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On Construction of Stochastic Pricing Operator from Asset Market Data

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Constructing IMRSs from asset market data has an advantage, that is we need not rely on troublesome consumption data. When we observe a subset of the market, however, there are two problems. First, the constructed IMRSs may not price the other assets. Second, we may also end up insufficient number of factors. These two difficulties lead to mis-pricing of APT. We find an IMRS constructed from only stock market data does not price the Government long-maturity bonds. Using both the stock returns and the one-month Treasury bill returns, this mis-pricing disappears. We also find that five factors extracted from both the stock returns and the Treasury bill return satisfies the condition for APT implied by the Euler equation.

I Introduction

There have been accumulated literature on theoretical and empirical asset pricing models, but no single theory seems to be successful in explaining observed asset prices. The equity premium puzzle of Mehra and Prescott (1985), for example, claims that the returns on the Standard and Poor index fluctuate too much compared with the smooth movement of consumptions. They use a model of a representative household who maximizes expected utility, which is additively separable in time and has a constant relative risk aversion coefficient. These specifications are unnecessarily restrictive, and several studies remove such restrictions. Constantiniedes (1990) studies non-separable utility which incorporates habit persistence. Epstein and Zin (1989) analyze a generalization of time-additive expected utility. Ferson and Constantiniedes (1991) and Epstein and Zin (1991) are empirical studies in these lines. They report supportive evidences for the improved models.[1] But the results are

[1] In conjunction with time-separability of utility, it is also of interest how durable consumption goods are. If you eat pizza this noon, it would affect your marginal utility of supper tonight. See Ferson and Constantiniedes (1991) and Heaton (1990) also.

also sensitive to the choice of consumption measure and of instrumental variables.

If we decide not to use consumption data, what can we learn about asset pricing theories only from the asset market data? Hansen and Jagannathan (1991) start with the Euler equation, and derive a mean-variance frontier for stochastic pricing operators.[2] In an economy where a representative consumer derives utility from a single-good consumption, the stochastic pricing operator corresponds to the intertemporal marginal rate of substitutions (IMRSs) of the consumer. However we emphasize here that their setting is more general than the representative consumer economy, and that the IMRSs of the representative consumer is one of interpretations for a stochastic pricing operator. Rather a stochastic pricing operator represents an asset pricing theory, and Hansen and Jagannathan's bound applies to all theories.[3]

Since Hansen and Jagannathan's stochastic pricing operator is formed from asset returns, it critically depends on the assets used in the construction. Then observability of the asset market becomes an issue.[4] In this paper we study this problem in the context of the arbitrage pricing theory (APT) of Ross (1976). As is with other APT literature, we introduce a linear factor structure in asset returns, and derive a condition which relates the Euler equation to the APT. We see that a stochastic pricing operator constructed from a subset of assets does not price the

[2] Snow(1991) derives other moments than the first and the second.

[3] Chamberlain (1983) and Hansen and Richard (1987) show that a stochastic pricing operator M exists and represents a pricing model if the set of asset payoffs P is complete and linear, a price function π is linear and continuous, and there exists p^0 in P such that *prob* ($\pi(p^0)$) $= 0$. The stochastic pricing operator M then satisfies Euler equation $\pi(p) = E(pM)$ for all p in P.

If we put a different stochastic pricing operator in the right hand side of the above equation, it gives a different price for asset payoffs. In this sense it represents a pricing model.

In this paper all payoffs are divided by its price. We will treat gross returns thus defined. Therefore we have following type of the Euler equation in the main text; $1 = E(ZM)$, where $Z := p \,/\, \pi(p)$.

[4] Roll(1977) criticizes empirical contents of the Capital Asset Pricing Model (CAPM). Since the CAPM implies that the market portfolio lies on the mean-variance frontier, it is not a test of the CAPM when researchers test whether *their* market portfolio is on the mean-variance frontier. Such "market" portfolios consist of marketed financial assets in most cases. But the true market portfolio would include large components like human capital and land. Researchers have no way to claim their market portfolio is the true market.

omitted assets in general. If we don't extract some factors for some reasons, for example, because of their tiny effect on the assets in the subset, the APT does not hold either even if the Euler equation holds.

