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# On Dynamic Effects of the Number of Players in a Commons Game : A Tragedy of Nutria in Okayama

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## 1 Introduction

Before the World War II, nutria (*Myocastor coypus*), large semi–aquatic rodents indigenous to South America, were imported to Japan for munitional fur farming industry. During and after the war time, however, a lot of nutria were released to the wild, either intentionally or not intentionally. In Okayama prefecture, the first wild–grown nutria was identified in Chayamachi, Kurashiki City, in 1944[1]. Nutria have survived up to now in many areas in Okayama, and nowadays they are recognized to cause damages to agricultural products due to their herbivorous activity[2]. Many local governments in Japan have their own mitigation policies against such 'invasive' alien species. In fact, Okayama City, for instance, conducts a kind of *nutria control program* consisting of an incentive payment of not more than 1,000 yen per one nutria (cf. 4,000 yen per wild boar)[3].

Given such an incentive scheme for the nutria control program of Okayama City, the situation falls into the category of the commons problem. There is pervasive literature on various versions of the commons problem or the tragedy of commons. See Hardin (1969), Levhari and Mirman (1980), Fudenberg and Tirole (1991), and Brander and Taylor (1998), just to name a few. The population of nutria in a habitat as a 'commons' can be thought of as a renewable resource stock which is accessible to all the (licensed) trappers, who are the players of the commons game, and for which the trappers struggle. This paper will try to model this situation in a simplest possible way so that the resulting model becomes tractable for further modifications and extensions in the future.

One notable feature of our model is the introduction of some kind of externality to payoff : the cost function is assumed to be not only increasing in the total amount of harvest (as usually postulated) but also decreasing in the current resource stock (i.e., the current size of population of nutria). The intuition behind the last part of this assumption is clear. That just says : 'trapping one out of a thousand of nutria is easier and hence less costly than

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1 See e.g.「岡山県大百科事典　上」, pp. 558–559.

2 For some statistics for agricultural damages caused by wild animals in Chugoku and Shikoku areas, see the website of Chugoku–Shikoku Noseikyoku（中国四国農政局）: http : //www.chushi.maff.go.jp/. See also the website of the Ministry of the Environment of Japan for the Invasive Alien Species Act and its related topics : http : //www.env.go.jp/.

3 There are some additional requirements for a nutria trapper to receive the incentive payment. For more details, see the website of Okayama City : http : //www.city.okayama.okayama.jp/keiei/yuugai/yuugai.htm/.

trapping one out of ten'. The population of nutria grows depending on its current level net of the amount of harvest by the trappers. This interaction between nutria and trappers can lead to oscillating and even chaotic population dynamics. The reason why oscillation tends to occur is rather simple. As the size of population of nutria gets smaller by trapping, the cost of trapping nutria gets higher. Since the incentive payment per nutria is assumed to be fixed, the increase of costs makes the trapping less lucrative for the trappers and prevents them to hunt nutria. This in turn stimulates the nutria population to grow again. As the population of nutria recovers its abundance, the trappers resume trapping, which may significantly reduce the population of nutria again. And the story repeats itself.

In this paper, we focus, among other things, on the impact of the change in the number of trappers on the population dynamics of nutria. Roughly speaking, in some typical cases, the fewer trappers are present in the commons, the more likely to be 'simple' the dynamics of population will be, as the population of nutria is assumed to grow exponentially (rather than to endogenously fluctuate) by itself without any human activity. As the number of trappers increases, however, fluctuating population dynamics tend to occur due to the effect of the 'tragedy of commons' (i.e., overexploitation of resources) together with the aforementioned effect of externalities involved in the cost function. As the number of trappers increases further, the population of nutria tends to end up getting harvested away, with some possible complicated transient motions.

