configurations P5a as shown in Figure 6 (a) for $m = 18$; in this case, we have

$C_{18}(x) = Cl(\{\sigma_{20}\}) \cup Cl(\{\sigma_{21}\})$

where $\sigma_{20}$ and $\sigma_{21}$ are discrete 2-polyhedra such that $Sk(\sigma_{20}) = \{z_1, z_2, z_3\}$ and $Sk(\sigma_{21}) = \{z_1, z_3, z_4\}$, and

$C_{18}(y) = Cl(\{\sigma_{22}\}) \cup Cl(\{\sigma_{23}\})$

where $\sigma_{22}$ and $\sigma_{23}$ are discrete 2-polyhedra such that $Sk(\sigma_{22}) = \{z_1, z_2, z_4\}$ and $Sk(\sigma_{23}) = \{z_2, z_3, z_4\}$. Thus, $C_{18}(x, y)$ obtained by (10) does not construct a discrete polyhedral complex. In order to obtain $C_{18}(x, y)$ as a discrete polyhedral complex, we therefore replace discrete polyhedral complexes $C_{18}(x)$ and $C_{18}(y)$ from Figure 6 (a) to (b) so that

$C_{18}(x) = Cl(\{\sigma_{10}\}) \cup Cl(\{\sigma_{11}\})$

$C_{18}(y) = Cl(\{\sigma_{11}\}) \cup Cl(\{\sigma_{12}\})$

where $Sk(\sigma_{10}) = \{z_1, z_2\}$, $Sk(\sigma_{11}) = \{z_2, z_3\}$ and $Sk(\sigma_{12}) = \{z_3, z_4\}$.

Consequently, setting $C_m(x)$ for each $x \in Z^3$ referring to Table 3 with taking account of the additional replacement of Figure 6 for $m = 18$, we uniquely obtain $C_m$ by (9) for any $m = 6, 18, 26$ from any $V \subset Z^3$.

4.1.2 Step 2: Construction of a Pure Discrete 3-Complex

Assume that the dimension of $C_m$ is three. Let $G$ be the set of all discrete 3-polyhedra in $C_m$. In order to obtain a pure discrete 3-complex $O_m$ from $C_m$, we remove all discrete $n$-polyhedra which are not included in any discrete 3-polyhedra in $C_m$ for every $n < 3$, such that

$O_m = Cl(G)$. (11)
If \( C_m \) is less than three dimensions, \( G \) is empty and thus \( O_m \) is also empty. This occurs when \( C_m \) contains only discrete 0-, 1- and 2-polyhedra and have no discrete 3-polyhedron. We consider that \( C_m \) \( \setminus O_m \) each of whose element has less than three dimensions is caused by the limited resolution of a digital image. If we would like to see the part \( C_m \setminus O_m \) in the higher dimensions, then we increase the resolution of a digital image at the part. From (11), it is clear that \( O_m \) is uniquely obtained from \( C_m \). Examples of the procedure for obtaining \( O_{26} \) from \( C_{26} \) are seen in Figures 4 (from (b) to (a)) and 5 (from (b) to (c)).

We can also obtain \( O_m \) directly from \( V \) without considering \( C_m \), such that

\[
O_m = \bigcup_{x \in \mathbb{Z}^3} O_m(x)
\]

where \( O_m(x) \) is a pure discrete 3-complex at each unit cubic region \( D(x) \). Each \( O_m(x) \) is easily obtained by referring to Table 5 instead of Table 3 for \( C_m(x) \). We easily create Table 5 by making \( C_m(x) \) in Table 3 to be pure. Note that \( O_{18}(x) \) will be replaced as an empty set if the 1-point configurations at \( D(x) \) and its adjacent \( D(y) \) are as illustrated in Figure 6.

4.1.3 Step 3: Boundary Extraction of a 3D Pure Complex

From Definition 3, the boundary \( \partial O_m \) of \( O_m \) is derived from the set \( H \) of discrete 2-polyhedra in \( O_m \) each of which is a face of exactly one discrete 3-polyhedron in \( O_m \). Because \( H \) is uniquely obtained from \( O_m \), \( \partial O_m \) is also uniquely obtained from \( O_m \). From the above procedure, we consequently obtain the following proposition.