Using monthly stock and bond data, we examine these effects. We test whether a pricing operator constructed from a subset of assets price the other assets. We will find that a pricing operator constructed from stock returns doesn't price the Government bond returns, but that a pricing operator constructed from stock and the Treasury Bill returns do price the Government bond returns. We further test the condition under which the Euler equation implies the APT, and get supportive result for the APT.

Section II reviews the Hansen-Jagannathan's construction of a stochastic pricing operator from the asset market data. With a factor structure which is standard in the APT literature, we derive a condition that relates the Euler equation to the APT. Section III considers the problems arising from observing a subset of the asset market. Section IV reports the empirical results. Section V concludes this paper.

II Stochastic Pricing Operator

II.1 Construction from financial assets data

Under weak conditions on the price function and the payoff space, which is explained in the footnote 3, there exists a random variable M, called a stochastic pricing operator. It satisfies the so called Euler equation;

$$E(MZ) = 1, \tag{1}$$

where Z is gross return of any asset, and E denotes expectation. As mentioned before, the stochastic pricing operator M can be interpreted as the IMRS of a representative consumer. Throughout the paper, we consider only gross returns, as if all assets are priced \$ 1 today and will bring uncertain income Z in the next period.[5] In Hansen and Richard (1987), the expectation in the Euler equation (1) is conditional expectation with respect to the information set at time t, but we consider unconditional expectation hereafter.

[5] We assume that prices of the assets never become zero.

A number of assets, N*, is large and can be infinity. A researcher observes a proper subset of them, and N (< N*) is the finite number of assets we observe. Hansen and Jagannathan (1991) suggest to construct M by finding an N dimensional vector W such that

$$E(ZZ'W) = \mathbf{1}_N. \tag{2}$$

Then

$$M := Z'W = Z'[E(ZZ')]^{-1}\mathbf{1}_N \tag{3}$$

is a candidate of IMRSs.[6]

II.2 Factor structure, APT, and Euler equation

The linear K-factor structure for asset return Z is given by;

$$Z_N = \mathbf{a}_N + B_{N\times K}\,\mathbf{f}_K + \varepsilon_N, \tag{4}$$
$$E(f) = E(\varepsilon) = \mathbf{0},\ E(ff') = I_{K\times K},\ E(f\varepsilon') = \mathbf{0}.$$

B is a N x K factor loading matrix, f is a K x 1 vector of factors, **a** is a N x 1 vector of the means of gross return Z. As usual we assume $\Omega = E(\varepsilon\varepsilon')$ has K bounded eigenvalues.

Using the factor structure (4), the pricing operator (3) is written as

$$M = \mathbf{a}^m + (b_1^m, \ldots, b_K^m)f + \varepsilon^m, \tag{5}$$

where $\mathbf{a}^m = W'\mathbf{a}$, $(b_1^m, \ldots, b_K^m) = W'B$, $\varepsilon^m = W'\varepsilon$. Then the equations (4) imply $E(\varepsilon^m) = 0$, and $E(\varepsilon^m f') = W'E(\varepsilon f') = 0$.

The APT says that, when idiosyncratic risk ε's can be diversified away, there is a linear relationship between the mean of assets' returns **a** and their factor loadings **B**, under absence of arbitrage. This paper focuses on the exact APT relationship;

$$\exists\lambda_0, \lambda_1, \ldots, \lambda_K \text{ such that } \mathbf{a} = \lambda_0\mathbf{1}_N + B(\lambda_1, \ldots, \lambda_K)'. \tag{6}$$

Under the factor structure assumption (4), what are the conditions that the Euler equation leads to the APT? To illustrate the basic idea,

[6]This M is a projection of stochastic pricing operators on the set spanned by asset returns. In this paper we do not discriminate pricing operators if their projectons are the same. We also assume the inverse in (3) exists.

consider a special case. Suppose there exist K portfolios which perfectly mimic the K factors without idiosyncratic risk, and idiosyncratic risks can be totally diversified away. For convenience, we regard the first K assets in the Z vector are such 'basic' portfolios, although they are actually portfolios consisting of N* primitive assets. Since these K portfolios span the asset returns, we may construct IMRSs from these K portfolios. In this case, the IMRS (3) is written as

$$M = \mathbf{a}^m + \sum_{j=1}^{j=K} b_j^m \, f_j. \tag{7}$$

Note that this expression does not involve any idiosyncratic risk because of the assumption made here.

