# Abstract cauchy problems for quasi-linear evolution equations in the sense of hadamard

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# ABSTRACT CAUCHY PROBLEMS FOR QUASI-LINEAR EVOLUTION EQUATIONS IN THE SENSE OF HADAMARD

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## 1. *Introduction*

This paper is concerned with abstract Cauchy problems for quasi-linear evolution equations in the sense of Hadamard. We start with three real Banach spaces

$$Y \subset E \subset X \tag{1.1}$$

such that the inclusions are *continuous* and *dense*, and

(H1) *there exists an isomorphism $S$ of $Y$ onto $X$.*

Let $D$ be a *closed, bounded* subset of $Y$, and consider the abstract Cauchy problem for the quasi-linear evolution equation

$$u'(t) = A(t, u(t))u(t) \quad \text{for } t \in [0, T], \tag{QE}$$

in the space $X$, where $\{A(t, w) : (t, w) \in [0, T] \times D\}$ is a family of closed linear operators in $X$ satisfying the following condition.

(H2) *The domain $D(A(t, w))$ of $A(t, w)$ contains $Y$ and $A(t, w)(Y) \subset E$ for $(t, w) \in [0, T] \times D$. The family $\{A(t, w) : (t, w) \in [0, T] \times D\}$ is strongly continuous in $B(Y, E)$.*

We introduce the notion of well-posedness of the Cauchy problem for (QE) in the sense of Hadamard. Let $X_0$ be another real Banach space continuously embedded in $X$ such that

$$X_0 \cap S^{-1}(X_0) \text{ is dense in } X_0, \tag{1.2}$$

and let $D_0$ be a subset of $X_0$ such that

$$D_0 \subset D \quad \text{and} \quad S(D_0) \subset X_0. \tag{1.3}$$

The Cauchy problem for (QE) is said to be *well-posed in the sense of Hadamard*, if classical solutions $u(t)$ exist for initial data $u(0)$ in the set $D_0$ and depend continuously on their initial data in the following sense. There exists $M > 0$ such that

$$\|u(t) - \widehat{u}(t)\|_X \leqslant M\|u(0) - \widehat{u}(0)\|_{X_0} \quad \text{for } t \in [0, T].$$

Here by a *classical solution* to the Cauchy problem for (QE) we mean a function in the class $C([0, T]; D) \cap C^1([0, T]; E)$ satisfying (QE) and the initial condition.

The abstract Cauchy problem in the sense described above does not seem to have been studied in full generality. For the autonomous case where $A(t, w)$ is

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independent of $(t, w)$, there exists a vast literature on Hadamard well-posed problems. (For instance, see Krein and Khazan [**10**] and Fattorini [**6**].) Hadamard well-posed problems have recently been studied using the theory of integrated semigroups or regularized semigroups. The theory of integrated semigroups was studied intensively by Arendt [**1**] and Neubrander [**11**], and that of regularized semigroups was initiated by Da Prato [**3**] and renewed by Davies and Pang [**4**]. For more information, we refer the reader to Arendt *et al.* [**2**] and deLaubenfels [**5**]. Some results obtained there were extended to the non-autonomous case by introducing the notion of regularized evolution operators and giving a generation theorem for regularized evolution operators in [**14**]. The generation theorem was used in [**16**] to study a class of quasi-linear evolution equations of second order which includes the abstract Kirchhoff equations of degenerate type. This is a recent development closely related with the theory of regularized semigroups or integrated semigroups.

For the quasi-linear case, it is well known that Kato's theory [**8**, **9**] is widely applicable to well-posed problems in the usual sense. The above-mentioned result [**16**] does not contain Kato's result, although it has an example to which Kato's theorem cannot be applied. Our purpose is to establish a theorem generalizing Kato's theorem as well as some results on abstract Cauchy problems closely related with regularized semigroups or integrated semigroups. Section 6 presents the relationship between our result and Kato's theorem.

To attain our objective, a new type of stability condition is introduced in § 2, from the viewpoint of the theory of finite difference approximations. Such an attempt was made in the non-autonomous case with $X = E = X_0$, and an important and fundamental result due to Kato [**9**] was improved in [**15**]. In the present formulation, the space $E$ is not always equal to $X_0$, so that some difficulties arise. For example, the solution of the difference equation associated with (QE) cannot be represented by the time-ordered products of resolvents, so that a different method from the previous one [**15**] is needed to prove the convergence in $Y$ of the solution of the finite difference equation associated with (QE) (see § 5). A new approach is also required in extending a classical solution to a larger interval (see Proposition 4.4).

Unlike the theory of quasi-contractive non-linear semigroups, we need the construction of approximate solutions with 'nice' properties, since our purpose is to find a solution in the class $C([0, T]; D)$, and a detailed reason is provided in a paragraph following Lemma 3.1. On the basis of a key estimate (Lemma 3.1), several types of mild solution are considered and their continuous dependence on initial data is given in § 3. Section 4 contains the convergence theorem of approximate solutions and the existence of several types of mild solution. The main result in this paper is provided by Theorem 5.6, which has the possibility of applying to the global solvability of concrete problems (see § 7) as well as an application to the local solvability of a degenerate Kirchhoff equation with non-linear perturbation (see § 8).

