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Approximation Formulas for the Lower Confidence Limits of Process Capability Indices

Yasushi Nagata
(Faculty of Economics, Okayama University)
Hidekazu Nagahata
(Faculty of Education, Okayama University)

1. Introduction

The process capability indices are very often used in the activities of quality control. Especially, in the processes of Japanese manufacturing industries, a lot of researchers and/or workers often evaluate those indices to check whether their processes are satisfactory or not.

Assuming the population distribution of the quality characteristic is normal one $N(\mu, \sigma^2)$ with mean μ and variance σ^2, the process capability indices are defined as follows:

(i) when there is only the upper specification limit S_U

$$CPU = \frac{S_U - \mu}{3\sigma} \qquad (1.1)$$

(ii) when there is only the lower specification limit S_L

$$CPL = \frac{\mu - S_L}{3\sigma} \qquad (1.2)$$

(iii) when there are both the upper and the lower specification limits S_U, S_L

(1) $$CP = \frac{S_U - S_L}{6\sigma} \tag{1. 3}$$

(2) $$CPK = \min\left\{\frac{S_U - \mu}{3\sigma}, \frac{\mu - S_L}{3\sigma}\right\}$$

$$= \min\{CPU, CPL\}. \tag{1. 4}$$

These indices are based on the ratio of the width between specifications (or the corresponding width) and the population dispersion. For instance, suppose CPU = 1 in eq. (1. 1). Then $S_U - \mu = 3\sigma$, which implies that the distance between population mean μ and the upper specification limit S_U is 3σ. As the products of which quality characteristics are greater than the limit S_U are defective, the proportion of nonconforming of the process is 0.0013.

It should be noted that CP in eq. (1. 3) is useful only when the population mean μ coincides with the middle point between S_U and S_L or when it is easy to adjust the bias. In other situations CPK in eq. (1. 4) is adequate.

Recently, various types of indices have been proposed, but these are not in general use at present. Therefore, in this paper we confine ourselves to discussing the indices in eqs. (1. 1) ~ (1. 4).

As abovementioned, in the evaluation of the process capability indices the value "1" is one of the common standard aims. But following more detailed criteria are generally adopted (see Ishikawa [4]):

(a) if (the value of the index) $\geqq$ 1.33, the process capability is satisfactory;

(b) if 1.33 > (the value of the index) $\geqq$ 1.00, the process capability is not bad.

(c) if (the value of the index) < 1.00, the process capability is bad.

Since these indices contain unknown parameters μ and σ, we usually take a random sample of n measurements $x_1, x_2, \cdots\cdots, x_n$ and estimate μ and σ as

$$\hat{\mu} = \bar{x} = \sum_{i=1}^{n} x_i / n \qquad (1.5)$$

$$\hat{\sigma} = s = \{\sum_{i=1}^{n} (x_i - \bar{x})^2 / (n-1)\}^{1/2}. \qquad (1.6)$$

And substituting these quantities into eqs. (1.1) ~ (1.4), we obtain estimators of the process capability indices

$$\widehat{CPU} = \frac{S_U - \bar{x}}{3s} \qquad (1.7)$$

$$\widehat{CPL} = \frac{\bar{x} - S_L}{3s} \qquad (1.8)$$

$$\widehat{CP} = \frac{S_U - S_L}{6s} \qquad (1.9)$$

$$\widehat{CPK} = \min\left\{\frac{S_U - \bar{x}}{3s}, \frac{\bar{x} - S_L}{3s}\right\}. \qquad (1.10)$$

It has been the usual way to calculate these estimators and to apply them to the above criteria (a)~(c). However, we should note that the above criteria are set out for the true values of process capability indices in eqs. (1.1) ~ (1.4) and that it is not inadequate to apply the estimated version (1.7) ~ (1.10) to these criteria directly. Estimators have dispersions. For example, even if we obtain $\widehat{CPU} = 1.50$, it might be a good value by chance.