Proposition 2 Given a finite subset \( V \subset \mathbb{Z}^3 \), the combinatorial boundary \( \partial O_m \) is uniquely obtained for any \( m \)-neighborhood system, \( m = 6, 18, 26 \).

4.2 Practical Algorithm of Combinatorial Boundary Tracking

For practical use, we present an effective algorithm of generating \( \partial O_m \) directly from \( V \) by referring to Table 6, which is a similar table used for the marching cubes method [22, 31], for each neighborhood system. The comparison between the marching cubes method and our method is discussed in [12].

We obtain Table 6 from Table 3 in the following. First we see only discrete 2-polyhedra of \( C_m(x) \) at each unit cubic region \( D_m(x) \) because \( \partial O_m \) is a pure discrete 2-complex; \( \partial O_m \) does not contain more than three-dimensional discrete convex polyhedra and less than two-dimensional discrete convex polyhedra which are not faces of any discrete 2-polyhedra. We then classify each discrete 2-polyhedron \( \sigma \) of \( C_m(x) \) in Table 3 into four types:

<table>
<thead>
<tr>
<th>Table 5: Three-dimensional polyhedral decomposition ( O_m(x) ) corresponding to the configuration of 1-points in a unit cubic region ( D(x) ) for (a) ( m = 6 ), (b) ( m = 18 ) and (c) ( m = 26 ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
</tr>
<tr>
<td>(b)</td>
</tr>
<tr>
<td>(c)</td>
</tr>
</tbody>
</table>
1. there exists a discrete 3-polyhedron \( \delta \) in \( C_m(x) \) such that
\[
\sigma = \delta \cap \{ x \in \mathbb{R}^3 : a \cdot x = z \}
\]
where \( a \cdot x \leq z \) is valid for \( \delta \), i.e., \( \sigma \) is a face of \( \delta \), and

1.1. the equation
\[
\{ x \in \mathbb{R}^3 : a \cdot x \leq z \} \cap D(x) = D(x)
\]
holds, i.e., \( \sigma \) is located at a face of a unit cube \( D(x) \);

1.2. equation (12) does not hold, i.e., \( \sigma \) is located inside a unit cube \( D(x) \);

2. there is no discrete 3-polyhedron \( \delta \) like the above, and

2.1. either of the equations
\[
\{ x \in \mathbb{R}^3 : a \cdot x \leq z \} \cap D(x) = D(x), \tag{13}
\]
\[
\{ x \in \mathbb{R}^3 : a \cdot x \geq z \} \cap D(x) = D(x), \tag{14}
\]
where \( \sigma \cap \{ x \in \mathbb{R}^3 : a \cdot x = z \} = \sigma \)
holds, i.e., \( \sigma \) is located at a face of a unit cube \( D(x) \);

2.2. both (13) and (14) do not hold, i.e., \( \sigma \) is located inside a unit cube \( D(x) \).

For each \( C_m(x) \), we obtain the set of discrete 2-polyhedra of each type such as \( TP_{11}(x) \), \( TP_{12}(x) \), \( TP_{21}(x) \) and \( TP_{22}(x) \). We then set
\[
I_m(x) = \bigcup_{\sigma \in TP_{11}(x)} Cl(\sigma),
\]
\[
J_m(x) = \bigcup_{\sigma \in TP_{22}(x)} Cl(\sigma).
\]

For \( \sigma \in TP_{11}(x) \), \( \sigma \) does not belong to \( \partial O_m \) if there exists a discrete 3-polyhedron \( \delta \) at a unit cube \( D(y) \) adjacent to \( D(x) \) such that \( \sigma \) is a face of \( \delta \). If there is no such \( \delta \) at \( D(y) \), then \( \sigma \in TP_{21}(y) \). Thus, we have \( \sigma \in J_m(y) \) at the adjacent \( D(y) \). For \( \sigma \in TP_{22}(x) \), \( \sigma \in C_m \setminus O_m \) and thus \( \sigma \notin \partial O_m \). For obtaining \( \partial O_m \), therefore, we need only \( J_m(x) \) and \( I_m(x) \) for every \( x \in \mathbb{Z}^3 \). In Table 6, we illustrate a pure discrete 2-complex
\[
T_m(x) = J_m(x) \cup I_m(x) \tag{15}
\]
The arrow of every \( \sigma \) in Table 6 indicates the side where the half space \( \{ x \in \mathbb{R}^3 : a \cdot x > z \} \) exists; roughly speaking, it is oriented to the exterior of \( \partial O_m \) and is useful for visualization as a normal vector of each \( \sigma \). Note that either \( J_m(x) \) or \( I_m(x) \) is empty for any \( T_m(x) \) in Table 6 except for the configuration P5a of the 18-neighborhood system.