Under this assumption (7), the Euler equation (1) with the factor structure (4) implies the exact APT relation (6). To show this, let $W^m = (w_1^m, \ldots, w_K^m, 0, \ldots, 0)$be a weight for M. Then the Euler equation $E(MZ) = E(ZZ'W^m) = \mathbf{1}_N$ is written as $(\mathbf{aa}' + BB' + \Omega)W^m = \mathbf{1}_N$. Since the mimicking portfolio assumption says the first K columns of Ω are zero, i.e., $\Omega_{N\times N} = \begin{bmatrix} 0_{K\times K} & 0_{K\times N-K} \\ 0_{N-K\times K} & \Omega^*_{N-K\times N-K} \end{bmatrix}$, we have $\Omega W^m = \mathbf{0}$. Therefore $\mathbf{a}(\mathbf{a}'W^m) + B(B'W^m) = \mathbf{1}$ holds, which implies APT (6) by defining $\lambda_0 = -1 \,/\, \mathbf{a}'W^m$, and $(\lambda_1, \ldots, \lambda_K) = \lambda_0 \, B'W^m$.

As is seen from this, uncorrelated idiosyncratic risk of a pricing operator with the entire assets,

$$\Omega W^m = (E(\varepsilon^m \varepsilon^1), \ldots, E(\varepsilon^m \varepsilon^N))' = \mathbf{0}, \tag{8}$$

is a sufficient condition for APT under (1) and (4). This condition will be discussed and tested in the following sections.

III APT failures

III.1 Failure in extracting factors leads to failure of APT

Recent APT studies report evidence against APT in pricing some stock portfolios. For example, Connor and Korajczyk (1988) use an asymptotic principal components technique on NYSE and AMEX monthly stock data and find mispricing in a small sized portfolios. Lehmann and Modest (1988) also report evidence against APT for size-sorted portfolio. Note that the pricing operator M satisfies the Euler equation for any portfolio X;

$$E[M(X'Z)] = E[(X'Z)M] = X'E[ZZ'W^m] = X'\mathbf{1}_N = 1.$$

Therefore, these reports against APT in pricing some portfolios suggests failure of (8), under the maintained assumptions (1) and (4).

What are situations that invalidate (8)? When we do not observe the entire market and use a subset of the assets to extract factors, this would happen most probably. If we use a subset, we tends to omit some factors that do not have significant factor loadings to the assets in the subset. If there are omitted factors and are included in the idiosyncratic disturbances, the condition (8) is no longer satisfied.

To see this, we write the factor model in terms of two sets of factors ;

$$\begin{aligned} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} &= \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \\ &= \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} f_1 + \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \end{aligned} \tag{9}$$

where $\eta_j = B_{j2}f_2 + \varepsilon_j$, j=1,2. The subscripts represent a sub-vector of an appropriate dimension. Similarly let denote the pricing operator as

$$\begin{aligned} M &= a_m + \begin{bmatrix} b_1^{m'} & b_2^{m'} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \varepsilon^m \\ &= a^m + b_1^{m'} f_1 + \eta^m, \end{aligned} \tag{10}$$

where $\eta^m = b_2^{m'} f_2 + \varepsilon^m$.

Suppose (8) holds for the true disturbance ε's, but a researcher omits the second set f_2 of factors. For the researcher's eyes, the terms η's in (9) and (10) are the disturbances. The pseudo-disturbaces η satisfy the condition in (4), but there are non-zero correlations between these idiosyncratic terms of the asset and of the pricing operator. That is

$$E\left(\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \eta^m \right) \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

even if $E(\varepsilon^m \varepsilon_j) = 0$ for j=1,2. Thus if we mis-specify the factor structure and some true factors are treated as disturbances, then the Euler equation doesn't imply the APT.

Suppose we omit the factor f_2 because we do not use the assets Z_2. Our concern here is to test the APT for the first set of assets. Under the maintained assumption $E(\varepsilon^m \varepsilon_j) = 0$, for j=1,2, the Euler equation (1) implies

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{-1}{a^m} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} -b_1^m/a^m \\ -b_2^m/a^m \end{bmatrix}.$$

This shows that the exact APT for the first set of assets holds only if the factor loading $B_{12} = \mathbf{0}$. Thus the factor loading of the omitted factor must be exactly zero for the asset concerned being priced by the APT. Otherwise, the exact APT holds only for special portfolio X satisfying $XB_{12} = \mathbf{0}$. But such X may not exist.

