

Harrod's Model of Growth Cycle : A Note

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1. Introduction

In Harrod (1939), we can read the following paragraph :
“Space forbids an application of this method of analysis to the successive phases of the trade cycle. In the course of it the values expressed by the symbols on the right-hand side of the equation undergo considerable change. As actual growth departs upwards or downwards from the warranted level, the warranted rate itself moves, and may chase the actual rate in either direction. (Harrod (1939), pp.28–29).

Fujimoto (1994) presented a simple model in which the number of equilibrium relations assumed, explicit or implicit, are kept to the minimum, and showed that instability persists. In the model, the capital coefficient and the desired saving ratio are constant. Then Kominami (1995) worked out a numerical example in which a simple model *a la* Harrod displays cyclical movements, making the desired saving ratio variable through time. Recently Namba(1995) showed the instability of equilibrium growth in a model which incorporates micro-foundations for some of macro relations together with the price level, the wage rate and the profit rate.

The purpose of this note is to reformulate Fujimoto's model with the

desired saving ratio being variable, and conduct a preliminary study of simulations.

2. A Simple Model

The symbols we employ are :

Y_t^e : expected or normal gross output at the beginning of period t .

Y_t : actual gross output at the period t .

K_t : capital stock at the beginning of period t .

δ : rate of depreciation of capital stock(assumed to be constant).

C^* : desired capital-output ratio (assumed to be constant).

g_t^e : desired rate of accumulation of K at the beginning of period t .

g_t : actual rate of accumulation at the period t .

I_t^e : desired gross investment at the beginning of period t .

I_t : actual gross investment at the period t .

D_t^e : expected demand for goods and services at the period t .

s_t^e : desired (gross) saving ratio at the period t .

s_t : actual (gross) saving ratio at the period t .

u_t : degree of utilization of capital stock at the period t .

In the below, subscript t will be dropped when the explanation is about a period in general. Note that the rate of accumulation is used in the gross sense, and the terms expected and desired are loosely interchangeable.

Now suppose at the beginning of period t , the magnitudes K , g^e , and s^e are given. The expected gross investment is determined by

$$I^e = g^e K. \quad (1)$$

The expected gross output is

$$Y^e = K/C^* \quad (2)$$

The scheduled demand is then given as

$$D^e = (1 - s^*)Y^e + I^e. \quad (3)$$

We now make the assumption that

$$Y = \text{mid}_1(Y^e, D^e). \quad (4)$$

The $\text{mid}(x, y)$ function returns a value which is an intermediate point on the segment between x and y . (See Fujimoto (1994) for more explanation.)

Below this type of function shall be called a *mid* function. The degree of utilization of capital stock u is measured by

$$u = Y/Y^e. \quad (5)$$

The actual gross investment is determined by

$$I = \text{mid}_2(I^e, s^*Y^e). \quad (6)$$

The function $\text{mid}_2(\cdot, \cdot)$ is another mid function. Note here that

$$I^e > s^*Y^e \quad \text{if and only if} \quad D^e > Y^e.$$

From the *ex post* identity, the actual gross accumulation rate and saving ratio can be calculated as

$$g = I/K, \quad \text{and}$$

$$s = I/Y.$$

Eq.(1) is a modified version of accelerator principle, while eq.(6) is a modified multiplier principle. In the literature published so far, it has been normally assumed that the expected investment is realized. The above equations determines all the variables at the period t . The dynamic equations are described by the following.

$$\begin{cases} K_{t+1} = K_t(1 - \delta) + I_t, \\ g_{t+1}^e = g_t^e + f(u_t, D_t^e/Y_t^e), \text{ and} \\ s_{t+1}^* = \text{mid}_3(s_t^*, s) + a \cdot (g_t - g_t^e). \end{cases}$$

Here the function $f(u, v)$ is such that it is defined on the set

$$\{(u, v) \mid u, v \geq 1 \quad \text{or} \quad 1 \geq u, v \geq 0\},$$

$f(1, 1) = 0$ and is non-decreasing in each variable. The coefficient a in the last equation of the adjustment system is a given positive constant.