## 2. *Stability conditions*

In this section we introduce a new stability condition from the viewpoint of the theory of finite difference approximations. We begin by making the following hypothesis, which is an analogue of the range condition in the quasi-contractive semigroup theory, because it guarantees the existence of solutions of the finite difference equation for (QE).

(H3) *There exists* $\lambda_0 > 0$ *such that if* $x_0 \in D_0$ *and* $\{t_k\}_{k=1}^i$ *is a finite sequence with* $0 = t_0 < t_1 < \ldots < t_i \leqslant T$ *and* $t_k - t_{k-1} \leqslant \lambda_0$ *for* $k = 1, 2, \ldots, i$, *then there exists a sequence* $\{x_k\}_{k=1}^i$ *in* $D$ *such that*

$$(x_k - x_{k-1})/(t_k - t_{k-1}) = A(t_{k-1}, x_{k-1})x_k \quad \text{for } k = 1, 2, \ldots, i. \tag{2.1}$$

Under assumption (H3), we consider the notion of a stable family of operators in the following sense: a family $\{\mathfrak{A}(t, w) : (t, w) \in [0, T] \times D\}$ of closed linear operators in $X$ is said to be *stable* if there exist $\lambda_0 > 0$, $1 \leqslant p < \infty$ and $M \geqslant 1$ such that to each finite sequence $\{t_k\}_{k=1}^i$ with $0 = t_0 < t_1 < \ldots < t_i \leqslant T$ and $t_k - t_{k-1} \leqslant \lambda_0$ for $k = 1, 2, \ldots, i$, each finite sequence $\{x_k\}_{k=1}^i$ in $D$ satisfying (2.1) with $x_0 \in D_0$, each $v_0 \in X_0$ and each finite sequence $\{f_k\}_{k=1}^i$ in $E$, there corresponds a unique sequence $\{v_k\}_{k=1}^i$ such that

$$(v_k - v_{k-1})/(t_k - t_{k-1}) = \mathfrak{A}(t_{k-1}, x_{k-1})v_k + f_k \quad \text{for } k = 1, 2, \ldots, i, \tag{2.2}$$

and the unique solution $\{v_k\}_{k=1}^i$ of (2.2) satisfies

$$\|v_i\|_X^p \leqslant M\left(\|v_0\|_{X_0}^p + \sum_{k=1}^{i} (t_k - t_{k-1})\|f_k\|_E^p\right). \tag{2.3}$$

Such a stable family $\{\mathfrak{A}(t, w) : (t, w) \in [0, T] \times D\}$ is denoted by

$$\{\mathfrak{A}(t, w) : (t, w) \in [0, T] \times D\} \in S((X, X_0, E), D_0, M, p, \lambda_0).$$

In a special case where $X = X_0 = E$ we give a sufficient condition for stability, which is useful for dealing with the global solvability of the inhomogeneous Kirchhoff equation with linear dissipation in § 7.

PROPOSITION 2.1. *Let* $\{A(t, w) : (t, w) \in [0, T] \times D\}$ *be a family of closed linear operators in* $X$ *satisfying condition* (H3) *and the following two conditions.*

(i) *There exists* $h_0 > 0$ *such that* $R(I - hA(t, w)) = X$ *for* $h \in (0, h_0]$ *and* $(t, w) \in [0, T] \times D$.

(ii) *There exists a family* $\{\|\cdot\|_{(t,w)} : (t, w) \in [0, T] \times D\}$ *of norms in* $X$ *such that the following conditions are satisfied:*

(N1) *there exist* $M \geqslant m > 0$ *such that* $m\|x\|_X \leqslant \|x\|_{(t,w)} \leqslant M\|x\|_X$ *for* $x \in X$ *and* $(t, w) \in [0, T] \times D$;

(N2) *there exist* $1 \leqslant p < \infty$ *and* $\omega \geqslant 0$ *such that*

$$\|x\|_{(t+h,w_h)}^p \leqslant \|x - hA(t, w)x\|_{(t,w)}^p + h\omega\|x\|_{(t+h,w_h)}^p \tag{2.4}$$

*for* $(t, w) \in [0, T] \times D$, $x \in D(A(t, w))$, $h \in (0, h_0]$ *with* $t + h \leqslant T$ *and* $w_h \in D$ *satisfying* $w_h - hA(t, w)w_h = w$.

*Then there exists* $M_p > 0$ *such that*

$$\{A(t, w) : (t, w) \in [0, T] \times D\} \in S((X, X, X), D_0, M_p \exp((2\omega + 1)T), p, \lambda_0)$$

*where* $\lambda_0 > 0$ *is a constant such that* $\lambda_0 \leqslant h_0$ *and* $(2\omega + 1)\lambda_0 \leqslant 1$.