We then see that
\[
I_m(x) \subset \partial O_m. \tag{16}
\]
For each discrete 2-polyhedron \( \sigma \in J_m(x) \), if \( \sigma \in J_m(y) \) at an adjacent unit cube \( D(y) \) to \( D(x) \) as shown in Figure 7,
\[
\sigma \in C_m \setminus O_m, \tag{17}
\]
and otherwise
\[
\sigma \in \partial O_m, \tag{18}
\]
Thus, we need to verify (18) for each \( \sigma \in J_m(x) \) for constructing \( \partial O_m \), while every \( \sigma \in I_m(x) \) is always in \( \partial O_m \) from (16). Such verification is achieved in step 1.3 in Algorithm 1. We also mention that the special treatment for the case illustrated in Figure 6 which occurs only for the 18-neighborhood system is considered in step 1.2 in Algorithm 1.
Algorithm 1

input: A subset \( V \) of \( W = \{(i,j,k) \in \mathbb{Z}^3 : 1 \leq i \leq L, 1 \leq j \leq M, 1 \leq k \leq N\} \).

output: A combinatorial boundary \( \partial \mathcal{O}_m \) for each \( m = 6, 18, 26 \).

begin

1. for \( 1 \leq k \leq N - 1 \) do
   for \( 1 \leq j \leq M - 1 \) do
     for \( 1 \leq i \leq L - 1 \) do
       1.1 for \( x = (i,j,k) \), obtain \( T_m(x) \) by referring to Table 6;
       1.2 if \( m = 18 \), check if each pair of \( T_m(x) \) and \( T_m(y) \) for \( y = (i-1,j,k), (i,j-1,k), (i,j,k-1) \) is in the case as illustrated in Figure 6 (a); if so, replace \( T_m(x) \) and \( T_m(y) \) from Figure 6 (a) to (b);
       1.3 if \( J_m(x) \subseteq T_m(x) \) is not empty, then check for each \( \sigma \in J_m(x) \) if \( \sigma \in J_m(y) \) where \( y = (i-1,j,k), (i,j-1,k), (i,j,k-1) \) as shown in Figure 7; if so, replace \( T_m(x) \) and \( T_m(y) \) with \( Cl(T_m(x) \setminus Cl(\{\sigma\})) \) and \( Cl(T_m(y) \setminus Cl(\{\sigma\})) \);

2. obtain \( \partial \mathcal{O}_m = \bigcup_{x \in W} T_m(x) \).

end

From Algorithm 1, it is obvious that we obtain the combinatorial boundary \( \partial \mathcal{O}_m \) for each \( m = 6, 18, 26 \) from any finite set \( V \subset \mathbb{Z}^3 \). Some results of combinatorial boundaries \( \partial \mathcal{O}_m \) for \( m = 6, 18, 26 \) with their inputs \( V \) are shown in Figures 8 and 9.

5. Relations between Borders and Combinatorial Boundaries

We already introduced the notion of the skeleton \( Sk(\sigma) \) of a discrete convex polyhedron \( \sigma \) such as the set of the vertices of \( \sigma \) [3]. Let \( \partial \mathcal{O}_m \) be the combinatorial boundary obtained by Algorithm 1 from a given \( V \subset \mathbb{Z}^3 \) for \( m = 6, 18, 26 \). We call the union of the skeletons of all discrete convex polyhedra of \( \partial \mathcal{O}_m \) the skeleton of \( \partial \mathcal{O}_m \) and it is denoted by \( Sk(\partial \mathcal{O}_m) \). We have the following relations between the skeleton \( Sk(\partial \mathcal{O}_m) \) and the border \( Br_{m'}(V) \) of (8). The relations are the discrete version of the relation (5) in \( \mathbb{R}^3 \).