First note that the above model has the warranted rate of gross accumulation at the period t , $g_t^* \equiv s_t^*/C^*$. If $g_t^e = g_t^*$, this rate g_t^* and s_t^* will persist into the future, and each expected rate is realized with $u_t = 1$. The actual rate of growth in gross output and that of capital stock are the same and equal to $g_t^* - \delta$. The intrinsic instability of the model was shown in Fujimoto(1994) provided the desired saving ratio remains constant. What will happen if the desired saving ratio is allowed to change, and the warranted rate is set to chase the actual rate of accumulation? In this note we present some results obtained in simulation studies on a personal computer.

3. A Simulation

To render our model still more tractable, we take up a simplest form of mid functions. The functions used are :

$$\text{mid}_i(x, y) = m_i \cdot x + (1 - m_i) \cdot y,$$

where m_i is a given positive constant coefficient (less than one) in each mid function. We also take up a simple form for the function $f(u, v)$:

$$f(u, v) = a1 \cdot (u - 1) + a2 \cdot (v - 1),$$

where $a1$, and $a2$ are fixed positive constants.

We also put the upper and lower limits for the planned rate of accumulation g^e . This rate g^e has to be in a given segment $[g_{\min}, g_{\max}]$. In Fig.1, a result is shown. The constants are :

$$C^* = 3.0, \quad \delta = 0.0125, \quad g_{\max} = 0.08, \quad g_{\min} = -0.05, \\ m_1 = 0.5, \quad m_2 = 0.5, \quad m_3 = 0.5, \quad a1 = 0.2, \quad a2 = 0.1, \quad a = 0.5,$$

The initial values are :

$$K_0 = 10.0, \quad g_0^c = 0.05, \quad s_0^* = 0.12.$$

After all these complications, the instability still survive. To have cyclical movements, the desired saving ratio should be allowed to vary more *wildly*. So, the adjustment equation for s_t^* is revised as :

$$s_{t+1}^* = \text{mid}_3 (s_t^*, s_t) + a \cdot g_t. \quad (7)$$

Thus, there is no longer the warranted rate as defined by Harrod.

In Fig.1, the darker cyclical curve in the middle shows the series of the annual gross output Y_t . The less darker curve nearby is the series of Y_t^c . From Fig.1, we can see the model displays a fairly regular cycles of period 25 years or so. This period is not very different from the period found in the simulation studies of Goodwin's growth cycle model.

(The program for simulation is written in Visual Basic(Microsoft Corp.) ver.1.0, and given in Appendices. This program can be run on Visual Basic ver.2.0 up to ver.4.0 without modification, with suitable forms defined.)

4. Remarks

In Fig.2, another simulation result is presented. This is obtained by reducing the depreciation rate from 0.0125 to 0.005. The model economy has expanded and got out of the window. The same result can be obtained if we raise the minimum accumulation rate from -0.05 to -0.03 , while keeping the depreciation rate fixed at 0.0125.

Fig.3 tells us that the period of cycles can be changed drastically by modifying the coefficients m_i . In the result shown in Fig.3, we modified as :

$$m_3 = 0.85, \quad \text{and } g_{\min} = -0.04.$$

The fact that m_3 is nearer to unity implies that in revising the desired saving ratio, the actual accumulation rate has less influence. Fig.3 realizes a period as long as 50 years, which is comparable to Kondratiev cycles.

One may make the desired capital-output ratio variable as time goes on. And eq.(2) is a little awkward in the presence of continuous over- or under utilization of capital. More importantly, eq.(7) is rather *ad hoc*, and excludes the existence of the warranted rate of growth. This should be modified. For example, we include the natural rate of growth, g^n , to reformulate eq.(7) :

$$s_{t+1}^* = \text{mid}_3(s_t^*, s_t) + a \cdot (g_t - g^n). \quad (8)$$

Now the system has the balanced growth at the warranted rate, $g_t^w = s_t^*/C^*$, provided that this warranted rate is equal to the natural rate. That the natural rate plays a substantial role in the short-run also is not unrealistic. Since we have adopted simple linear mid functions, the formulation is not different from simple linear adaptive adjustment. So we may try

$$s_{t+1}^* = s_t^* + b \cdot (s_t - s_t^*), \quad (9)$$

where b is now allowed to be greater than unity.

One more problem to tackle with is the non-negativity of various coefficients, to which we paid no attention while Harrod (1939) was careful enough on this matter.