## 2 Description of the model

Let us consider a habitat populated by nutria of population size $N_t$ at the beginning of time $t$. Time is discrete and extends from zero to infinity. We suppose that there are $n$ licensed trappers of nutria who each have equal access to that habitat. Let $s_{i,t}$ denote the amount of nutria trapped by trapper $i\ (i = 1, 2, \cdots, n)$ in time $t$. The total amount of trapped nutria in time $t$, $S_t$, is thus given by

$$S_t = \sum_{i=1}^{n} s_{i,t}. \tag{1}$$

Let $C(S_t, N_t)$ be the total (or social) cost function of harvesting $S_t$, given the size of population of nutria $N_t$. We assume that the total cost function satisfies the following conditions :

(i) $C : R_+^2 \rightarrow R$ is sufficiently smooth ;

(ii) $\partial C / \partial S_t > 0$ and $\partial^2 C / \partial S_t^2 > 0$ (strictly increasing and convex in $S_t$) ;

(iii) $\partial C / \partial N_t < 0$ (strictly decreasing in $N_t$).

Suppose that the authority pays $P$ yen per unit of nutria (e.g. nutria tail) to a trapper. Furthermore, the trappers are assumed myopic in the sense that they do not care about their future payoffs. This assumption implies that each trapper indexed by $i$ solves an optimization problem period by period. The optimization problem for trapper $i$ in time $t$ is thus formulated as follows :

$$\max_{s_{i,t}\geq 0} \qquad Ps_i - \frac{s_{i,t}}{S_t} C\left(S_t, N_t\right) \tag{2}$$

subject to

$$N_t \geq S_t. \tag{3}$$

For the symmetric interior Cournot–Nash solutions, the first–order conditions are given by

$$P = \left(1 - \frac{s_{i,t}}{S_t}\right) \frac{C\left(S_t, N_t\right)}{S_t} + \frac{s_{i,t}}{S_t} \frac{\partial C\left(S_t, N_t\right)}{\partial S_t}, \qquad i = 1, 2, \cdots, n. \tag{4}$$

Summing up eqs. (4) from $i = 1$ to $n$ yields

$$P = \lambda \frac{C\left(S_t, N_t\right)}{S_t} + (1 - \lambda) \frac{\partial C\left(S_t, N_t\right)}{\partial S_t}, \tag{5}$$

where

$$\lambda = \frac{n - 1}{n}. \tag{6}$$

Note that $\lambda \in [0, 1]$. The parameter $\lambda$ can be interpreted as the *intensity of competitiveness*. As $\lambda$ goes to zero, the 'market' gets more monopolistic. On the contrary, $\lambda$ being close to unity means that the market is highly competitive. In particular, the special case of $\lambda = 0$ corresponds to the cooperative case. For simplicity of analysis, we assume that $\lambda$ can take any real number in the unit interval $[0, 1]$.

To obtain a concrete and simple model, we assume that the total cost function takes the following quadratic form :

$$C\left(S_t, N_t\right) = \frac{S_t(1 + S_t)}{N_t} \tag{7}$$

Given (7), the interior solution $S_t^*$ for the arranged first–order condition (5) is given by

$$S_t^* = \frac{PN_t - 1}{2 - \lambda}. \tag{8}$$

Now suppose that the population of nutria evolves according to the following law of motion :

$$N_{t+1} = \beta\left[N_t - S_t\right], \qquad \beta > 1 \tag{9}$$

Note that if there is no human intervention to the nutria habitat, i.e., if $S_t = 0$ for every $t \geq 0$, then the population explodes for any initial condition $N_0 > 0$. Combining the interior solution $S_t^*$ in (8) and the corner solution for (2) with the law of motion in (9) gives the dynamics for $\{N_t\}_{t=0}^{\infty}$ :

$$N_{t+1} = \beta\left[N_t - \min\left\{X_t, \max\left\{\frac{PN_t - 1}{2-\lambda}, 0\right\}\right\}\right] \tag{10}$$

$$= \max\{0, \min\{L(N_t), \beta N_t\}\} =: f(N_t),$$

where

$$L(N_t) = \frac{\beta\left[1 + (2-\lambda-P)N_t\right]}{2-\lambda}. \tag{11}$$

Evidently, for any parameter set, $f$ has the property that :

(i) $f : R_+ \to R_+$ is continuous and piecewise linear ;

(ii) $f(0) = 0$,

(iii) there is a kink at $1/P$.