III.2 A pricing operator may not price some assets if it is constructed from a subset

The above failure of APT is surely due to non-observability of factors. However if we could observe *the entire market*, this would be little harm. The problem arises from the fact that we do not observe it. In the construction of IMRSs from the asset market data, there is the same sort of problem. The constructed IMRS prices the assets used for the construction, but it is not clear whether the IMRS prices the other assets than those used for the construction.

Unfortunately, the answer is negative in general. Suppose that Z_1 and Z_2 constitute the market, and we contract an IMRS as a linear combination of Z_1, a subset at hand. The construction is to find a weight W_1^m such that

$$E\left[Z_1 Z_1'\right] W_1^m = 1. \tag{11}$$

It is an empirical question whether the IMRS thus constructed prices assets in Z_2. Suppose it becomes possible to observe Z_2. For the weight satisfying (11), the Euler equation for both Z_1 and Z_2 is

$$E\left[\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} (Z_1' Z_2') \begin{pmatrix} W_1^m \\ 0 \end{pmatrix}\right] = E\begin{bmatrix} Z_1 Z_1' W_1^m \\ Z_2 Z_1' W_1^m \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The equality for the first set Z_1 is satisfied because of (11). However we don't know whether the equality for the second set Z_2 holds. If the IMRS $Z_1'W_1^m$ doesn't price the other assets at all, its usefulness would be limited.

In sum it is very important to ask whether the range of assets are sufficiently large to test pricing theories. Failure to extract factors leads to failure of APT even if the Euler equation holds. Using small subset of the market leads to failure of the Euler equation for assets outside of the subset. In this sense, it is curious why the APT studies previously mentioned do not use assets other than stocks.

IV Empirical Results

IV.1 Results for the Euler equation

In this subsection, we conduct the following empirical exercise. Using stock data, we construct a pricing operator (3), and see whether it satisfies (1) for stock portfolio that are not used in the construction of (3). Especially we test the pricing of the portfolios, for which some APT studies report mispricing. Moreover we see whether it satisfies (1) for assets other than those used for the construction of IMRS. We use the Government bonds with maturities one to five years as the other assets.

The stock data is monthly returns on the New York and American Stock Exchanges (NYSE/AMEX), taken from the Center for Research in Securities Prices (CRSP). Since (3) involves a calculation of an inverse of N x N matrix, we form 20 portfolios for the purpose of feasible construction of M. We follow the portfolio formation strategy of McCulloch and Rossi(1990), except for observation frequency; they use weekly data.

More precisely, using monthly CRSP file from 1962 to 1990, we construct 20 size-sorted and 20 dividend-yield-sorted portfolios for 1963 to 1990. Among the stocks that are consecutively listed on NYSE or AMEX for a year, we assign a rank according to the size or the dividend yields at the end of the previous year. Then we form equally weighted 20 portfolios according to the rank. Each portfolio consists of at least 80 stocks for size-sorted case, and 60 for dividend-yield-sorted case. We do not require stocks to be listed consecutively for 29 years. This portfolio construction procedure avoids survival biases, and gives better hope

TABLE 1. Does an IMRS price stock portfolio?

A. IMRS from dividend-yield-sorted portfolios

Priced Asset	$\overline{ZM}$	$\sigma(\overline{ZM})$	F-stat	P-value
Size 1 stock	1.425	18.978	1.003	0.489
Size 2 stock	0.546	6.868	1.003	0.489
Size 3 stock	0.188	3.180	1.001	0.496
Size 4 stock	0.246	2.940	1.004	0.485
Size 5 stock	0.010	2.124	0.999	0.504
Size 6 stock	0.122	1.968	1.001	0.496
Size 7 stock	0.100	1.940	1.000	0.500
Size 8 stock	0.093	1.833	1.000	0.500
Size 9 stock	0.044	1.626	0.998	0.507
Size 10 stock	0.038	1.474	0.998	0.507