*Proof.* Let $0 = t_0 < t_1 < \ldots < t_i \leqslant T$ and $h_k := t_k - t_{k-1} \leqslant \lambda_0$ for $1 \leqslant k \leqslant i$, and let $x_0 \in D_0$ and $\{x_k\}_{k=1}^{i}$ be a sequence in $D$ satisfying (2.1). Let $v_0 \in X_0$ and $\{f_k\}_{k=1}^{i}$ be a sequence in $E$. Then it follows from condition (i) that the finite difference equation (2.2) with $\mathfrak{A} = A$ has a solution $\{v_k\}_{k=1}^{i}$.

To prove (2.3) we need to show that there exists $C_p > 0$ such that

$$\|u + hv\|_{(t,w)}^{p} \leqslant (1+h)\|u\|_{(t,w)}^{p} + C_p h \|v\|_{(t,w)}^{p}$$

for $(t,w) \in [0,T] \times D$, $0 \leqslant h \leqslant 1$ and $u, v \in X$. Since the function

$$\theta \to (\|u\|_{(t,w)} + \theta h \|v\|_{(t,w)})^p / p$$

is convex on $[0,1]$, we have

$$\begin{aligned}(\|u+hv\|_{(t,w)}^{p} - \|u\|_{(t,w)}^{p})/p &\leqslant (\|u\|_{(t,w)} + h\|v\|_{(t,w)})^{p-1} h \|v\|_{(t,w)} \\ &\leqslant 2^{p-1}(\|u\|_{(t,w)}^{p-1} h \|v\|_{(t,w)} + h\|v\|_{(t,w)}^{p}).\end{aligned}$$

Here we have used the inequality $(a+b)^r \leqslant (2b)^r \leqslant 2^r(a^r + b^r)$ for $0 \leqslant a \leqslant b < \infty$ and $r \geqslant 0$. The desired inequality is obtained by Young's inequality.

We use the inequality shown above, (2.2) with $\mathfrak{A} = A$, and (2.4) with $(t,w) = (t_{k-1}, x_{k-1})$, $x = v_k$, $h = h_k$ and $w_h = x_k$ to find that

$$(1 - h_k \omega)\|v_k\|_{(t_k,x_k)}^{p} \leqslant (1+h_k)\|v_{k-1}\|_{(t_{k-1},x_{k-1})}^{p} + h_k C_p \|f_k\|_{(t_{k-1},x_{k-1})}^{p}$$

for $1 \leqslant k \leqslant i$. This recursive inequality gives

$$\prod_{k=1}^{i} (1 - h_k\omega)(1+h_k)^{-1} \|v_i\|_{(t_i,x_i)}^{p} \leqslant \|v_0\|_{(t_0,x_0)}^{p} + \sum_{k=1}^{i} h_k C_p \|f_k\|_{(t_{k-1},x_{k-1})}^{p}.$$

The desired inequality (2.3) is proved by condition (N1) and the fact $(1-t)^{-1} \leqslant \exp(2t)$ for $t \in [0, \frac{1}{2}]$. In a way similar to the above argument, the uniqueness of solutions follows from condition (N2). □

Remark 2.1. (1) Proposition 2.1 asserts that the stability condition follows from the range condition (i) together with the dissipativity condition (ii) in a special case where $X = X_0 = E$. An idea similar to this is found in the paper by Hughes *et al.* [**7**], but it is different from ours in that they used a family $\{\|\cdot\|_{(t,w)}\}$ of equivalent norms in $X$, depending Lipschitz continuously on $(t,w)$.

(2) As will be seen in § 7, the notion of dissipativity conditions corresponds to energy inequalities in the case of concrete problems. An advantage of our result is that the strong convergence of approximate solutions in the underlying space is proved under such dissipativity conditions.

Now, we state basic hypotheses on the family $\{A(t,w) : (t,w) \in [0,T] \times D\}$ appearing in (QE).

(H4) *There exist* $1 \leqslant p < \infty$ *and* $M \geqslant 1$ *such that*

$$\{A(t,w) : (t,w) \in [0,T] \times D\} \in S((X, X_0, E), D_0, M, p, \lambda_0).$$

(H5) *There exists* $L_A > 0$ *such that* $\|A(t,w) - A(t,z)\|_{Y,E} \leqslant L_A \|w - z\|_X$ *for* $(t,w), (t,z) \in [0,T] \times D$.