Theorem 1 The border \( Br_{m'}(V) \) and the skeleton \( Sk(\partial \mathcal{O}_m) \) of the combinatorial boundary \( \partial \mathcal{O}_m \) obtained from a finite subset \( V \subset \mathbb{Z}^3 \) have the relations such that

\[
Br_6(V) = Sk(\partial \mathcal{O}_{26}) \cup (Sk(C_{26}) \setminus Sk(O_{26})) \quad (19)
\]

Figure 8: (a) An digitized sphere and its combinatorial boundaries for (b) 6-, (c) 18- and (d) 26-neighborhood systems.
\[ Br_{18}(V) = Sk(\partial O_6) \cup (Sk(C_6) \setminus Sk(O_6)) \setminus A_{(18,6)}, \]
\[ Br_{26}(V) = Sk(\partial O_6) \cup (Sk(C_6) \setminus Sk(O_6)). \]

where

\[ A_{(m', m)} = \bigcup_{x \in \mathbb{Z}^3} A_{(m', m)}(x) \]

so that \( A_{(m', m)}(x) \) is given as the set of points at a unit cube \( D(x) \) only when \( D(x) \) has a 1-point configuration \( P5a \) or \( P7 \) only for \( (m', m) = (6, 18) \) or \( (18, 6) \), respectively, as shown in Table 7. Note that \( A_{(6, 18)}(x) \) for the configuration \( P5a \) is empty if it has no adjacent unit cube whose configuration is also \( P5a \) as shown in Figure 6.

A pair \( (m', m) \) of neighborhood systems which is considered in Theorem 1 is \( (6, 26), (6, 18), (18, 6) \) or \( (26, 6) \) and similar pairs \( (m', m) \) are also seen in the relations between \( Br_{m'}(V) \) and its \( m \)-connectivity mentioned in section 2.2.16, 18. For proving Theorem 1, we need the following two lemmas.

Lemma 1 For a unit cubic region \( D(x) \), setting

\[ CubeBr_{m'}(V; x) = \{ y \in V \cap D(x) : D(x) \cap N_{m'}(y) \cap \bar{V} \neq \emptyset \} \]

for each \( m' = 6, 18, 26 \), we have

\[ Br_{m'}(V) = \bigcup_{x \in \mathbb{Z}^3} CubeBr_{m'}(V; x). \]

(Proof) Since

\[ N_{m'}(y) \cap \bar{V} = \bigcup_{x \in \mathbb{Z}^3} (D(x) \cap N_{m'}(y) \cap \bar{V}), \]

we obtain (24) from (8) and (23). (Q.E.D.)

The points in \( CubeBr_{m'}(V; x) \) are illustrated for every possible configuration of 1-points in \( D(x) \) in Table 7.

Lemma 2 At each unit cubic region \( D(x) \) for \( x \in \mathbb{Z}^3 \), setting \( T_m(x) \) to be a discrete 2-complex given by Table 6, \( C_m(x) \) to be a discrete polyhedral complex given by Table 3 and \( O_m(x) \) to be a pure discrete 3-complex of \( C_m(x) \) by Table 5, we have

\[ Sk(\partial O_m) \cup (Sk(C_m) \setminus Sk(O_m)) = \bigcup_{x \in \mathbb{Z}^3} (Sk(T_m(x)) \cup (Sk(C_m(x)) \setminus Sk(O_m(x)) \setminus Sk(T_m(x)))) . \]

(Proof) Let us consider the two discrete 2-subcomplexes \( J_m(x) \) and \( I_m(x) \) such as (15) for each \( T_m(x) \). From (16),

\[ Sk(I_m(x)) \subset Sk(\partial O_m). \]
The points of $Sk(I_m(x))$ are illustrated in Table 7. Let us consider a vertex $z \in Sk(J_m(x)) \setminus Sk(I_m(y))$ for any $y$. If $z$ is a vertex of $\sigma$ of (17),

$$z \in Sk(C_m) \setminus Sk(O_m)$$  \hspace{1cm} (27)

and if $z$ is a vertex of $\sigma$ of (18),

$$z \in Sk(\partial O_m).$$  \hspace{1cm} (28)

