Global Comparative Statics for Models with Hicksian Imperfect Stability

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1. Introduction

In a recent article of mine in this journal, Fujimoto (1996) showed the powerfullness of a proposition due to Gale and Nikaido (1965) (see also Theorem 20.5 in Nikaido (1968, p.371)) in establishing some results in comparative statics in the large. This proposition enables us to deal with the cases in which more than one equation are perturbed, and also those in which patterns of parametrical changes are not so simple. One more point in studying global shifts is that the older system plays no role in deriving comparative differences. Only the 'new' system after the shifts matters.

Most textbooks explain only comparative statics in the small, and due attention has not been paid to the contributions by Morishima (1964). This is partly because local results are wrongly believed to be extendible to global ones, for example, in a way similar to analytic continuation along a curve. While accumulating small changes, some of the original assumptions may cease to be valid. Besides local analysis has a drawback: we can consider only the present system and small continuous perturbations to the system. This simply means we are left unable to
handle drastic changes that cannot be expressed as a series of small changes satisfying some required properties.

This note is to construct a bridge between local analysis and global one, and to establish a proposition that can be used to prove a set of global versions of comparative statics results for models with Hicksian imperfect stability. In section 2, an example is given to show that consecutive use of a local result cannot lead to the corresponding global proposition. In section 3, our main theorems are presented, whose structure resembles that of Gale–Nikaido's. Section 4 contains some applications. The final section is for some remarks.

2. An Example

Consider a system of simultaneous equations with two variables $x_1$ and $x_2$, and a parameter of real scalar $\alpha$. (Let $\varepsilon$ be a small positive constant.)

$$
\begin{align*}
\begin{cases}
f_i(x_1, x_2) - \alpha &= 0 \\
f_2(x_1, x_2) - \varepsilon \cdot \alpha \cdot (\alpha - 1) \cdot x_1 \cdot x_2 &= 0
\end{cases}
\end{align*}
$$

Suppose originally $\alpha = 0$, and we have a solution vector $x^* = (x_1^*, x_2^*)$. Now we wish to find out changes in this vector as $\alpha$ gets larger. When $\alpha = 0$ or 1, the bottom equation is satisfied by the current solution. If local results tell us the direction of changes only under the assumption that the bottom equation is satisfied by the current solution even after a shift in the parameter, they fail to provide a global extension by a simple patchwork of neighbourhoods. This is because the bottom equation no longer holds when $\alpha$ takes off from the value zero.
3. Main Theorems

Let us take up the following system of equations:

\[
\begin{align*}
    f_1(x_1, x_2, \ldots, x_n) &= 0 \\
    f_2(x_1, x_2, \ldots, x_n) &= 0 \\
    \quad \vdots \\
    f_n(x_1, x_2, \ldots, x_n) &= 0
\end{align*}
\]

(1)

Let the Jacobian matrix of the vector function on a given domain \( D, f \equiv (f_1, f_2, \ldots, f_n)' \), with respect to \( x \equiv (x_1, x_2, \ldots, x_n)' \) be denoted by \( J(x) \). (A prime means the transposition.) The cofactor of this Jacobian with respect to the \((i,j)\) element is written as \( |J_{ij}(x)| \). We assume:

Ass.1. Each \( f_i \) is continuously differentiable on \( D \) in every variable.

Ass.2. \( |J(x)| > 0 \) and \( |J_{11}(x)| > 0 \) for any \( x \in D \).

Ass.3. The inverse image of the segment \([f(x^*), f(x^')]\) contains a curve \( C \) in \( D \) which connects \( x^c \) and \( x^* \).

Ass.4. For an \( x^* \in D, f(x^*) = 0 \).

Ass.5. For an \( x^c \in D, f_i(x^c) < 0 \) and \( f_i(x^c) = 0 \) for \( i = 2, \ldots, n \).

Note that Ass.3 is weaker than requiring the univalence of \( f \) on \( D \).

Now we can show

**Theorem 1.** Given the assumptions Ass.1 to Ass.5, \( x_0^c < x_1^* \).

**Proof.** We consider an imaginary system of equations with a real scalar \( \alpha \) as a parameter:

\[
\begin{align*}
    f_1(x_1, x_2, \ldots, x_n) - \alpha &= 0 \\
    f_2(x_1, x_2, \ldots, x_n) &= 0 \\
    \quad \vdots \\
    f_n(x_1, x_2, \ldots, x_n) &= 0
\end{align*}
\]

(*)

When \( \alpha = f_i(x^c) \), the system (*) has a solution \( x^c \), while it has a
solution $x^*$ with $\alpha = 0$. The curve $C$ postulated in Ass.3 neither intersects itself nor reaches $f(x^*)$, which is the origin, midway between $x^0$ and $x^*$ because of local univalency guaranteed by Ass.2. Thus we can apply successively, starting from $\alpha = f_1(x^0)$, a series of the well known local result to this system: $dx_1/d\alpha = |J_{11}(\alpha)|/|J(\alpha)| > 0$. Therefore we have the desired result.