(H6) *There exist an open, bounded subset $W$ of $Y$ satisfying $D \subset W$ and a strongly continuous family $\{B(t,w) : (t,w) \in [0,T] \times W\}$ in $B(X,E)$ satisfying condition* (B) *below such that*

$$SA(t,w)S^{-1} = A(t,w) + B(t,w) \quad \text{for } (t,w) \in [0,T] \times D. \tag{2.5}$$

(B) *There exists $L_B > 0$ such that $\|B(t,w) - B(t,z)\|_{X,E} \leqslant L_B \|w - z\|_Y$ for $(t,w), (t,z) \in [0,T] \times W$.*

Remark 2.2. (1) In a special case where $X = X_0 = E$, condition (H6) was introduced by Kato [**8**], and used successfully by Sanekata [**13**] to prove that a successive approximate solution converges in the 'nice' space $Y$, without the reflexivity assumption of underlying spaces. Condition (2.5) means that

$$\{x \in X : S^{-1}x \in D(A(t,w)) \text{ and } A(t,w)S^{-1}x \in Y\} = D(A(t,w))$$

and

$$SA(t,w)S^{-1}x = A(t,w)x + B(t,w)x \quad \text{for } x \in D(A(t,w)).$$

This condition corresponds to the notion of commutators in the case of concrete problems, and so it is not difficult to check this condition in most applications. Recently, Kato [**9**] introduced a new concept of 'intertwining condition' instead of this condition, so that his abstract theory can be applied to concrete problems in the space of continuous functions.

(2) Since $D$ and $W$ are bounded subsets of $Y$, the following assertions follow from the strong continuity of the operators and conditions (H5) and (B): there exists $M_A > 0$ such that $\|A(t,w)\|_{Y,E} \leqslant M_A$ for $(t,w) \in [0,T] \times D$, and there exists $M_B > 0$ such that $\|B(t,w)\|_{X,E} \leqslant M_B$ for $(t,w) \in [0,T] \times W$.

(3) There exists a strongly continuous family $\{\overline{A}(t,w) : (t,w) \in [0,T] \times \widetilde{D}\}$ in $B(Y,E)$ such that $\overline{A}(t,w) = A(t,w)$ for $(t,w) \in [0,T] \times D$, where $\widetilde{D}$ denotes the closure of $D$ in $X$. This fact follows from conditions (H2) and (H5).

(4) Under conditions (H2) through (H5) we see that a solution of (2.1) is unique. Indeed, let $\{y_k\}_{k=1}^{i}$ be another solution of (2.1) with $y_0 = x_0$ and set $z_k = x_k - y_k$ for $0 \leqslant k \leqslant i$. Since $z_0 = 0$ and

$$(z_k - z_{k-1})/(t_k - t_{k-1}) = A(t_{k-1}, x_{k-1})z_k + (A(t_{k-1}, x_{k-1}) - A(t_{k-1}, y_{k-1}))y_k$$

for $1 \leqslant k \leqslant i$, we have by (H4) and (H5),

$$\|z_k\|_X^p \leqslant M \sum_{l=1}^{k} (t_l - t_{l-1}) L_A^p \|z_{l-1}\|_X^p \big( \max_{0 \leqslant l \leqslant i} \|y_l\|_Y^p \big)$$

for $1 \leqslant k \leqslant i$. It is shown inductively that $z_k = 0$ for $1 \leqslant k \leqslant i$.

From § 2 to § 5 we assume conditions (H1) through (H6).

Let $\overline{M} = 2^{p-1} M \exp(2^p M M_B^p T)$ and let $\varepsilon_0 \in (0, \lambda_0]$ be a number satisfying $2^p M M_B^p \varepsilon_0 \leqslant 1$. The following proposition asserts that the stability condition (H4) is preserved under the perturbation of a uniformly bounded family in $B(X,E)$.

Proposition 2.2. *The family $\{A(t,w) + B(t,w) : (t,w) \in [0,T] \times D\}$ belongs to the class $S((X, X_0, E), D_0, \overline{M}, p, \varepsilon_0)$.*

*Proof.* Let $0 = t_0 < t_1 < \ldots < t_i \leqslant T$ and $t_k - t_{k-1} \leqslant \varepsilon_0$ for $1 \leqslant k \leqslant i$, and let $x_0 \in D_0$ and $\{x_k\}_{k=1}^i$ be a sequence in $D$ satisfying (2.1). Let $v_0 \in X_0$ and $\{f_k\}_{k=1}^i$ be a finite sequence in $E$. We first show that the difference equation

$$(v_k - v_{k-1})/(t_k - t_{k-1}) = A(t_{k-1}, x_{k-1})v_k + B(t_{k-1}, x_{k-1})v_k + f_k \tag{2.6}$$

has a solution $\{v_k\}_{k=1}^i$ in $X$. For this purpose, let $1 \leqslant l \leqslant i$ and assume that a sequence $\{v_k\}_{k=1}^{l-1}$ has been chosen so that (2.6) is satisfied with $1 \leqslant k \leqslant l-1$. Then we set $v^{(0)} = v_{l-1}$ and define $v^{(n)}$ inductively by a unique element satisfying the equation

$$(v^{(n)} - v_{l-1})/(t_l - t_{l-1}) = A(t_{l-1}, x_{l-1})v^{(n)} + B(t_{l-1}, x_{l-1})v^{(n-1)} + f_l \tag{2.7}$$

for $n = 1, 2, \ldots$. This definition makes sense. Indeed, for the sequence $\{g_k\}_{k=1}^l$ in $E$ defined by $g_k = B(t_{k-1}, x_{k-1})v_k + f_k$ for $1 \leqslant k \leqslant l-1$ and $g_l = B(t_{l-1}, x_{l-1})v^{(n-1)} + f_l$, consider the difference equation $(u_k - u_{k-1})/(t_k - t_{k-1}) = A(t_{k-1}, x_{k-1})u_k + g_k$ for $1 \leqslant k \leqslant l$, where $u_0 = v_0 \in X_0$. Then it has a solution $\{v_1, \ldots, v_{l-1}, v^{(n)}\}$ by the stability condition (H4) and the hypothesis of induction.