B. IMRS from size-sorted portfolios

Priced Asset	$\overline{ZM}$	$\sigma(\overline{ZM})$	F-stat	P-value
Div 1 stock	-0.080	1.266	1.001	0.496
Div 2 stock	0.006	1.372	0.997	0.511
Div 3 stock	0.015	1.404	0.997	0.511
Div 4 stock	0.019	1.327	0.997	0.511
Div 5 stock	0.016	1.306	0.997	0.511
Div 6 stock	0.012	1.266	0.997	0.511
Div 7 stock	0.027	1.283	0.997	0.511
Div 8 stock	0.025	1.253	0.997	0.511
Div 9 stock	0.033	1.258	0.998	0.507
Div 10 stock	0.015	1.248	0.997	0.511

$$\overline{ZM} = \frac{1}{T}\sum_{t=1}^{T}(Z_t M_t - 1),$$

$$\sigma(\overline{ZM}) = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}(Z_t M_t - 1 - \overline{ZM})^2}.$$

See the equation (12) for the F-statistic.

that the central limit theorem be applicable. We get the real return by subtracting CPI rate of change.

Table 1 reports the result of a test whether an IMRS constructed from size-sorted portfolios prices dividend-yield-sorted portfolios, and vice versa. We use sample moments in the calculation (3). The test statistics reported in Tables are

$$\frac{1}{T}\sum_{t=1}^{T}(Z_tM_t-1)^2\Big/\frac{1}{T-1}\sum_{t=1}^{T}(Z_tM_t-1-\overline{ZM})^2, \tag{12}$$

where $\overline{ZM}$ = 1/T $\sum_{t=1}^{T}(Z_tM_t-1)$. It follows F(T,T-1) if the pricing errors Z_tM_t - 1 have independent normal distributions. The estimation period is January 1963 to December 1989, so T is 324.

The results say an IMRS constructed from stock portfolio prices well other stock portfolios. This is not surprising, because we will not lose so much information by using portfolios instead of the primitive assets if there is a factor structure (4). We may lose efficiency, though.

We then test whether IMRSs constructed from stock portfolio price the Government bonds. We use the discount government bonds with one, two, three, four, and five years to maturity, in the Fama-Bliss file from 1963 to 1989. The result in Table 2 shows that neither of IMRSs price the government bonds.

What happens if we include the Treasury bill returns when constructing IMRSs? We add U.S. Treasury Bill total return in CRSP indices file, which is a one-bill portfolio containing the shortest-term bill having not less than one month to maturity.

This changes the result drastically. Table 3 reports the IMRSs constructed from the stock portfolios and Treasury bill portfolio price the Government bonds as well as stock portfolios. Note that bonds return we use in the construction of IMRS is of short maturity, while the Government bonds we use in the pricing are of longer maturities.

IV.2 Results for APT

As we see in Section II, the Euler equation leads to the exact APT under the condition (8). That is, idiosyncratic risk of IMRSs is uncorrelated with idiosyncratic risks of the assets. To test this condition, we first extract five factors from the previous data set: 20 dividend-yield-sorted

TABLE 2. Do IMRS's price Government bonds?

A. IMRS from dividend-yields-sorted portfolios

Priced Asset	$\overline{ZM}$	$\sigma(\overline{ZM})$	F-stat	P-value
1 Yr bond	-0.288	0.497	1.333	0.0044
2 Yr bond	-0.288	0.498	1.329	0.0048
3 Yr bond	-0.286	0.498	1.326	0.0051
4 Yr bond	-0.285	0.499	1.324	0.0053
5 Yr bond	-0.285	0.499	1.323	0.0054

B. IMRS from size-sorted portfolios

Priced Asset	$\overline{ZM}$	$\sigma(\overline{ZM})$	F-stat	P-value
1 Yr bond	-0.293	0.451	1.419	0.0007
2 Yr bond	-0.292	0.451	1.415	0.0008
3 Yr bond	-0.291	0.452	1.411	0.0009
4 Yr bond	-0.290	0.452	1.408	0.0009
5 Yr bond	-0.290	0.453	1.407	0.0009

stock portfolios and U.S. Treasury bill. We don't have any theory to specify the number of factors, so we employ the one that previous studies have used. Factor estimation method is based on Maximum Likelihood.