When the number of data n is large enough, it might not lead to misunderstanding to neglect the dispersions of the estimators. However, in these days it is necessary to control various kinds of variables, which presses the researchers and/or workers to evaluate the process promptly. In such situations only small sample size observations (like $n = 10$ or 20) are taken. When the number of the data is small, we should consider the dispersions of the estimators. And constructing a confidence interval is one of the standard methods for taking account of the dispersions.

From this point of view, Chou et al.[2] gave tables of the lower confidence limits for the process capability indices in eqs. (1. 1) $\sim$ (1. 4). And Kushler and Hurley[5] compares the performances of several lower confidence intervals numerically. On the other hand, Bissell[1] and Nagata[6] proposed approximate two-sided confidence intervals for CPU and CPL defined in eqs. (1. 1) and (1. 2), which are simple and are very accurate. Zhang et al.[9] constructed the two-sided confidence interval of CPK in eq. (1. 4). Furthermore, Nagata and Nagahata[7] proposed simpler methods than that of Zhang et al.[9] and investigated the procedures in detail.

We consider that it is more natural to construct a two-sided one than a lower one. We, of course, recognize that the lower confidence limit is more important than the upper one, but we claim that it is more convenient and informative to report both limits and width of the confidence interval with the point estimator. It is also noticeable that since the two-sided confidence intervals are usually constructed for a fraction of defective and a variance even though as to those parameters upper confidence limits are more important than the lower ones.

However, we do not intend to deny completely the use of the lower confidence limit. In some situations it might be convenient to report the least confident value of the index as an output. We also found that the lower confidence limits in the table given by Chou et al.[2] are produced by wrong calculation. Thus, in this paper we will derive simple and accurate approximation formulas for the lower confidence limits of the indices in eqs. (1. 1), (1. 2) and (1. 4). We will proceed the discussion as in Nagata and Nagahata[7].

Confidence limits of index in eq. (1. 3) are easily obtained by transforming usual ones for the variance σ^2, so we will not discuss any more about this index.

In Section 2 we will prepare some notations and several relations between them, which will be needed in Section 3. In Section 3 we will derive some approximation formulas for lower confidence limits and discuss the performances of them numerically.

2. Notations and their relations

In this section we introduce some notations and their relations briefly, which are needed in Section 3. See Nagata and Nagahata[7] for details.

Define the constants A and B as

$$A = \frac{2}{f}\frac{\Gamma((f+1)/2)}{\Gamma(f/2)}, \quad B = 1 - A^2$$

$$\text{and } f = n - 1, \qquad (2.\ 1)$$

where $\Gamma(\cdot)$ is a gamma function, n is the number of data and f is the degree of freedom. Note that A and B are the mean and the variance of s/σ, respectively.

Let us consider CPU at first. We want to find $t(\alpha)$ such that

$$\Pr(\widehat{CPU} \leqq t(\alpha)) \doteqdot 1 - \alpha. \qquad (2.\ 2)$$

That is, $t(\alpha)$ is an approximate upper 100α percent point of distribution of $\widehat{CPU}$. Approximating the distribution of s/σ to N (A, B), we obtain

$$t(\alpha)=\frac{A\cdot CPU + z(\alpha)\{B\cdot CPU^2 + (A^2 - Bz(\alpha)^2)/(9n)\}^{1/2}}{A^2 - Bz(\alpha)^2}, \tag{2.3}$$

where $z(\alpha)$ is the upper 100α percent point of N (0, 1).

Now, define

$$f(a\,;b,c\,;A,B)=\frac{Ab + a\{Bb^2 + (A^2 - Ba^2)c/9\}^{1/2}}{A^2 - Ba^2}, \tag{2.4}$$

then $t(\alpha)$ in eq. (2.3) is rewritten as

$$t(\alpha) = f(z(\alpha)\,;CPU, 1/n\,;A,B).$$

Furthermore, define as

$$g(a\,;x,c\,;A,B) = Ax + a\{Bx^2 + c/9\}^{1/2}. \tag{2.5}$$

We obtain the following relation between eq. (2.4) and eq. (2.5).