Now, we set $w^{(n)} := v^{(n)} - v^{(n-1)}$ for $n = 1, 2, \ldots$. Then the sequence $\{z_k\}_{k=0}^l$ defined by $z_k = 0$ for $0 \leqslant k \leqslant l-1$ and $z_l = w^{(n)}$ is a solution of the equation

$$(z_k - z_{k-1})/(t_k - t_{k-1}) = A(t_{k-1}, x_{k-1})z_k + F_k \quad \text{for } 1 \leqslant k \leqslant l,$$

where $F_k = 0$ for $1 \leqslant k \leqslant l-1$ and $F_l = B(t_{l-1}, x_{l-1})w^{(n-1)} \in E$. By condition (H4) again we have

$$\|w^{(n)}\|_X^p \leqslant M(t_l - t_{l-1})\|B(t_{l-1}, x_{l-1})w^{(n-1)}\|_E^p \leqslant MM_B^p \varepsilon_0 \|w^{(n-1)}\|_X^p,$$

which implies $\|v^{(n)} - v^{(n-1)}\|_X \leqslant (\frac{1}{2})^{n-1}\|v^{(1)} - v^{(0)}\|_X$ for $n \geqslant 1$, by the choice of $\varepsilon_0$. By a standard technique we see that the sequence $\{v^{(n)}\}$ is convergent in $X$ as $n \to \infty$. We denote the limit $\lim_{n\to\infty} v^{(n)}$ by $v_l$. Since $B(t_{l-1}, x_{l-1}) \in B(X, E)$ we deduce from (2.7) that the sequence $\{A(t_{l-1}, x_{l-1})v^{(n)}\}$ converges to the element $(v_l - v_{l-1})/(t_l - t_{l-1}) - (B(t_{l-1}, x_{l-1})v_l + f_l)$ as $n \to \infty$. The closedness of $A(t_{l-1}, x_{l-1})$ implies that the equation (2.6) is satisfied with $k = l$. The desired claim is thus proved by induction. By the stability condition (H4), the solution $\{v_k\}_{k=1}^i$ of (2.6) must satisfy the estimate

$$\|v_l\|_X^p \leqslant M\left(\|v_0\|_{X_0}^p + \sum_{k=1}^{l}(t_k - t_{k-1})\|B(t_{k-1}, x_{k-1})v_k + f_k\|_E^p\right)$$

for $1 \leqslant l \leqslant i$. Since $\|B(t_{k-1}, x_{k-1})v_k + f_k\|_E^p \leqslant 2^{p-1}(M_B^p\|v_k\|_X^p + \|f_k\|_E^p)$ by Remark 2.2(2) and Hölder's inequality, the desired inequality (2.3) with $M = \overline{M}$ is obtained by Lemma 2.4 below.

Finally, we prove the uniqueness of solutions of (2.6). To do this, let $\{\widehat{v}_k\}_{k=1}^i$ be another solution of (2.6) and set $z_k := v_k - \widehat{v}_k$ for $1 \leqslant k \leqslant i$. Then the sequence $\{z_k\}_{k=1}^i$ is a solution of the difference equation

$$(z_k - z_{k-1})/(t_k - t_{k-1}) = A(t_{k-1}, x_{k-1})z_k + B(t_{k-1}, x_{k-1})z_k \quad \text{for } 1 \leqslant k \leqslant i.$$

Since $z_0 = 0$, we have by condition (H4), $\|z_l\|_X^p \leqslant M\sum_{k=1}^l (t_k - t_{k-1})M_B^p\|z_k\|_X^p$ for $1 \leqslant l \leqslant i$. Lemma 2.4 implies that $z_k = 0$ for $1 \leqslant k \leqslant i$. □

We next give a result on uniform boundedness in $Y$ of solutions of finite difference equations to be used in later arguments.