Having extracted factors f_j's, we then run regressions on the estimated factors to estimate the constant a^p and the factor loadings b_j^p;

$$Z_p = a^p + \sum_{j=1}^{j=5} b_j^p f_j + \varepsilon^p, \tag{13}$$

where p represents a portfolio. Table 4a reports regression results for stock portfolios and for the Government bonds. It reports the results for the smallest (Size 1), the middle (Size 10), and the largest (Size 20) portfolios only, but the result for the remaining seventeen are similar. The five factor model looks well fitted for stock portfolios. As for the Government bonds, which are not used in factor extraction process, only Factor 5 is significant. The results for the portfolios of the Government bonds with two to four years to maturity are similar, and not reported. We form estimated idiosyncratic risks $\hat{\varepsilon^p}$ as the residuals of these regressions.

TABLE 3. Do IMRSs price Government bonds and stock?

A. IMRS from dividend-yield-sorted portfolios and Treasury Bill

Priced Asset	$\overline{ZM}$	$\sigma(\overline{ZM})$	F-stat	P-value
1 Yr bond	-0.00216	0.469	0.997	0.511
2 Yr bond	-0.00095	0.469	0.997	0.511
3 Yr bond	1.24D-06	0.470	0.997	0.511
4 Yr bond	0.00072	0.470	0.997	0.511
5 Yr bond	0.00111	0.470	0.997	0.511
Size 1 stock	-1.177	80.113	0.997	0.511
Size 2 stock	-0.747	27.759	0.998	0.507
Size 3 stock	-0.457	17.283	0.998	0.507
Size 4 stock	-0.025	7.558	0.997	0.511
Size 5 stock	0.008	5.600	0.997	0.511
Size 6 stock	0.035	3.973	0.997	0.511
Size 7 stock	0.069	4.347	0.997	0.511
Size 8 stock	0.095	4.093	0.998	0.507
Size 9 stock	-0.008	2.705	0.997	0.511
Size 10 stock	0.011	2.952	0.997	0.511

B. IMRS from size-sorted portfolios and Treasury Bill

Priced Asset	$\overline{ZM}$	$\sigma(\overline{ZM})$	F-stat	P-value
1 Yr bond	0.00461	0.335	0.997	0.511
2 Yr bond	0.00602	0.336	0.997	0.511
3 Yr bond	0.00709	0.336	0.997	0.511
4 Yr bond	0.00791	0.337	0.997	0.511
5 Yr bond	0.00831	0.337	0.998	0.507
Div 1 stock	-0.089	1.049	1.004	0.485
Div 2 stock	-0.052	1.108	0.999	0.504
Div 3 stock	-0.052	1.138	0.999	0.504
Div 4 stock	-0.034	1.103	0.998	0.507
Div 5 stock	-0.005	0.897	0.997	0.511
Div 6 stock	0.001	0.876	0.997	0.511
Div 7 stock	0.015	0.933	0.997	0.511
Div 8 stock	0.016	0.914	0.997	0.511
Div 9 stock	0.032	0.856	0.998	0.507
Div 10 stock	0.026	0.791	0.998	0.507

TABLE 4a. Do the factors extracted from dividend-yield-sorted portfolios and Treasury Bill explain size-sorted portfolios and the Government bonds?
(OLS coefficient, standard error in parenthesis)

Asset priced	Independent variables Constant	Factor1	Factor2	Factor3	Factor4	Factor5	R-square	Durbin-Watson
Size 1	18.39	187.8	-9.489	29.02	-12.22	-13.05	0.991	1.819
stock	(1.058)	(1.042)	(1.046)	(1.054)	(1.109)	(1.144)		
Size 10	2.066	9.921	-0.553	-0.535	-0.344	-0.303	0.996	1.884
stock	(0.034)	(0.033)	(0.033)	(0.034)	(0.035)	(0.037)		
Size 20	1.232	0.432	0.528	0.074	0.585	0.022	0.776	2.067
stock	(0.027)	(0.026)	(0.026)	(0.027)	(0.028)	(0.029)		
1 Yr	1.018	1.84e-4	2.64e-3	-1.10e-3	1.50e-3	7.54e-4	0.040	0.925
bond	(2.10e-3)	(2.07e-3)	(2.08e-3)	(2.09e-3)	(2.21e-3)	(2.27e-3)		
5 Yr	1.022	8.61e-4	3.07e-3	-5.10e-4	1.93e-3	8.39e-3	0.043	0.802
bond	(2.29e-3)	(2.26e-3)	(2.26e-3)	(2.28e-3)	(2.40e-3)	(2.48e-3)		