## 3 Dynamic analysis

Since $f$ in (10) is piecewise linear, the possible dynamic patterns of $f$ are relatively well known. Note first that the slope of the branch corresponding to $L$ in (11) is given by

$$L'(N) = \frac{\beta(2-\lambda-P)}{2-\lambda} \tag{12}$$

Claim 1 : *Suppose* $P \le 1$. *Then for any number of trappers, i.e., for any* $\lambda \in [0,1]$, *the sequence* $\{N_t\}$ *generated by (10) is monotonically increasing or decreasing for* $N_0 > 0$.

Proof : $f$ is monotonically increasing $\Leftrightarrow L'(N) \ge 0 \Leftrightarrow 2 - P \ge \lambda$. Since $\lambda \in [0,1]$ and $P \le 1$ by assumption, we have $2 - P \ge 1 \ge \lambda \ge 0$. □

The above proposition says that if the incentive payment $P$ is low enough, then the population dynamics for nutria are rather simple due to the monotonicity of $f$, regardless of the number of trappers. On the contrary, the population dynamics may be more complex when $P$ is high enough :

Claim 2 : *Suppose* $P > 2$. *Then for any number of trappers, i.e., for any* $\lambda \in [0,1]$, $f$ *cannot be monotonic. To be more specific*, $f'(N) = \beta > 1$ *for* $N \in (0, 1/P)$, *and* $f'(N) < 0$ for $N \in (1/P, 1/(P-2+\lambda))$.

Proof : Evident from the fact that $L'(N) < 0 \Leftrightarrow 2 - P < \lambda$. □

When the incentive payment is moderate, i.e., $1 < P \le 2$, then whether the map $f$ is monotonically increasing depends on the number of trappers :

Claim 3 : *Suppose* $1 < P \leq 2$. *Then there exists* $\lambda^c (= 2 - P) \in [0, 1]$ *such that for* $\lambda \in [0, \lambda^c]$, $f$ *is monotonically increasing and that for* $\lambda \in (\lambda^c, 1)$, $f$ *is not so*.

Proof : Again from sign $\{L'(N)\}$ =sign $\{2 - P - \lambda\}$, the assertion follows. □

In any case for $P > 1$, $f$ can be non–monotonic depending on the number of trappers, which is a symptom for complicated population dynamics. In fact, for large sets of parameter values, $f$ exhibits complicated dynamics. We first show that for a large set of plausible parameter values, $f$ exhibits 'observable chaos'. By 'observable chaos' here we mean that the map $f$ has a (unique) $f$–invariant probability measure absolutely continuous with respect to Lebesgue measure. For more mathematical treatments, see Lasota and Yorke (1973) and Li and Yorke (1978). Let

$$l(\beta, P) = 2 - \frac{\beta P}{1 + \beta} \quad \text{and} \tag{13}$$

$$r(B, P) = 2 - \frac{(\beta - 1)P}{\beta}. \tag{14}$$

Note that $l(\beta, P) < r(\beta, P)$ for any $\beta > 1$ and $P > 0$.

Proposition 1 (Observable Chaos) : *Let* $\beta$ *and* $P$ *be such that*

$$0 \leq l(\beta, P) < 1.$$

*If* $l(\beta, P) < \lambda < \min\{1, r(\beta, P)\}$, *then* $f$ *exhibits observable chaos*.

Proof : One can check that $l(\beta, P) < \lambda$ implies that $f$ is expanding, $|f'(N)| \geq \alpha > 1$, on the set on which $f$ is positive (whenever the derivative exists). Furthermore, $\lambda < r(\beta, P)$ implies that $f(c) < N^c$, where $c = 1/P$ is the critical point and $N^c(> 0)$ is a point such that $f(N) = 0$ for $N \geq N^c$ and $f(N) > 0$ for $N \in (0, N^c)$. Thus, we have $0 < f^2(c) < c < f(c) < N^c$ and $f(I) = I$ for $I = [f^2(c), f(c)]$. Hence, $f : I \to I$ is piecewise linear and expanding with a unique critical point $c$ in its interior of $I$. Consequently, Theorem 1 in Lasota and Yorke (1973) guarantees that $f : I \to I$ admits an absolutely continuous invariant measure, whose uniqueness follows immediately from the uniqueness of the critical point, due to Theorem 1 in Li and Yorke (1978). □

The above proposition ensures that for every 'economically meaningful' (in the sense of Lebesgue) initial condition in $I$, the population of nutria eventually keeps fluctuating aperiodically, without becoming extinct, within a certain set of states (which is in fact a union of one or more closed intervals in $I$, determined independently of almost every $x \in I$). For the later use, we will refer to the above situation as the case of *observable chaos*.