PROPOSITION 2.3. *Let* $\{t_k\}_{k=1}^i$ *be a sequence such that* $0 = t_0 < t_1 < \ldots < t_i \leqslant T$ *and* $t_k - t_{k-1} \leqslant \varepsilon_0$ *for* $1 \leqslant k \leqslant i$, *let* $x_0 \in D_0$ *and* $\{x_k\}_{k=1}^i$ *be a sequence in* $D$ *satisfying* (2.1). *Let* $\{f_k\}_{k=1}^i$ *be a sequence in* $Y$ *such that* $Sf_k \in E$ *for* $1 \leqslant k \leqslant i$. *Let* $v_0 \in Y \cap X_0$ *and* $Sv_0 \in X_0$, *and assume that* $\{v_k\}_{k=1}^i$ *is a solution of*

$$(v_k - v_{k-1})/(t_k - t_{k-1}) = A(t_{k-1}, x_{k-1})v_k + f_k \tag{2.8}$$

*for* $1 \leqslant k \leqslant i$. *Then we have* $v_k \in Y$ *for* $1 \leqslant k \leqslant i$,

$$(Sv_k - Sv_{k-1})/(t_k - t_{k-1}) = (A(t_{k-1}, x_{k-1}) + B(t_{k-1}, x_{k-1}))Sv_k + Sf_k \tag{2.9}$$

*for* $1 \leqslant k \leqslant i$ *and*

$$\|Sv_i\|_X^p \leqslant \overline{M}\bigg(\|Sv_0\|_{X_0}^p + \sum_{k=1}^{i}(t_k - t_{k-1})\|Sf_k\|_E^p\bigg). \tag{2.10}$$

*Proof.* By Proposition 2.2, the family $\{A(t,w) + B(t,w) : (t,w) \in [0,T] \times D\}$ belongs to the class $S((X, X_0, E), D_0, \overline{M}, p, \varepsilon_0)$. Let $w_0 = Sv_0$. Then there exists a unique solution $\{w_k\}_{k=1}^i$ of the difference equation

$$(w_k - w_{k-1})/(t_k - t_{k-1}) = (A(t_{k-1}, x_{k-1}) + B(t_{k-1}, x_{k-1}))w_k + Sf_k$$

for $1 \leqslant k \leqslant i$, satisfying the estimate $\|w_i\|_X^p \leqslant \overline{M}\big(\|w_0\|_{X_0}^p + \sum_{k=1}^i (t_k - t_{k-1})\|Sf_k\|_E^p\big)$ for $1 \leqslant k \leqslant i$. Now, we set $\widehat{v}_0 = v_0$ and $\widehat{v}_k = S^{-1}w_k$ for $1 \leqslant k \leqslant i$. Clearly, $\widehat{v}_k \in Y$ for $0 \leqslant k \leqslant i$, and (2.9) and (2.10) are satisfied with $v_k$ replaced by $\widehat{v}_k$. By (2.5) we see that $\{\widehat{v}_k\}_{k=1}^i$ is also a solution of (2.8). Since $\widehat{v}_0 = v_0$ we have $\widehat{v}_k = v_k$ for $1 \leqslant k \leqslant i$, by the uniqueness of solutions in condition (H4). □

A discrete version of Gronwall's inequality is given by the next lemma.

LEMMA 2.4. *Let* $\{a_l\}_{l=1}^i$ *and* $\{h_l\}_{l=1}^i$ *be two sequences of non-negative numbers, let* $\{\alpha_l\}_{l=1}^i$ *be a non-decreasing sequence of non-negative numbers, and let* $\beta \geqslant 0$. *Assume that* $a_l \leqslant \alpha_l + \beta\sum_{k=1}^l h_k a_k$ *for* $1 \leqslant l \leqslant i$. *If* $\beta h_l \in [0, \frac{1}{2}]$ *for* $1 \leqslant l \leqslant i$, *then we have* $a_l \leqslant \exp\big(2\beta\sum_{k=1}^l h_k\big)\alpha_l$ *for* $1 \leqslant l \leqslant i$.

*Proof.* Let $1 \leqslant l \leqslant i$, and set $b_0 = \alpha_l$ and $b_j = \alpha_l + \beta\sum_{k=1}^j h_k a_k$ for $1 \leqslant j \leqslant l$. Then we have $a_j \leqslant \alpha_j + \beta\sum_{k=1}^j h_k a_k \leqslant \alpha_l + \beta\sum_{k=1}^j h_k a_k = b_j$ for $1 \leqslant j \leqslant l$, by using the non-decreasing property of $\{\alpha_l\}$. Since $b_j - b_{j-1} = \beta h_j a_j \leqslant \beta h_j b_j$ for $1 \leqslant j \leqslant l$, we have $b_j \leqslant \exp(2\beta h_j)b_{j-1}$ for $1 \leqslant j \leqslant l$. It follows that $b_j \leqslant \exp(2\beta\sum_{k=1}^j h_k)b_0$ for $1 \leqslant j \leqslant l$, which implies that $a_j \leqslant \exp(2\beta\sum_{k=1}^j h_k)\alpha_l$ for $1 \leqslant j \leqslant l$. The desired inequality is obtained by setting $j = l$. □

## 3. *Several types of mild solutions and a key estimate*

We begin by giving a key estimate with some comments.