Thus,

$$Sk(J_m(x)) \setminus Sk(I_m(y)) \subset Sk(\partial O_m) \cup (Sk(C_m) \setminus Sk(O_m))$$  \hspace{1cm} (29)

for any $x$ and $y$. The points of $Sk(J_m(x)) \setminus Sk(I_m(y))$ are also shown in Table 7. For each discrete convex polyhedron $\sigma \in C_m(x) \setminus O_m(x) \setminus T_m(x)$, if $\sigma \in T_m(y)$ at other unit cube $D(y)$ adjacent to $D(x)$, we have (17) or (18), and otherwise we have (17). If a vertex $z$ of $Sk(C_m(x)) \setminus Sk(O_m(x)) \setminus Sk(T_m(x))$ is a vertex of $\sigma$ of (18), we have (28), and if $z$ is a vertex of $\sigma$ of (17), we have (27). Thus,

$$Sk(C_m(x)) \setminus Sk(O_m(x)) \setminus Sk(T_m(x)) \subset Sk(\partial O_m) \cup (Sk(C_m) \setminus Sk(O_m))$$  \hspace{1cm} (30)

The points of $Sk(C_m(x)) \setminus Sk(O_m(x)) \setminus Sk(T_m(x))$ are also shown in Table 7.

From (26), (29), (30) and the relation

$$Sk(T_m(x)) = Sk(I_m(x)) \cup (Sk(J_m(x)) \setminus Sk(I_m(x))),$$

derived from (15), we have

$$Sk(T_m(x)) \cup (Sk(C_m(x)) \setminus Sk(O_m(x)) \setminus Sk(T_m(x))) \subset Sk(\partial O_m) \cup (Sk(C_m) \setminus Sk(O_m))$$

for each $x \in \mathbb{Z}^3$, and thus

$$\bigcup_{x \in \mathbb{Z}^3} (Sk(T_m(x)) \cup (Sk(C_m(x)) \setminus Sk(O_m(x)) \setminus Sk(T_m(x))))$$

$$\subset Sk(\partial O_m) \cup (Sk(C_m) \setminus Sk(O_m)).$$  \hspace{1cm} (32)

Now we verify if there exists a point

$$z \in Sk(\partial O_m) \cup (Sk(C_m) \setminus Sk(O_m))$$

such that

$$z \notin Sk(T_m(x)) \cup (Sk(C_m(x)) \setminus Sk(O_m(x)) \setminus Sk(T_m(x)))$$  \hspace{1cm} (33)

for any $x \in \mathbb{Z}^3$. Considering a point $z \in V \cap D(x)$ which satisfies (33) for any $x \in \mathbb{Z}^3$, we see that

$$z \in Sk(O_m) \setminus Sk(\partial O_m)$$

from Tables 3, 6 and 7, namely,

$$z \notin Sk(\partial O_m) \cup (Sk(C_m) \setminus Sk(O_m))$$

Table 7: For each 1-point configuration of a unit cube $D(x)$, the configurations of points of $CubeB_{m'}(V; x)$ for $m' = 6, 18, 26$, $Sk(I_m(x))$, $Sk(J_m(x)) \setminus Sk(I_m(y))$ for any $y$ such that $y \neq x$ and $Sk(C_m(x)) \setminus Sk(O_m(x))$ for $m = 6, 18, 26$ are shown with $A_{(m', m)}(x)$ for the adjustment in the cases of $(m', m) = (6, 18), (18, 6)$. 

<table>
<thead>
<tr>
<th>point</th>
<th>$m' = 6$</th>
<th>$m' = 18$</th>
<th>$m' = 26$</th>
<th>$m = 6$</th>
<th>$m = 18$</th>
<th>$m = 26$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-point</td>
<td>CubeB_{6}</td>
<td>CubeB_{18}</td>
<td>CubeB_{26}</td>
<td>CubeB_{6}</td>
<td>CubeB_{18}</td>
<td>CubeB_{26}</td>
</tr>
</tbody>
</table>
because
\[ \text{sk}(C_m) = (\text{sk}(\partial O_m)) \cup (\text{sk}(O_m) \setminus \text{sk}(\partial O_m)) \]
and
\[ \text{sk}(\partial O_m) \cup (\text{sk}(C_m) \setminus \text{sk}(O_m)) \cap (\text{sk}(O_m) \setminus \text{sk}(\partial O_m)) = \emptyset. \]

Therefore, if \( z \in \text{sk}(\partial O_m) \cup (\text{sk}(C_m) \setminus \text{sk}(O_m)) \), then
\[ z \in \text{sk}(T_m(x)) \cup (\text{sk}(C_m(x)) \setminus \text{sk}(O_m(x))) \setminus \text{sk}(T_m(x)) \]
and it contradicts (33). Thus, from (32) we obtain (25). (Q.E.D.)