Proposition 2. 1

Assume a, c, A and B are all positive constants and $A^2 - Ba^2 > 0$. Then

$$x \leqq f(a\,;b,c\,;A,B) \text{ if and only if } b \geqq g(-a\,;x,c\,;A,B). \tag{2.6}$$

From Proposition 2.1 note that

$$\begin{aligned} 1-\alpha &\doteqdot \Pr(\widehat{CPU} \leqq f(z(\alpha)\,;CPU, 1/n\,;A,B)) \\ &= \Pr(g(-z(\alpha);\widehat{CPU}, 1/n\,;A,B) \leqq CPU), \end{aligned} \tag{2.7}$$

which yields a one-sided confidence interval of CPU as

$$J = (g(-z(\alpha)\,;\widehat{CPU}, 1/n\,;A,B),\ \infty). \tag{2.8}$$

And the coverage probabilities of this interval can be evaluated by the first equation of (2.7) numerically. Specifically, since $3\sqrt{n}\,\widehat{CPU}$ is distributed as a noncentral t-distribution with f degrees of freedom and noncentrality parameter $3\sqrt{n}\,CPU$, we can calculate the coverage probabilties by Owen's [8] method.

As to the case of CPL we can proceed as the exactly same way as the case of CPU.

Next we will consider the case of CPK. Since the distribution of $\widehat{CPK}$ is much more complicated, we have to make an additional device. But fundamentally, we will treat the similar confidence interval as in eq. (2.8)

$$J' = (g(-z(\alpha);\widehat{CPK}, 1/n;A,B),\ \infty), \qquad (2.9)$$

and evaluate the coverage probabilities by

$$\Pr(\widehat{CPK} \leqq f(z(\alpha);CPK, 1/n;A,B)) \qquad (2.10)$$

numerically. In this case we can also use Owen's [8] method specifically.

The bias and the variance of the statistic $\min\{S_U - \bar{x},\ \bar{x} - S_L\}$ in eq. (1.10) are functions of not only μ and σ but of other quantity, which leads the case of CPK to be more complicated than that of CPU.

Define T and d by

$$T = (S_U + S_L)/2,\quad d = (T - \mu)/\sigma. \qquad (2.11)$$

It can be shown that the bias and the variance of $\min\{S_U-\bar{x},\ \bar{x}-S_L\}$ are even functions of d, the bias is increasing and approaches to 0 as $|d| \to \infty$ and the variance is increasing and goes to $1/n$ as $|d| \to \infty$.

Let $|d|$ be sufficiently large and $d < 0$. That is, let CPU be much smaller than CPL. Then $\widehat{CPK}$ is equal to $\widehat{CPU}$ with probability one. Therefore, the distribution of $\widehat{CPK}$ converges to one of $\widehat{CPU}$ as $|d| \to \infty$. This fact will be used to construct the confidence interval of CPK in

Section 3.

3 . Approximation formulas for lower confidence limits of CPU, CPL and CPK

First, let us consider the procedure for CPU. We will think the following confidence interval:

$$J_0 = (g(-z(\alpha)\,;\,\widehat{CPU},\,1/n\,;\,1,\,1/(2f)),\ \infty)$$
$$= (\widehat{CPU} - z(\alpha)\{\widehat{CPU}^2/(2f) + 1/(9n)\}^{1/2},\ \infty). \qquad (3.\ 1)$$

This is obtained simply by putting $z(\alpha)$ instead of $z(\alpha/2)$ in the lower limit of the two-sided confidence interval I_0 (in eq. (20) of Nagata and Nagahata [7]).

For nominal levels $1-\alpha = 0.950(z(\alpha) = 1.645)$, CPU = 0.40 (0.30) 2.50, and n = 10, 20, 30, 50, 100, the coverage probabilities Pr(CPU $\in J_0$)'s are given Table 1.

We observe from Table 1 that the coverage probabilities of J_0 are slightly smaller than the nominal value 0.950. We have observed that for $1-\alpha = 0.900$ the coverage probabilities are further below the nominal value.