LEMMA 3.1. *Let* $0 < \tau \leqslant T$. *Let* $\Delta = \{0 = t_0 < t_1 < \ldots < t_N\}$ *be a partition of* $[0, t_N]$ *such that* $t_N \leqslant \tau$. *Let* $u_0 \in D_0$ *and* $w_0 \in Y \cap X_0$, *and let* $\{\varepsilon_i\}_{i=1}^N$ *and* $\{g_i\}_{i=1}^N$ *be sequences in* $E$. *Assume that* $\{u_i\}_{i=1}^N$ *and* $\{w_i\}_{i=1}^N$ *are a sequence in* $D$ *satisfying*

(3.1) *and a sequence in* $Y$ *satisfying* (3.2) *respectively:*

$$(u_i - u_{i-1})/(t_i - t_{i-1}) = A(t_{i-1}, u_{i-1})u_i + \varepsilon_i \quad \text{for } i = 1, 2, \ldots, N, \tag{3.1}$$

$$(w_i - w_{i-1})/(t_i - t_{i-1}) = A(t_{i-1}, u_{i-1})w_i + g_i \quad \text{for } i = 1, 2, \ldots, N. \tag{3.2}$$

*Let us define two step functions* $u$ *and* $w$ *by*

$$u(t) = \begin{cases} u_{i-1} & \text{for } t \in [t_{i-1}, t_i) \text{ and } i = 1, 2, \ldots, N, \\ u_N & \text{for } t = t_N, \end{cases} \tag{3.3}$$

$$w(t) = \begin{cases} w_{i-1} & \text{for } t \in [t_{i-1}, t_i) \text{ and } i = 1, 2, \ldots, N, \\ w_N & \text{for } t = t_N. \end{cases} \tag{3.4}$$

*Let* $P = \{0 = s_0 < s_1 < \ldots < s_K = t_N\}$ *be a partition of* $[0, t_N]$ *which is a refinement of* $\Delta$, *and let* $v_0 \in X_0$, $z_0 \in X_0$ *and* $h \in C([0, \tau]; E)$. *Let* $\{v_j\}_{j=1}^K$ *and* $\{z_j\}_{j=1}^K$ *be sequences in* $D$ *and* $X$ *respectively such that*

$$(v_j - v_{j-1})/(s_j - s_{j-1}) = A(s_{j-1}, v_{j-1})v_j \quad \text{for } j = 1, 2, \ldots, K,$$

$$(z_j - z_{j-1})/(s_j - s_{j-1}) = A(s_{j-1}, v_{j-1})z_j + h(s_{j-1}) \quad \text{for } j = 1, 2, \ldots, K,$$

*and define two functions* $v$ *and* $z$ *similarly to* $u$ *and* $w$ *in* (3.3) *and* (3.4). *Then we have*

$$\|u(t) - v(t)\|_X \leqslant C(\|u_0 - v_0\|_{X_0} + \delta(\alpha + \beta + \gamma)), \tag{3.5}$$

$$\begin{aligned} \|w(t) - z(t)\|_X \leqslant\, & C(\|w_0 - z_0\|_{X_0} + \|g - h\|_{L^p(0,\tau;E)}) + \delta(\widetilde{\alpha} + \widetilde{\beta} + \widetilde{\gamma} + \rho_h(|\Delta|)) \\ & + C\widetilde{M}_Y(\|u_0 - v_0\|_{X_0} + \delta(\alpha + \beta + \gamma)) \end{aligned} \tag{3.6}$$

*for* $t \in [0, t_N]$, *where* $C$ *denotes various constants,* $\delta : [0, \infty) \to [0, \infty)$ *stands for various non-decreasing, continuous functions such that* $\delta(0) = 0$, $\rho_h$ *is the modulus of continuity of* $h$, $|\Delta| = \max_{1 \leqslant i \leqslant N}(t_i - t_{i-1})$, $g \in L^p(0, \tau; E)$ *and the symbols* $\widetilde{M}_Y, \alpha, \widetilde{\alpha}, \beta, \widetilde{\beta}, \gamma$ *and* $\widetilde{\gamma}$ *are defined by*

$$\widetilde{M}_Y = \max_{0 \leqslant i \leqslant N} \|w_i\|_Y,$$

$$\alpha = \max_{1 \leqslant i \leqslant N} \|\varepsilon_i\|_E, \quad \widetilde{\alpha}^p = \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \|g(t) - g_i\|_E^p \, dt,$$

$$\beta = \max_{1 \leqslant i \leqslant N} (\sup\{\|(A(t, u_{i-1}) - A(t_{i-1}, u_{i-1}))u_{i-1}\|_E : t \in [t_{i-1}, t_i]\}),$$

$$\widetilde{\beta} = \max_{1 \leqslant i \leqslant N} (\sup\{\|(A(t, u_{i-1}) - A(t_{i-1}, u_{i-1}))w_{i-1}\|_E : t \in [t_{i-1}, t_i]\}),$$

$$\gamma = \max_{1 \leqslant i \leqslant N} \|u_i - u_{i-1}\|_Y, \quad \widetilde{\gamma} = \max_{1 \leqslant i \leqslant N} \|w_i - w_{i-1}\|_Y.$$