(Proof of Theorem 1) For \((m', m) = (6, 26), (26, 6)\), we have

\[ \text{CubeBr}_m(V; x) = \text{sk}(T_m(x)) \cup (\text{sk}(C_m(x)) \setminus \text{sk}(O_m(x))) \setminus \text{sk}(T_m(x)) \]
for each \( x \in Z^3 \) from (31) and Table 7. Thus, from Lemmas 1 and 2, we obtain (19) and (22).

For \((m', m) = (6, 18)\), if we have the case as shown in Figure 6 for the configuration P5a of \( D(x) \), we see that

\[ \text{CubeBr}_m(V; x) = \text{sk}(T_m(x)) \cup (\text{sk}(C_m(x)) \setminus \text{sk}(O_m(x))) \setminus \text{sk}(T_m(x)) \]
from Table 7, and otherwise we have (34). Thus, we obtain (20).

For \((m', m) = (18, 6)\), for the configuration P7 of \( D(x) \), we see that

\[ \text{CubeBr}_m(V; x) = \text{sk}(T_m(x)) \cup (\text{sk}(C_m(x)) \setminus \text{sk}(O_m(x))) \setminus \text{sk}(T_m(x)) \]
from Table 7, and otherwise we have (34). Thus, we obtain (21). (Q.E.D.)

The relations (19), (20), (21) and (22) of Theorem 1 for \( Z^3 \) correspond to the relation (5) for \( R^n \) shown in [23]. The difference between them is that there is an additional term which is the second term for the union in the right side of each equation of (19), (20), (21) and (22) while there is no such additional term in (5). The second term \( \text{sk}(C_m) \setminus \text{sk}(O_m) \) is a set of vertices which are not included in any discrete 3-polyhedra but included in less than three-dimensional discrete convex polyhedra of \( C_m \). In Figure 5, we can see \( \text{sk}(C_m) \setminus \text{sk}(O_m) \) such as the most right point which is included in a discrete 1-polyhedron in (b) but does not appear in (c). Because we have a step for removing the dimension reduction parts, namely, obtaining \( O_m \) from \( C_m \) such as a procedure from (b) to (c) in Figure 5, we need to add the second term \( \text{sk}(C_m) \setminus \text{sk}(O_m) \) to compare with the border points \( Br_m(V) \) based on general topology. Note that not only discrete 1-polyhedra but also discrete 2-polyhedra may exists in \( C_m \setminus O_m \). Therefore, for the 3-dimensional border tracking, we have two possibilities of the dimensions for dimension reduction parts \( C_m \setminus O_m \), i.e. one or two dimensions, while we have only one possibility of the dimensions, i.e. one dimension, for the 2-dimensional border tracking. This difference has caused the difficulty of 3-dimensional border tracking problem as we already mentioned in Subsection 2.2.1.

The third terms \( A(6, 18) \) and \( A(18, 6) \) which only appear in (20) and (21) respectively show the difference between \( \text{sk}(\partial O_{18}) \) and \( \text{sk}(\partial O_{26}) \) of (19) and (20) and the difference between \( Br_{18}(V) \) and \( Br_{26}(V) \) of (21) and (22), respectively.