In our next paper, we will consider the above points as well as the economic meanings of changes in various parameters, and also the implications of government intervention through public spending after incorporating Harrods idea on this point.

Our tentative conclusion in this simulation study is that pursuing Harrods ideas to create trade cycles with a variable desired saving ratio

may not be successful : a new twist seems necessary as can be seen in eqs. (7)–(9). A variable desired capital-output ratio may be of help.

References

- Fujimoto, T.(1994) : Harrodian Instability Principle in Full Disequilibrium, *Okayama Economic Review*, Vol.25(4), pp.154–164.
- Harrod, R.F. (1939) : An Essay in Dynamic Theory, *Economic Journal*, Vol. 49, pp.14–33.
- Kominami, Atsuko (1995) : A Study on Harrod's Instability Principle (in Japanese), MA Thesis (Dept. of Econ., Univ. of Okayama).
- Namba, Y. (1995) : On the Instability of Equilibrium Growth (in Japanese), *Studies in Contemporary Economics* (Western Association of Economics), Vol.4, pp.43–57.

Appendix 1. Codes for Form1 (Form1 = Figure)

```

Sub Command 1_Click ()      ***Control Button : Start
    FRAME
    Harrod
    Beep
    Text2.Visible = -1 , *TRUE : Text 11 Box MultiLine property : TRUE
    Text2.Text = "***Ended. ***" + Chr$(13) + Chr$(10) + "Push ""End""
button."
End Sub

```

```

Sub Command 2_Click ()      ***Control Button : Pause
    Form3.Show 1 *** 1 = Show a form as a modal one.
    Form1.SetFocus
End Sub

```

```

Sub Command 3_Click ()      ***Control Button : End
    End
End Sub

```

```

Sub FRAME ()                '===Subroutine FRAME
    Picture1.ScaleLeft = 0
    Picture1.ScaleWidth = 500
    Picture1.ScaleTop = 500
    Picture1.ScaleHeight = -500
    Picture1.Cls
End Sub                      '===End of Subroutine FRAME

```

Appendix 2. Codes for Form3 (Form3 = Dialogue after Pausing)

```

Sub Command1_Click ()      ***Control Button : Continue .
    Unload Form3
End Sub

Sub Command2_Click ()      ***Control Button : End after Pausing
    End
End Sub

```

Appendix 3. "HARROD. BAS"

```

'A Model of Trade Cycle due to R.Harrod : 1995-2-14(Tue).
Declare Function BitBlt Lib "GDI" (ByVal destHdc%, ByVal x2%,
    _____ByVal y2%, ByVal w%, ByVal h%, ByVal srcHdc%,
    _____ByVal srcx%, ByVal srcy%, ByVal Rop As Long) As Integer
    *** _____ continuation of a line.

Const SRCCOPY = &HCC0029
Const SRCAND = &H8800C6
Const SRCINVERT = &H660046
Declare Function TextOut Lib "GDI" (ByVal hdc%, ByVal X%,
    _____ByVal Y%, ByVal lpString As String, ByVal n%) As Integer
Declare Function SetBkMode Lib "GDI" (ByVal hdc%,
    _____ByVal nBkMode%) As Integer

Const Transparent = 1
Const Opaque = 2  /*Background erased first.

/*constants for Harrods Model */
    Const Cd = 3.0  /*desired capital output ratio */
    Const m1 = .5  /*the coefficient for the function mid 1*/

```

```

Const m2 = .5    /*the coefficient for the function mid 2 */
Const m3 = .5    /*the coefficient for the function mid 3 */
Const a1 = .2    /*the coef. of adjustment in Ge : Utilization*/
Const a2 = .1    /*the coef. of adjustment in Ge : Demand/Supply */
Const asv = .5   /*the coefficient of adjustment in Sd */
Const eps = .0001 /*to avoid round-off errors in functions*/
Const Gmax = .08  /*upper limit for Get
Const Gmin = -.05 /*lower limit for Get
Const Dep = .0125 /*Rate of Depreciation

```

```
Sub Harrod () '---1996-5-5(Sun)
```

```

/****** *)
/*A Model of Trade Cycle a la R.Harrod (EJ, [1939]. *)
/*      for Visual Basic v.1.0 : *)
/*Q 1. Change parametrical values, and find out their effects on *)
/*      the periods. *)
/*Q 2. Note the effect of symmetry of upper & lower limits of *)
/*      expected rate of growth. Break the symmetry, and. . . *)
/* Q 3. Set also limits for the expected saving ratio. *)
/****** */