We remark here that there are slight differences between the one-sided case and the two-sided case. The coverage

Table 1 Coverage probabilities Pr(CPU $\in J_0$)'s
($1-\alpha=0.950$, $z(\alpha)=1.645$)

	n				
CPU	10	20	30	50	100
0.40	.946	.946	.947	.948	.948
0.70	.947	.947	.947	.948	.948
1.00	.947	.947	.947	.948	.948
1.30	.947	.947	.948	.948	.948
1.60	.947	.947	.948	.948	.949
1.90	.947	.947	.948	.948	.949
2.20	.947	.947	.948	.948	.949
2.50	.948	.947	.948	.948	.949

probabiities of the two-sided confidence interval I_0 proposed in Nagata and Nagahata [7] are larger than the nominal levels from their Tables 1 and 2. But these facts and the results of Table 1 in this paper do not contradict each other. In the case of two-sided confidence interval, the coverage probabilities are computed as

$$\begin{aligned} &\Pr(\mathrm{CPU} \in I_0) \\ &\quad = \Pr(\widehat{\mathrm{CPU}} \leqq f(z(\alpha/2); \mathrm{CPU}, 1/n; 1, 1/(2f))) \\ &\qquad - \Pr(\widehat{\mathrm{CPU}} \leqq f(-z(\alpha/2); \mathrm{CPU}, 1/n; 1, 1/(2f))), \end{aligned}$$

so that the effects of underestimation for the probabilities are well cancelled out.

The results of Table 1 seems to be good in practice, but let us make coverage probabilities equal or greater than the nominal values. From the facts described in Section 2,

$$\begin{aligned} 1-\alpha &\doteqdot \Pr(\widehat{\mathrm{CPU}} \leqq f(z(\alpha); \mathrm{CPU}, 1/n; A, B)) \\ &= \Pr(g(-z(\alpha); \widehat{\mathrm{CPU}}, 1/n; A, B) \leqq \mathrm{CPU}) \\ &= \Pr(\mathrm{CPU} \in (A \cdot \widehat{\mathrm{CPU}} - z(\alpha)\{B \cdot \widehat{\mathrm{CPU}}^2 + 1/(9n)\}^{1/2}, \infty)). \end{aligned}$$

Setting $B = 1/(2f)$ as before, we will change the value of A. Enkawa[3] derived $A = \{1 - 1/(2f)\}^{1/2}$ and $A = \{1 - 1/(3f)\}^{1/2}$ as the approximate values of A. Referring to his results, setting

$$A = \{1 - c/f\}^{1/2},$$

we have examined various values of c, and have selected the value $c = 2/5$ which renders the coverage probabilities most favorable numerically. Hence, we obtain the procedure

$$\begin{aligned} J_1 &= (g(-z(\alpha); \widehat{\mathrm{CPU}}, 1/n; \{1-2/(5f)\}^{1/2}, 1/(2f)), \infty) \\ &= (\{1-2/(5f)\}^{1/2}\,\widehat{\mathrm{CPU}} - z(\alpha)\{\widehat{\mathrm{CPU}}^2/(2f) + 1/(9n)\}^{1/2}, \infty). \end{aligned} \quad (3.2)$$

For $1-\alpha = 0.950$ ($z(\alpha) = 1.645$), 0.900 ($z(\alpha) = 1.282$), CPU = 0.40 (0.30) 2.50, and n = 10, 20, 30, 50, 100, the coverage probabilities Pr(CPU $\in J_1$)'s are given in Tables 2 and 3.

Table 2 Coverage probabilities Pr(CPU $\in J_1$)'s
($1-\alpha=0.950$, $z(\alpha)=1.645$)

CPU	n: 10	20	30	50	100
0.40	.955	.952	.951	.951	.950
0.70	.957	.953	.952	.952	.951
1.00	.958	.954	.953	.952	.951
1.30	.958	.954	.953	.952	.951
1.60	.958	.955	.953	.952	.951
1.90	.959	.955	.953	.952	.952
2.20	.959	.955	.953	.952	.952
2.50	.959	.955	.953	.952	.952

We observe from Tables 2 and 3 that the coverage probabilities are equal or greater than the nominal values. Similar results are obtained for $1-\alpha=0.990$.