6. Conclusions and Discussions

In this paper, we gave a solution to one of the central problems in three-dimensional image analysis; “is it possible to give a triangulation of border points \( Br_m(V) \) such that all vertices of triangulated surfaces are border points and adjacent vertices are m-neighboring for \( m = 6, 18, 26 \)?” Our answer is “yes.” We also succeed to present Algorithm 1 which gives such a triangulated surface \( \partial O_m \) from any finite subset \( V \subset Z^3 \). We insists that the calculation time is linear to the size of \( V \), i.e. the size of a 3-dimensional digital image, and it is the same as that of the set operation (8) for obtaining \( Br_m(V) \) from \( V \) even if our algorithm provides a pure discrete 2-polyhedron \( \partial O_m \) which contains not only a point set \( \text{sk}(\partial O_m) \) but also the combinatorial topological structures of \( \partial O_m \). Theorem 1 which indicates discrete versions of the relation (5) shows that \( \partial O_m \) becomes a triangulation of \( Br_m(V) \) if we choose a good pair such as \((m, m') = (6, 18), (6, 26), (18, 6), (26, 6)\). Note that there may be extra points of \( \text{sk}(O_{m'}) \setminus \text{sk}(O_m) \) if \( Br_m(V) \) contains some lattice points where we cannot put any discrete 3-polyhedron because of their configurations such as the configuration around the right point of Figure 5 (b). Our discrete polyhedral complex is useful to analyse the reasons why we have to ignore such points, i.e. points of \( \text{sk}(O_{m'}) \setminus \text{sk}(O_m) \) for triangulation of \( Br_m(V) \). It is also interesting that the possible pairs for \((m, m')\) are similar to the pairs \((\alpha, \beta)\) for \( \beta \)-connectness of \( \alpha \)-borders [18] as we mentioned in Section 2.2.1.
6.1 Improvement of the Combinatorial Boundary Tracking Algorithm

It may be also possible to present more effective combinatorial boundary tracking algorithms whose calculation time is linear to the number of border points if we succeed to investigate every possible local configurations of combinatorial boundaries. In fact, such an effective border tracking algorithm for three-dimensional digital image is already presented by using an algebraic-topology-based approach by using voxel faces [21], but only for manifold cases. As we already mentioned, our combinatorial boundaries can be also non-manifolds such as the illustrations in Figure 3. We thus need to extend the algorithm for non-manifold cases by using discrete polyhedral complexes.

6.2 Comparison with Other Polyhedral Complexes in $Z^n$

We took the combinatorial/algebraic-topology-based approach by using discrete polyhedral complexes for giving a solution to the triangulation problem. Due to the strong powers for topological problems in discrete spaces, similar complicial representations for a finite subset $V \subseteq Z^3$ are also seen in different literatures [14, 18, 29], for example. For our term of “discrete polyhedral complexes” $C_m$ for $V$, they use the different terms: “cellular complexes” [14], “continuous analogs” [18] and “polyhedra” [29]. Because their aims are different, the ways of obtaining $C_m$ from $V$ are also different.

“Continuous analogs” are presented for defining a digital fundamental group whose concept is used for three-dimensional thinning. During three-dimensional thinning, they need to preserve a “digital topology” whose criteria are given by using the concepts of connectedness and of a digital fundamental group. For a digital fundamental group, they need to consider a region of interest and also its complement, and therefore consider topologies for the whole $Z^3$, not only for $V \subseteq Z^3$ as we do in this paper. In [18], one example for a set of continuous analogs is presented. They are different from our discrete polyhedral complexes in the geometric sense; for example, some continuous analogs may have augmented points which are not lattice points but centroids of lattice cubes as their vertices. On the other hand, if we consider discrete polyhedral complexes $C_m(V)$ and $C_m'(V)$ choosing some pairs for $(m, m')$ for $V$ and $V'$, then we do not know if they satisfy the conditions of continuous analogs or not. Because such discussion is beyond the subjects of this paper, we leave it for our future work.

Even if the aims in [14, 29] are different from ours such as calculation of topological equivalence between two different subsets of $Z^3$ [29], we see that “cellular complexes” [14] and “polyhedra” [29] are the same as our discrete polyhedral complexes $C_6(V)$ for the 6-neighborhood system. This is because the shapes of discrete convex polyhedra for $m = 6$ such as cubes, squares, unit line segments, etc. can be seen in lattice grids and they are straightforward to topologize $Z^3$. In fact, if we topologize $Z^3$ instead of $V \subseteq Z^3$ in the same way of $C_6(V)$, i.e. $C_6(Z^3)$, we see Khalimsky space [13] which is well known in digital image analysis. In [15] it is also shown that Khalimsky space is homeomorphic to Kovalevsky’s finite topology [20] for the case $Z^2$.

References