```

```
***Get, as cannot be used because they are reserved.
```

```
/* variables*/
```

```

Dim K0, Kt, Ge 0, Gext As Double
Dim Gt As Double /*actual rate of (gross) accumulation*/
Dim Iet As Double /*expected gross investment at period t*/
Dim It As Double /*actual investment at period t*/

```

Dim Yet As Double '/*expected gross output*/
 Dim Yt As Double '/*actual gross output*/
 Dim Det As Double '/*expected demand at period t*/
 Dim Ut As Double '/*degree of utilization of Kt*/
 Dim Sd0, Sd As Double '/*desired saving ratio*/
 Dim St As Double '/*actual saving ratio at period t*/
 Dim PH, PV As Integer

K0 = 10# '/*capital at the beginning*/
 Ge0 = .05 '/*planned rate of accumulation at the beginning*/
 Sd0 = .12 '/*desired saving ratio*/

Form1.Text1.Text = Str\$(Gmax) : Form1.Text3.Text = Str\$(Gmin)
 Form1.Text4.Text = Str\$(Dep)

Form1.Picture1.Line (0, 250)-(500, 250), QBColor(15) '/*time-axis
 Form1.Picture1.Line (0, 350)-(500, 350), QBColor(10) '/*10% in green
 Form1.Picture1.Line (0, 150)-(500, 150), QBColor(10) '/*-10%

Form1.Picture1.Line (100, 0)-(100, 500), QBColor(9) '/*100 th year
 Kt = K0 : Gext = Ge0 : Sd = Sd0

For i% = 1 To 500 '*****Iteration for 500 years

Iet = Gext * Kt
 Yet = Kt / Cd
 Det = (1# - Sd) * Yet + Iet

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$Yt = \text{mid1}(Yet, Det)$

$Ut = Yt / Yet$

$It = \text{mid2}(Iet, Sd * Yet)$

`/* It :=Sd*Yt ; /* /* *** savings realized as scheduled ! */`

$Gt = It / Kt$

$St = It / Yt$

$PH = i\%$

$PV = (Yet * 10\#) + 250$

`Form1.Picture1.PSet (PH, PV), QBColor(13) /*Yet in magenta */`

$PV = (It * 1000\# / Kt) + 250$

`Form1.Picture1.PSet (PH, PV), QBColor(11) /*Gt in cyan =11 */`

$PV = (Gext * 1000\#) + 250$

`Form1.Picture1.PSet (PH, PV), QBColor(12) /*Get in red =12 */`

$PV = (Yt * 10\#) + 250$

`Form1.Picture1.PSet (PH, PV), QBColor(14) /*Yt in yellow =14 */`

`***to The Year After`

$Kt = Kt * (1\# - Dep) + It$

$Gext = Gext + Gadjust(Ut, Det / Yet)$

`If (Gext > Gmax) Then Gext = Gmax /*Limits to Get */`

`If (Gext < Gmin) Then Gext = Gmin`

$Sd = \text{mid3}(Sd, St) + asv * Gt$

`X% = DoEvents() /*To Accept a Mouse_Click while printing to a Form !`

`Next i%`

`*****The Bottom of Iteration for 500 years`

End Sub

Function mid1 (X As Double, Y As Double)

 If (Abs(X - Y) < eps) Then

 mid1 = X

 Else

 mid1 = m1 * X + (1# - m1) * Y

 End If

End Function

Function mid2 (X As Double, Y As Double)

 If (Abs(X - Y) < eps) Then

 mid2 = X

 Else

 mid2 = m2 * X + (1# - m2) * Y

 End If

End Function

Function mid3 (X As Double, Y As Double)

 If (Abs(X - Y) < eps) Then

 mid3 = X

 Else

 mid3 = m3 * X + (1# - m3) * Y

 End If

End Function

Function Gadjust (X As Double, Y As Double)