Next we consider the following system (2) with a parameter $\alpha$.

$$
\begin{align*}
&f_1(x_1, x_2, \ldots, x_n; \alpha) = 0 \\
&f_2(x_1, x_2, \ldots, x_n; \alpha) = 0 \\
&\quad \ldots \\
&f_n(x_1, x_2, \ldots, x_n; \alpha) = 0
\end{align*}
$$

The parameter $\alpha$ in this system is a vector, and may be of an infinite dimension, and can be a function on a certain domain. We suppose that when $\alpha = \alpha^0$, the above system has a solution vector $x^0$, and when $\alpha = \alpha^*$, we have a solution $x^*$. It is not difficult to see the following holds.

**Theorem 2.** Suppose the system (2) with $\alpha = \alpha^*$ satisfies the assumptions Ass.1 to Ass.5, then $x_i^* < x_i^0$.

This is not a tautology. By virtue of our simple two-stage approach, we can now deal with such systems as involve parameters in a very complicated way so far as the assumptions Ass.1 to Ass.5 are verified.

### 4. Applications

Not many words are necessary to suggest possible applications of the above theorems to economic models. First of all, in models of general equilibrium, $f_i(x)$ is regarded as the excess supply function of commodity $i$, while $x_j$ stands for the price of commodity $j$. Supposing suitable
differentiability of excess supply functions, we have

**Proposition 1.** In a model of general equilibrium with Hicksian imperfect stability, a taste shift from the numeraire to a commodity raises the price (in terms of the numeraire) of that commodity provided the vector of equilibrium relative prices is unique.

Note that Proposition 1 is valid *globally*. Under gross substitutability, the local result was obtained by Hicks (1939), and extended to the global one by Morishima. (See Morishima (1964) and also Arrow and Hahn (1971).). Fujimoto (1990) considered the case where the Jacobian matrix has dominant diagonality, which includes gross substitutability as a special case. Fujimoto (1990) gives also a proposition when a taste shifts from the numeraire to a group of commodities. This type of generalization is not possible with our Theorems 1 and 2 in this note. Shiomura (1995) takes up the case in which the Jacobian matrix is a $P$-matrix. His method, however, suggests that Hicksian imperfect stability is enough to establish a global version of a comparative statics result due to Hicks. Certainly to verify Ass.3 without assuming global univalence poses another difficult problem. Concavity, monotonicity, or homogeneity of given functions may help.

### 5. Remarks

Though our idea of two–stage method to establish Theorem 2 from Theorem 1 is simple, it then makes it possible to treat those systems where parameters enter in a complex manner. The number of parameters can be infinite. A change in a single parameter often has to be compensated by consequent changes in other parameters in order to keep
other equilibrium conditions undisturbed. So traditional local analysis should be generalized also in this respect.

The final remark is concerned with the fact that in Proposition 1 no assumptions are made on the system with the parameter $\alpha = \alpha^e$, say, the system before changes. This is to be anticipated when we derive Theorem 2 from Theorem 1 which involves only one system. When we consider global comparative statics, the focus is on the new system together with the new and the old equilibria. No role is played by the old system.

References


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This note is to provide a bridge between traditional local analysis for comparative statics and its global counterpart. Many economists vaguely believe that it is possible to obtain a global result by applying consecutively a series of local results. This belief is not well founded in models where parameters enter in a not-so-simple way. An example is given to show that local analysis is after all local.

In the proof of the first main theorem, a consecutive use of a well known local result is employed. Some necessary assumptions are explicitly stated. Then this theorem is applied to establish another main theorem in which a simple repetitive application of local analysis may break down because some required properties cease to hold. This two-stage approach seems to be useful in tackling with other types of equations.

As an application of our theorems, a general equilibrium model with Hicksian imperfect stability is taken up. A comparative statics result due to Hicks is extended to the case of global changes.

An interesting point to note is that when dealing with global comparative statics, the old system plays no explicit role. Only the new system matters together with the new and old equilibrium values. Understanding this point is important when we come to consider such real situations as involve technical changes in which new processes as well as new commodities turn up.