Table 3 Coverage probabilities Pr(CPU $\in J_1$)'s
($1-\alpha=0.900$, $z(\alpha)=1{:}282$)

CPU	n: 10	20	30	50	100
0.40	.902	.900	.900	.900	.900
0.70	.903	.901	.901	.900	.900
1.00	.904	.901	.901	.900	.900
1.30	.904	.902	.901	.901	.900
1.60	.904	.902	.901	.901	.900
1.90	.904	.902	.901	.901	.900
2.20	.904	.902	.901	.901	.900
2.50	.904	.902	.901	.901	.900

Therefore, we propose the procedure J_1 in eq. (3. 2) as an approximate one-sided confidence interval of CPU.

An approximate one-sided confidence interval of CPL is obtained by substituting $\widehat{CPL}$ for $\widehat{CPU}$ in J_1.

Next, let us consider the procedure for CPK. To begin with, we will think the following procedure

$$J_2 = (g(-z(\alpha);\widehat{CPK}, 1/n; 1, 1/(2f)), \infty)$$
$$= (\widehat{CPK} - z(\alpha)\{\widehat{CPK}^2/(2f) + 1/(9n)\}^{1/2}, \infty). \tag{3. 3}$$

Note that this procedure is of the same type as J_0 in eq. (3. 1) for CPU.

For n = 30, $1-\alpha=0.950$, CPK = 0.40 (0.30) 1.60 and $|d| = 0.00(0.10)1.00$, the coverage probabilities Pr(CPK $\in J_2$)'s of the procedure J_2 are given in Table 4.

It can be observed from Table 4 that when $|d| = 0$ the coverage probabilities of J_2 are considerably larger than the nominal value 0.950, and that they are decreasing in $|d|$ and converge to some values as $|d| \to \infty$. As we explained in Section 2, these limits are the values in Table 1 (n = 30).

Table 4 Coverage probabilities $\Pr(CPK \in J_2)$'s ($1-\alpha=0.950$, $z(\alpha)=1.645$, $n=30$)

$\lvert d \rvert$	CPK 0.40	0.70	1.00	1.30	1.60
0.00	.996	.986	.977	.972	.968
0.10	.987	.973	.966	.962	.959
0.20	.972	.961	.956	.954	.953
0.30	.958	.953	.951	.950	.949
0.40	.950	.949	.948	.948	.948
0.50	.947	.948	.948	.948	.948
0.60	.947	.947	.947	.948	.948
0.70	.947	.947	.947	.948	.948
0.80	.947	.947	.947	.948	.948
0.90	.947	.947	.947	.948	.948
1.00	.947	.947	.947	.948	.948

Now, we present the following proposition.

Proposition 3. 1

Let t be an arbitrary constant, then

$$\Pr(\widehat{CPK} \leqq t) \geqq \Pr(\widehat{CPU} \leqq t). \qquad (3.4)$$

Proof:

$$\begin{aligned}\Pr(\widehat{CPK} \leqq t) &= 1 - \Pr(\min(\widehat{CPU}, \widehat{CPL}) \geqq t) \\ &= 1 - \Pr(\widehat{CPU} \geqq t, \widehat{CPL} \geqq t) \\ &\geqq 1 - \Pr(\widehat{CPU} \geqq t) \\ &= \Pr(\widehat{CPU} \leqq t).\end{aligned}$$

Q. E. D.

Let CPK = CPU (i. e. $d < 0$) without loss of generality. Then we have the following relation between J_0 in eq. (3. 1) and J_2 in eq. (3. 3):

$$
\begin{aligned}
&\Pr(\mathrm{CPK} \in J_2)\\
&= \Pr(g(-z(\alpha);\widehat{\mathrm{CPK}}, 1/n; 1, 1/(2f)) \leqq \mathrm{CPK})\\
&= \Pr(\widehat{\mathrm{CPK}} \leqq f(z(\alpha);\mathrm{CPK}, 1/n; 1, 1/(2f)))\\
&= \Pr(\widehat{\mathrm{CPK}} \leqq f(z(\alpha);\mathrm{CPU}, 1/n; 1, 1/(2f)))\\
&\geqq \Pr(\widehat{\mathrm{CPU}} \leqq f(z(\alpha);\mathrm{CPU}, 1/n; 1, 1/(2f)))\\
&= \Pr(g(-z(\alpha);\widehat{\mathrm{CPU}}, 1/n; 1, 1/(2f)) \leqq \mathrm{CPU})\\
&= \Pr(\mathrm{CPU} \in J_0). \qquad (3.5)
\end{aligned}
$$

We can also confirm this relation by investigating Table 1 ($n = 30$) and Table 4.