TABLE 4b. Do the factors extracted from dividend-yield-sorted portfolios and Treasury Bill explain an IMRS ?
(OLS coefficient, standard error in parenthesis)

Asset priced	Independent variables Constant	Factor1	Factor2	Factor3	Factor4	Factor5	R-square	Durbin-Watson
IMRS	0.983	-0.126	-0.167	-0.019	-0.169	-0.042	0.345	1.949
	(0.021)	(0.021)	(0.021)	(0.021)	(0.022)	(0.022)		

TABLE 5. Does IMRS lead to APT ?
(F-statistic for $E(\hat{\varepsilon^j}\hat{\varepsilon^m}) = 0$. P-values in the parenthesis)

Size 1 stock	1.026 (0.407)	Size 11 stock	1.027 (0.404)
Size 2 stock	1.022 (0.421)	Size 12 stock	1.002 (0.493)
Size 3 stock	1.009 (0.467)	Size 13 stock	1.002 (0.493)
Size 4 stock	1.029 (0.397)	Size 14 stock	1.000 (0.500)
Size 5 stock	1.023 (0.418)	Size 15 stock	0.998 (0.507)
Size 6 stock	1.018 (0.435)	Size 16 stock	1.002 (0.493)
Size 7 stock	1.026 (0.407)	Size 17 stock	1.002 (0.493)
Size 8 stock	1.020 (0.428)	Size 18 stock	1.026 (0.407)
Size 9 stock	1.044 (0.347)	Size 19 stock	1.000 (0.500)
Size 10 stock	1.027 (0.404)	Size 20 stock	0.999 (0.503)
1 yr bond	0.997 (0.510)	2 yr bond	0.997 (0.510)
3 yr bond	0.997 (0.510)	4 yr bond	0.997 (0.510)
5 yr bond	0.997 (0.510)		

Using the IMRS we get in Table 3a for M above, we run the following regression to estimate a^m and b_j^m:

$$M = a^m + \sum_{j=1}^{j=5} b_j^m f_j + \varepsilon^m. \tag{14}$$

Table 4b reports the results. The R-square is not high, but most factors are significant. From this regression, we get an estimated idiosyncratic risk $\hat{\varepsilon}^m$.

Using the estimated disturbances in (13) and (14), we see whether the average of their product is zero, the key condition (8). Table 5 reports test statistics. They do not reject that the condition for the exact APT relation being implied from the Euler equation.

This is suggestive, because most APT studies using stock data reject APT for size sorted portfolios. For example, Connor and Korajczyk (1988), Lehmann and Modest (1988), report mispricing of APT on small-sized firms' portfolios, using NYSE and AMEX stock data. Although our test is not a direct test of APT, the results here indicates the exact APT would hold for size-sorted portfolios, if we include Treasury bill return series as well as stock series in factor extraction process.

V Concluding Remark

Constructing IMRSs from asset market data has an advantage, that is, we need not rely on troublesome consumption/expenditure data, but has a problem when we do not observe the entire market. When we want to fit a linear factor model to asset markets, we may also fail to extract enough factors because of the non-observability of the entire market. Reported mispricing of APT for size-sorted stock portfolios can be attributable to this, because they use only stock data in modelling a factor structure.

We first check whether IMRSs constructed from stock market data price the Government long-maturity bonds. We find they do not price them, but IMRSs constructed from stock market and Treasury bill market data do price them. Introducing a factor structure in the model links the Euler equation with the exact APT relation. Using extracted factors from the dividend-yield-sorted stock portfolios and the Treasury bill data, we see whether the constructed IMRS satisfies the condition for the APT being implied by the Euler equation. The results are affirmative for both size-sorted stock portfolios and for the Government bonds.

Since we don't reject the pricing equations, we might wonder the power of the test may not be strong. Exploiting conditional expectations in the Euler equation (3) would give us more power. Furthermore, traditional tests of the APT might have more power. More powerful tests are always desirable. A prospective research would be a test of APT using broader set of assets than stock and compare it with our lines of the test exploiting conditional expectations.

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