岡山大学経済学会雑誌31(4),2000, 277～283

A Univalence Theorem for Nonlinear Mappings : An Elementary Approach

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1. Introduction

It is Gale and Nikaido (1965) who studied the univalence of nonlinear mappings in a serious manner. They obtained a sufficient condition for maps with a rectangular region, using the concept of P-matrix. (See also Nikaido (1968).) Before their work, Samuelson (1953) made a conjecture that a map is univalent when the leading principal minors of the Jacobian matrix do not vanish everywhere on a given domain. Gale and Nikaido (1965, p. 82) gave a counter example to this conjecture of Samuelson.

Pearce (1967) claimed that the regularity of the Jacobian determinant for a set of homogeneous and concave functions would be sufficient for univalence. McKenzie (1967 a, 1967 b) raised counter examples against Pearce's assertion noting that the shape of the boundary of a given domain should be too restrictive to be applicable in the context of factor price equalization. Then, Kuga (1972) showed that the non-vanishing of the Jacobian is sufficient for one-to-one correspondence if smooth substitution between factors and strict

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concavity of production functions are satisfied on the whole non-negative orthant. In Mas-Colell (1979 a, 1979 b, 1985) he sharpened the result by Gale and Nikaido under an additional assumption of homogeneity. Especially Mas-Colell (1979 b) proved a proposition that the positivity of the Jacobian at interior points of the domain together with additional positivity of principal minor determinants on the boundary is sufficient for a map to be univalent. (See also Kehoe (1980).)

On the other hand, there are two geometrical conditions for univalence by Kuhn (1959) and McKenzie (1955). These conditions are discussed, and the result of McKenzie is generalized in Fujimoto and Ranade (1998).

The purpose of this note is to present a proposition more general than Gale and Nikaido (1965) with more restrictions on the domain of mappings, while employing an elementary method. Ours depends on a simple elimination procedure. Section 2 explains notation and states our main proposition. In Section 3 we prove this proposition. Finally Section 4 gives some remarks.

2. Proposition

Let R^n be the real Euclidean space of dimension n. We suppose n is greater than one. A given mapping

$$F \equiv (f_1, f_2, \ldots, f_n)',$$

where a prime shows the transposition of an n-tuple, transforms a domain D into R^n, and this domain is of the form

$$D \equiv R \times R \times \cdots \times R \times S,$$

where S is a convex set in R. We make the following assumptions.

A 1 : Each $f_i(x)$, for $i = 1, 2, \ldots, n$, is continuously differentiable on D with its total differential being

$$df_i = \sum_{j=1}^{n} f_{ij}(x) dx_j,$$

where x is $x = (x_1, x_2, \ldots, x_n)'$.

A 2 : The determinant of the Jacobian J of F

$$|J| \equiv \left| \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{pmatrix} \right|,$$

as well as its all leading (upper-left corner) principal minors do not vanish on D, and their determinantal values stay within a respective bounded interval, e. g., as in a condition $0 < m_{11} \leq f_{11} \leq M_{11}$ for some constants m_{11} and M_{11}.

Now we state our main

Proposition : Given the assumptions A 1 and A 2, the mapping F is univalent.

3. Proof

First we eliminate the variable x_1 from the system of equations

$$f(x) = c, \tag{1}$$

where c is a fixed element in the image set of $F(D)$. Because of the assumptions A 1 and A 2, we can identify the function

$$x_1 = g_1(x_2, x_3, \dots, x_n),$$

for any $x_{(1)} = (x_2, x_3, \dots, x_n)'$ in $R \times R \times \cdots R \times S \subset R^{n-1}$. Then this function $g_1(x_{(1)})$ is substituted into x_1 of the equations of system (1) other than the first one. Then we calculate the Jacobian of the reduced system using the implicit function theorem :

$$J_{(1)} = \begin{pmatrix} (f_{11} \cdot f_{12} - f_{21} \cdot f_{12})/f_{11} & (f_{11} \cdot f_{23} - f_{21} \cdot f_{13})/f_{11} & \cdots & \cdots \\ (f_{11} \cdot f_{32} - f_{31} \cdot f_{12})/f_{11} & (f_{11} \cdot f_{33} - f_{31} \cdot f_{13})/f_{11} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

because

$$\frac{\partial \varphi_i}{\partial x_j} = \frac{\partial f_i}{\partial x_j} + \frac{\partial f_i}{\partial x_1} \cdot \frac{\partial x_1}{\partial x_j},$$

where $\varphi_i(x_{(1)})$, $i = 2, 3, \dots, n$, is $f_i(x)$ after x_1 is eliminated. From this reduced form of the Jacobian, we instantly know that the reduced system also satisfies the assumptions A 1 and A 2. This is because the value of a particular leading principal minor of the reduced system is that of the corresponding leading principal minor of the original system with the first row and column revived with its suitable tail–cut, divided by f_{11}.

Now we can proceed in a similar manner to eliminate the second variable x_2, and continue to do so until finally we reach a single variable equation.

$$\varphi_n(x_n) = b_n.$$

The domain of φ_n is the set S which is assumed to be convex in R, and the sign of $\partial \varphi_n / \partial x_n$ is either positive or negative and fixed on S. So, there can be no two solutions in the outset.

This completes the proof.

4. Remarks

Our result is more general than the Gale and Nikaido theorem so far as the condition on signs of minors is concerned. In our case, the sign can be positive or negative provided its value is away from zero with a certain definite gap. Besides we require the constant sign only for the leading principal minors, not all of them. On the other hand, in our formulation the domain is to be unbounded except for the final variable after a suitable renumbering. It is noted that our result is a proper generalization of the linear case.

Since at least one variable is allowed to be in a bounded set, our proposition cannot be covered by the theorem due to Hadamard (1904, 1906) on univalence. (See also Ortega and Rheinboldt (1970, p. 137) for this theorem.)

It is nice if we can prove the theorem in Gale and Nikaido (1965) using our approach. That is, the very properties of a given mapping whose Jacobian is a P-matrix everywhere on the domain may allow us to use our method in this note on a rectangular region.

It is desirable to extend our proposition so that a possibly bounded convex domain can be allowed for. This problem as well as the above is not so easy because it is related to the univalence problem itself.

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This note is to extend a well-known theorem due to Gale and Nikaido on the univalence of nonlinear mappings. Our approach is based on a simple elimination method of variables, and the key proposition used is the implicit function theorem. In terms of the condition on signs of principal minors, our result is more general than that of Gale and Nikaido since the sign of a minor can be positive or negative. Besides we require the sign condition only for the leading principal minors.

On the other hand, the domain of mappings we can deal with has to be unbounded for all but one variable. In addition, the value of each principal minor must be in a finite range.

Some remarks are given in the final section.