From Table 4 it should be noted that the caverage probabilities of the confidence interval J for CPK vary in $|d|$. Therefore, it might be more reasonable to use

$$\mathrm{IPC}(J) = \inf_{|d|} \Pr(\mathrm{CPK} \in J) \qquad (3.6)$$

as an evaluation scale for the procedure J. By eq. (3.5) we have

$$\mathrm{IPC}(J_2) = \inf_{|d|} \Pr(\mathrm{CPK} \in J_2) \geqq \Pr(\mathrm{CPU} \in J_0).$$

Since $\lim_{|d| \to \infty} \Pr(\mathrm{CPK} \in J_2) = \Pr(\mathrm{CPU} \in J_0)$,

$$\mathrm{IPC}(J_2) = \Pr(\mathrm{CPU} \in J_0). \qquad (3.7)$$

That is, the values of IPC (J_2) are the same as those in Table 1.

Similarly, in order to construct the lower confidence interval of CPK of which IPC (J)'s are equal or greater than the nominal values, we have only to employ the same type as J_1 in eq. (3.2):

$$
\begin{aligned}
J_3 &= (g(-z(\alpha);\widehat{\mathrm{CPK}}, 1/n; \{1-2/(5f)\}^{1/2}, 1/(2f)),\ \infty)\\
&= (\{1-2/(5f)\}^{1/2}\,\widehat{\mathrm{CPK}} - z(\alpha)\{\widehat{\mathrm{CPK}}^2/(2f) + 1/(9n)\}^{1/2},\ \infty)
\end{aligned}
\qquad (3.8)
$$

We can see the values of IPC (J_3) in Tables 2 and 3.

Finally, we will compare the approximate lower confidence limits presented in J_1 and J_3 with the values in the tables of Chou, et al. [2]. In Table 4 of Chou, et al. the lower confidence limits for $1-\alpha=0.950$ are given. We can see that the approximate lower confidence limits with $z(\alpha)=1.645$ in J_1 are reasonably close to the values in their Table 4. And in thier Table 3 the values t(n, CPU) such that

$$\Pr(\widehat{CPU} \leqq t(n, CPU)) = 0.950 \qquad (3.9)$$

are given. Note that since

$$\begin{aligned}&\Pr(CPU \in J_1)\\&= \Pr(\widehat{CPU} \leqq f(1.645;\, CPU,\, 1/n;\, \{1-2/(5f)\}^{1/2},\ 1/(2f))),\end{aligned}$$

$f(1.645; CPU, 1/n; \{1-2/(5f)\}^{1/2}, 1/(2f))$ is an approximate value of t(n, CPU). Comparing these approximate values and those of t(n, CPU)'s, we can also see that they are quite close.

For the values for CPK, however, there are discrepancies between the results of Chou, et al. and ours. In their Table 5 the values of the lower confidence limits of CPK are given, which are uniformly smaller than their values for CPU. Therefore, the values in their Table 5 are smaller than those of the lower limits in J_3 (recall that the lower limits of J_1 and J_3 are same). And in Table 6 of Chou, et al. the values q(n, CPK)'s such that

$$\Pr(\widehat{CPK} \leqq q(n, CPK)) = 0.950 \qquad (3.10)$$

are given, and from their Tables 3 and 6 we observe

$$t(n, CPU) < q(n, CPK) \qquad (3.11)$$

uniformly. But these relations in eqs. (3.9) ~ (3.11) contradict with our Proposition 3.1. Therefore, we should remark that the results in Tables 5 and 6 of Chou, et al. [2] are too conservative.

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