Contrary to the case where endogenous perpetual fluctuations in the population dynamics are possible, there are other cases where the nutria become eventually exterminated by the trappers. For mathematical notations which are not defined here, see e.g. Robinson (1998).

Proposition 2 (Extermiantion with Transient chaos) : *Let* $\beta$ *and* $P$ *be such that*

$$0 \le r(\beta, P) < 1.$$

*If* $r(\beta, P) < \lambda < 1$, *then there exists an* $f-$*invariant compact set* $\Lambda \subset [0, N^c]$ *such that the restriction of* $f$ *to* $\Lambda$ *is topologically conjugate to the one−sided full−shift on two symbols. Here,*

$$\Lambda = \{N \in [0, N^c] \mid f^q(N) \in [0, N^c] \forall q \ge 0\}. \tag{15}$$

*Moreover, for any initial value* $N_0 \notin \Lambda$, *there exists an integer* $k = k(N_0) \ge 0$ *such that* $f^m(N_0) = 0$ for $m \ge k$.

Proof : Condition $r(\beta, P) < \lambda$ implies that $f(c) > N^c = 1/(P + \lambda - 2)$, where notations are as in the proof of the previous proposition. Recalling the unimodal piecewise linear form of $f$ on $[0, N^c]$, we see that there are points $N^l$ and $N^r$ $(0 < N^l < c < N^r < N^c)$ such that $f(N^l) = f(N^r) = N^c$. Let $I = [0, N^c]$, $I_0 = [0, N^l]$, and $I_1 = [N^r, N^c]$. Note that $I_i \subset I$, $I_0 \cap I_1 = \phi$, and $I_0 \cup I_1 \subset f(I_i)$ for $i = 0, 1$ (which is a 'horseshoe' condition). Furthermore, there is $\alpha > 1$ with $|f'(N)| \ge \alpha$ for $N \in I_i$ for $i = 0, 1$ (which implies hyperbolicity). Thus, by a usual argument (see e.g. Robinson 1998, Theorem 5.2), $\Lambda = \cap_{k=0}^{\infty} f^{-k}(I_0 \cup I_1)$ is a Cantor set with $f(\Lambda) = \Lambda$ and the above statement for the topological equivalence between $f|_\Lambda$ and symbolic dynamics follow. For the last statement, it suffices to notice that for $N \in I \setminus I_0 \cup I_1$, $f^k(N) = 0$ for $k \ge 2$, and that for $N \in I_0 \cup I_1 \setminus \Lambda$, there is, by construction of $\Lambda$, an integer $m \ge 1$ such that $f^m(N) \in I \setminus I_0 \cup I_1$. □

According to the above proposition, for an exceptional but infinite initial set $\Lambda$ (of measure zero), the population of nutria keeps fluctuating in a periodic or aperiodic manner, while for almost all cases the nutria eventually become extinct. However, because of continuity of $f$, trajectories close to but not on $\Lambda$ will behave like those on $\Lambda$, which causes the population to fluctuate for ralatively long time before it settles down to zero. Hereafter, we will refer to this situation as the case of *extermination with transient chaos*.

Now we turn attention to the qualitative as well as quantitative transition of the population dynamics as the number of trappers varies. As is expected, the number of trappers matters to the long–run behavior of the nutria population.

