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《研究ノート》

Some Remarks on the Univalence of Nonlinear Mappings

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1. Introduction

This is one of my research notes concerning the univalence of nonlinear mappings, and is to give some remarks on the related literature. First of all, what was presented in Fujimoto and Ranade[2] is actually a special case of the result given as an exercise (E 5.3–4) in Ortega and Rheinboldt[7, p. 140]. We apologize to the reader for having overlooked this. The proposition in Fujimoto[3], however, is not included in this exercise because mappings therein have nothing to do with differentiability nor even with continuity.

Second, the propositions in Bandyopadhyay and Biswas[1] are special cases of those in Moré[5], the latter allowing for functions which are decreasing with respect to some variables. Besides, the results in [1] are almost vacuous as explained in Section 2 below.

Section 3 discusses how to make a slight extension of Gale–Nikaido theorem to deal with a mapping whose Jacobian vanishes on a negligible subset of the domain. The homotopy invariance theorem is a help here.

In Section 4, we argue that the spaces under discussion can be discrete, and thus we can accommodate indivisible commodities and/or processes.

2. Discontinuity of Mappings and Dominant Diagonality

Consider the following example of a mapping f from R_+^2 into R_+^2 consisting of two element functions.

$$\begin{cases} f_1(x_1, x_2) = \begin{cases} 2x_1 + x_2 & \text{for } x_2 < \dfrac{11}{15} \\ 3x_1 + x_2 & \text{for } x_2 \geq \dfrac{11}{15} \end{cases} \\ f_2(x_1, x_2) = x_1 + 2x_2 \end{cases}$$

This mapping is clearly increasing, and continuously differentiable almost everywhere. Whenever the Jacobian matrix is available, it is strictly diagonally dominant. The following equation system, however,

$$\begin{cases} f_1(x_1, x_2) = 2 \\ f_2(x_1, x_2) = 2 \end{cases}$$

has two solutions $(2/3,\ 2/3)'$ and $(2/5,\ 4/5)'$. (See Figure 1.) What is wrong in Bandyopadhyay and Biswas[1]? When defining their functions's $P_i's$ [1, p. 439], we have to divide the domain into two subsets, and it is not easy to verify their condition WPD. In the above example, the mapping does not satisfy WPD, and so Theorem 1 in [1] is not wrong. And yet, since every function is nondecreasing in each variable, discontinuity should take place in such a manner as in the above example, and WPD cannot be satisfied.

Hence, Theorem 1 in [1] cannot handle discontinuity we may normally have in mind, and moreover it is difficult to verify their condition WPD when given functions display discontinuity.

On the other hand, Theorem 3.3 in Moré[5, p.365] is not vacuous even when functions are not continuous, because functions can be decreasing with respect to some variables. The existence of a solution, however, will become a tougher problem. (See Fig. 2.)

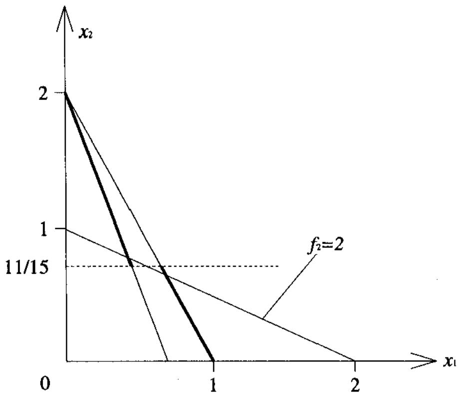

Fig. 1

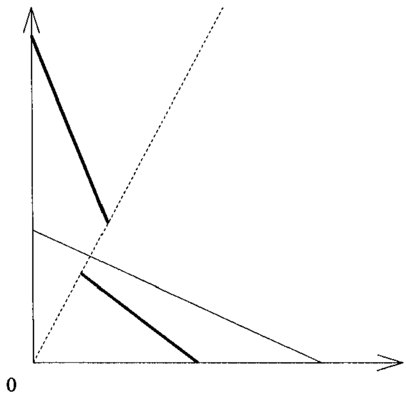

Fig. 2

3. Vanishing Jacobian

Gale and Nikaido [4, p.89] (also Nikaido [6, Theorem 20.8, p.377]) give a univalence theorem under a slightly weaker condition : minors can vanish. Now, we may wish to include the following example in [1].

$$\begin{cases} y_1 = (x_1 - 1)^3 + 2(x_1 + x_2) \\ y_2 = x_1 + x_2 \\ y_3 = x_2 + x_3 \end{cases}$$

The Jacobian itself vanishes when $x_1 = 1$. One of the simplest ways to cover this case is to employ the homotopy invariance theorem. We perturb the second function as $f_2^* = x_1 + x_2 + \varepsilon x_2$, where ε is a small positive scalar, and consider a new mapping f^* with the other two functions remaining the same, and a homotopy class $F(x,t) \equiv (1-t)f + tf^*$ for $t \in [0,1]$. Clearly any interior point of the image $f(R_+^3)$ does not touch the image of the boundary of R_+^3 by $F(x,t)$ while t changes over the unit interval. The image simply expands as t increases. Therefore, the homotopy invariance theorem tells the index is the same for two mappings while f^* has the Jacobian which is a P-matrix everywhere. Let us define the set $V \equiv \{x | x \in R_+^3$, the Jacobian vanishes at $x\}$, and its complement in R_+^3, V^c. We now know on the set V^c, the mapping f is injective, and if $f(x) = f(y)$ for $x \neq y$, these x and y should be in V. Thus if we can prove the original mapping f is injective on V, it is injective on the interior of R_+^3. To consider points on the boundary, we may expand the cone of nonnegative orthant slightly so that it includes the original cone in its interior as is discussed in [1, p.440], with the origin separately examined.

It is not difficult to generalize the above method. When the given mapping has almost everywhere the Jacobian which is a P-matrix, but vanishes on a subset of the domain, we add εx_i to the i-th element function, and have a non-vanishing P-matrix Jacobian everywhere. (Thanks are due to Dr J. Murai for his simple proof based on mathematical induction.) All we have to show is the injectiveness of the mapping on the subset.

4. Discrete Spaces

When we examine the proofs in [1] and [5], they really do not need the continuity of the domain space R^n. A given space can be discrete, and we can deal with the space like Z^n, where Z is the set of integers. For example, in [1] the inverse elements appear only on page 447 as d^*. These elements we can include as those in the quotient field of Z. The argument in [1] remains valid.

In Moré [5], the only part which requires continuity in establishing the univalence is Lemma 2.8[5, p.361]. That is, α appearing in its proof should satisfy $0 < \alpha < 1$. The proof, however, may avoid the use of α, and instead can use two integers m and n such that $m > n > 1$. In more detail, we can put $m|v_k| = n\sum_{j\neq k}|v_j|$, and define $x_k = m \cdot \operatorname{sgn} v_k$ and $x_j = -n \cdot \operatorname{sgn} v_j$ for $j \neq k$. The proof due to Moré can now proceed *mutatis mutandis*.

References

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