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```
If ((Abs(X - 1#) < eps) And (Abs(Y - 1#) < eps)) Then
  Gadjust = 0
Else
  Gadjust = a1 * (X - 1#) + a2 * (Y - 1#)
End If
End Function
```

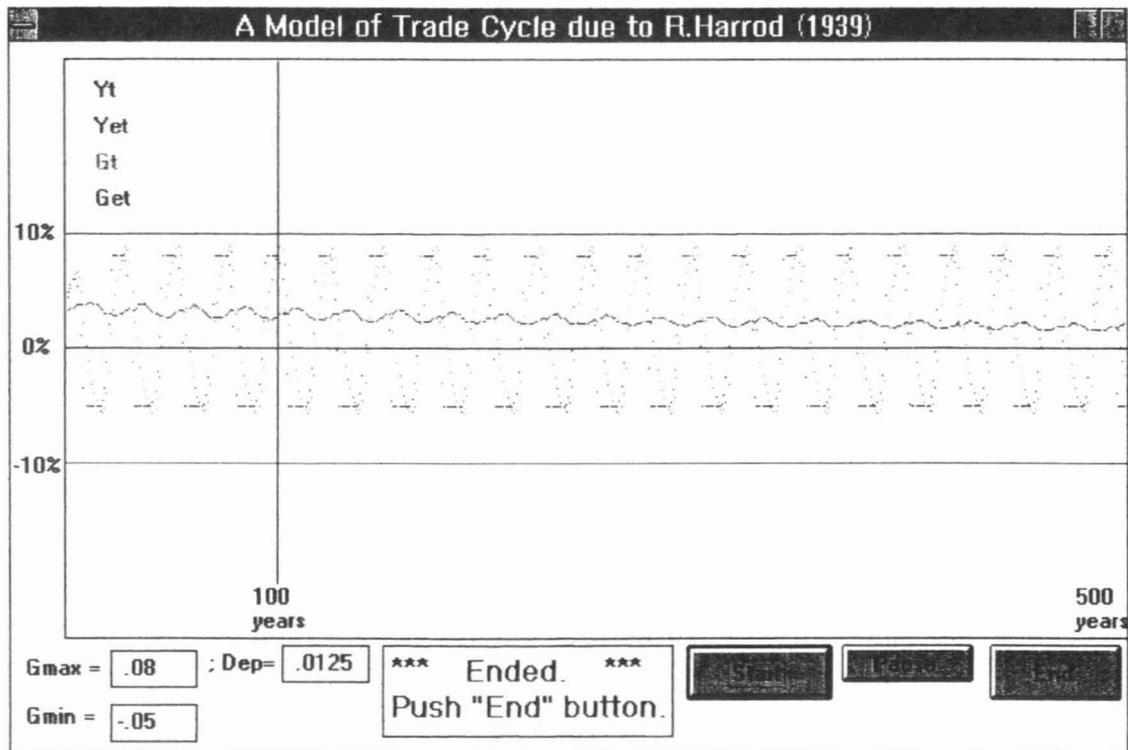


Fig. 1

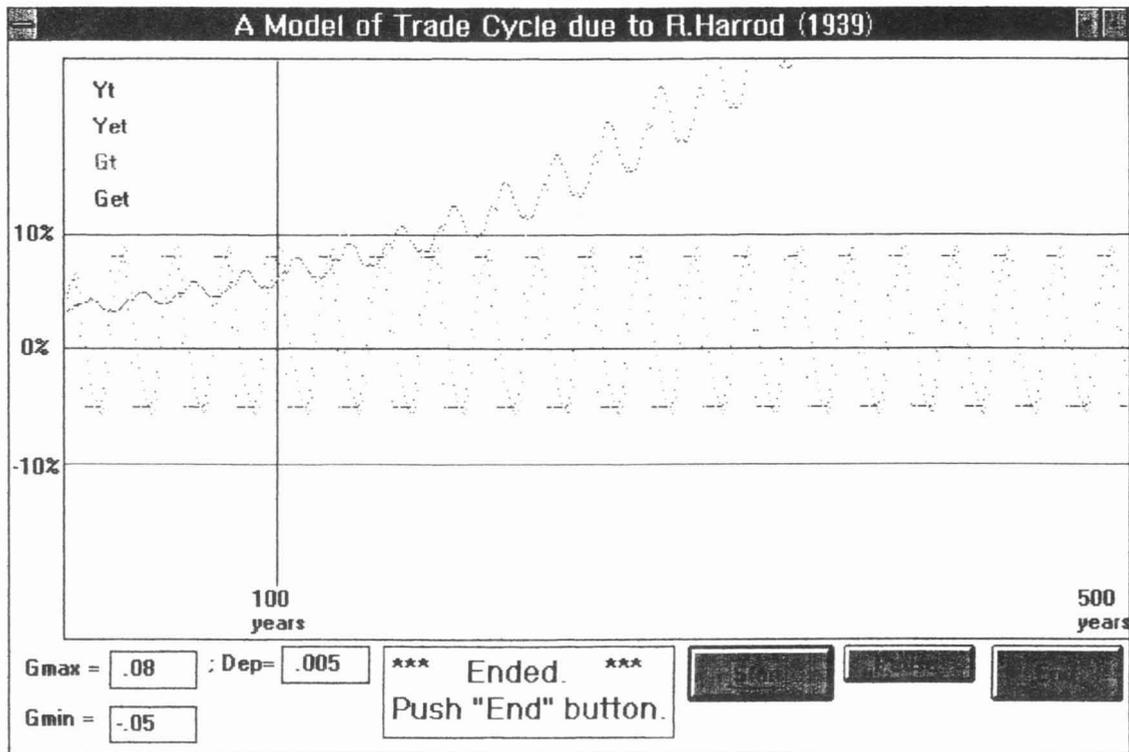


Fig. 2

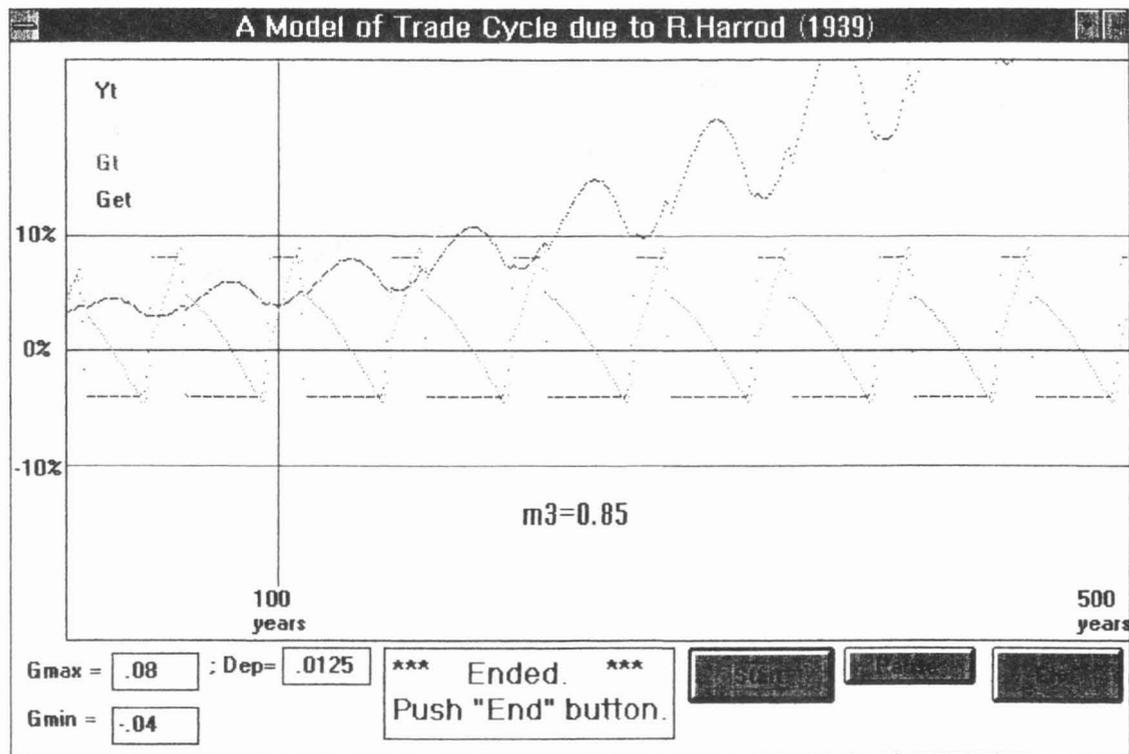


Fig. 3