Proposition 3 (Dynamic Effects of Changes in the Number of Trappers) : *Let* $P \in (1, 2)$ *and* $\beta > P/(P-1)$. *Then there exist two critical values of intensity of competitiveness given by* $\lambda^* = l(\beta, P)$ *and* $\lambda^{**} = r(\beta, P)$ *such that*

$$0 < \lambda^* < \lambda^{**} < 1 \tag{16}$$

with the property that

(i) *if* $\lambda \in [0, \lambda^*)$ *then the population explodes or converges to a steady state ;*

(ii) *if* $\lambda \in (\lambda^*, \lambda^{**})$ *then the case of 'observable chaos' occurs as in Proposition 1 ; and*

(iii) *if* $\lambda \in (\lambda^{**}, 1]$ *then the case of 'extermination with transient chaos' occurs as in Proposition 2.*

Proof : Noting that $\lim_{\beta\to+\infty} l(\beta, P) = \lim_{\beta\to+\infty} r(\beta, P) = 2 - P$ and that $l(\beta, P) < r(\beta, P)$ for any $\beta > 1$ and $P > 0$, we see that there is $\overline{\beta}$ such that the inequalities in (16) hold for $\beta > \overline{\beta}$. By simple computations, $\overline{\beta} = P/(P-1)$ suffices. For case (i), there are two subcases : (a) $L'(N) \geq 1$ and (b) $-1 < L'(N) < 1$. For (a), $f$ has no positive fixed point, and $f(N) > N$ for $N > 0$. Thus any trajectory diverges to infinity for $N_0 > 0$. For (b), there is a unique positive fixed point $\overline{N} = \beta/\big(\beta P + (1-\beta)(2-\lambda)\big)$, which is easily seen to be globally attracting. For cases (ii) and (iii), we can apply the results of Propositions 1 and 2, respectively. □

We can obtain some similar results even for $P \geq 2$, but we will omit it as it is straightforward.

## 4 Concluding remarks

We have modeled a commons situation in which $n$ trappers (players) have access to a habitat populated by nutria (renewable resource stock). Departing from usual assumptions, we have assumed, among other things, that the current stock of the renewable resource enters the cost function as a factor of externality. Our example has demonstrated that this type of externality may give rise to strong nonlinearities that can cause several types of complicated behaviors in the population dynamics of nutria interacting with non–cooperative trappers.

Furthermore, we have notably shown that an increase in the number of trappers can cause drastic changes in the qualitative as well as quantitative features of the population dynamics. More specifically, depending on the competitiveness in the nutria commons, the population may : grow unboundedly, converge to a steady state level, fluctuate chaotically, or getting exterminated in the end with some possible complex transient motions. By the same token, one could argue how an increase in the pecuniary incentive payment affects the population dynamics.

It is important to note that along a typical trajectory in the case of observable chaos, the level of population of nutria oscillates around the steady state. Suppose that for some given number of trappers, the steady state of $f$ is globally attracting. Suppose then that a new trapper now enters the commons so that $f$ undergoes a bifurcation to observable chaos, with the steady state becoming unstable. In this case, it is easily seen that the 'tragedy of the commons' does occur at the *steady state* level : that is, the increase in the number of trappers reduces the level of steady state population. It is, however, not obvious, due to overshooting, how such an increase in the number of trappers affects the *time average* level along a chaotically oscillating trajectory. In such dynamic situations, it is conjectured that an increase in the number of trappers might increase the level of resource stock on average. This conjecture is worth examining, but we will leave it for our future research.

Finally, as the recent establishment of the Invasive Alien Species Act in Japan indicates, there occur many ecological as well as agricultural problems caused by invasive alien species, besides nutria, such as mongoose, raccoon, black bass, snapping turtle and so on. Since these problems are much wider in scope, applying our simple model to more complex real practices require a more elaborate model structure. Our findings in the paper, however, provide at the very least some insights that should be considered in evaluating of policies for preventing adverse effects on ecosystems caused by invasive alien species.

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# On Dynamic Effects of the Number of Players in a Commons Game : A Tragedy of Nutria in Okayama

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This paper considers a dynamic commons game in relation with mitigation of invasive alien species such as nutria in Okayama. In our commons game, players (trappers) non–cooperatively seek to maximize their own payoff by extracting the renewable resource stock (nutria). One key assumption is that the cost of extraction of the resource is negatively related to the current stock level. For a low level of resource stock, the extraction cost is high, which makes the extraction less lucrative for the players and which in turn stimulates the renewable resource stock to regenerate more rapidly. As the resource stock reaches a high level, the reverse process will start, and this can cause oscillating behaviors. Our simple model proposed here exemplifies that an increase in the number of players can drastically change the qualitative as well as quantitative features of the dynamics for the renewable resource stock.