Unlike the generation of quasi-contractive non-linear semigroups, we need the construction of approximate solutions with ‘nice’ properties. In fact, our purpose is to find a solution $u$ in the class $C([0, T]; D)$, and so it is necessary to show that a sequence $\{u^\varepsilon(t)\}$ of approximate solutions converges in the space $Y$ as well as the underlying space $X$. By taking account of the fact that the limit function $v$ of the sequence $\{Su^\varepsilon(t)\}$ must be $Su(t)$ and formally satisfies the equation

$$v'(t) = A(t, u(t))v(t) + f^v(t) \quad \text{for } t \in [0, T], \tag{3.7}$$

where $f^v(t) = B(t, S^{-1}v(t))v(t)$ for $t \in [0, T]$, the function $v(t)$ is regarded as a fixed point of the operator $\Phi$ mapping $z$ to the unique limit of a sequence $\{w^\varepsilon(t)\}$ of approximate solutions of (3.7) with $f^v$ replaced by $f^z$. By the above insight similar to that due to Sanekata [**13**], the problem of whether the sequence $\{w^\varepsilon(t)\}$ converges in $X$ occurs in discussing the convergence in $Y$ of a sequence $\{u^\varepsilon(t)\}$ of approximate solutions. This is a reason why we need to study the construction and convergence of approximate solutions with 'nice' properties. By virtue of Lemma 3.1, we introduce the notion of approximate solutions to the Cauchy problem for (QE).

By (QE; $u_0$) we denote the Cauchy problem for (QE) with initial condition $u(0) = u_0$.

Let $u_0 \in D_0$ and $\varepsilon > 0$. If there exists a sequence $\{(t_i, u_i, \varepsilon_i)\}_{i=1}^N$ in $[0,\tau] \times D \times E$ satisfying (3.1) such that $0 = t_0 < t_1 < \ldots < t_N \leqslant \tau$, $t_i - t_{i-1} \leqslant \varepsilon$ for $1 \leqslant i \leqslant N$, $\tau - \varepsilon < t_N$, $\alpha \leqslant \varepsilon, \beta \leqslant \varepsilon$ and $\gamma \leqslant \varepsilon$, then a step function defined by (3.3) is called a *regular $\varepsilon$-approximate solution to* (QE; $u_0$) *on* $[0,\tau]$.

Let $u \in C([0,\tau]; X)$. If for each sufficiently small $\varepsilon > 0$ there exists a regular $\varepsilon$-approximate solution $u^\varepsilon$ to (QE; $u_0$) on $[0,\tau]$ such that $\|u^\varepsilon(t) - u(t)\|_X \leqslant \varepsilon$ for $t$ in the domain of $u^\varepsilon$, then the function $u$ is said to be a *regular mild solution to* (QE; $u_0$) *on* $[0,\tau]$.

PROPOSITION 3.2. *Let $u_0 \in D_0$. A classical solution $u$ to* (QE; $u_0$) *on* $[0,T]$ *is a regular mild solution to* (QE; $u_0$) *on* $[0,T]$.

*Proof.* Let $\Delta = \{0 = t_0 < t_1 < \ldots < t_N = T\}$ be a partition of $[0,T]$. If we set

$$u_i = u(t_i) \quad \text{and} \quad \varepsilon_i = \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} (A(t, u(t))u(t) - A(t_{i-1}, u_{i-1})u_i)\, dt$$

for $1 \leqslant i \leqslant N$, then (3.1) is satisfied. Since $(t,s) \to A(t, u(s))u(s)$ is continuous in $E$ on $[0,T] \times [0,T]$ and $u$ is continuous in $Y$ on $[0,T]$, the assertion is true. □

Let $u_0 \in D_0$ and $u$ be the unique regular mild solution to the Cauchy problem (QE; $u_0$) on $[0,\tau]$. (The uniqueness of regular mild solutions will be proved by Proposition 3.3.) To prove $u \in C([0,\tau]; Y)$, the notion of approximate solutions to the problem

$$\begin{cases} v'(t) = A(t, u(t))v(t) + f(t) & \text{for } t \in [0,\tau], \\ v(0) = x, \end{cases} \tag{CP; $x, f)^u$}$$

is also needed for each $x \in X_0$ and $f \in L^p(0,\tau; E)$.

(a) Let $\varepsilon > 0$. If there exist $w_0 \in Y \cap X_0$ and a sequence $\{(t_i, u_i, \varepsilon_i, w_i, g_i)\}_{i=1}^N$ in $[0,\tau] \times D \times E \times Y \times E$ satisfying (3.1) and (3.2) such that $Su_i \in D(A(t_{i-1}, u_{i-1}))$,

$$\xi_i := (Su_i - Su_{i-1})/(t_i - t_{i-1}) - (A(t_{i-1}, u_{i-1}) + B(t_{i-1}, u_{i-1}))Su_i \in E$$

and $\|\xi_i\|_E \leqslant \varepsilon$ for $1 \leqslant i \leqslant N$, such that a step function defined by (3.3) is a regular $\varepsilon$-approximate solution to (QE; $u_0$) on $[0,\tau]$ whose difference from $u$ is less than or equal to $\varepsilon$ and such that $\|w_0 - x\|_{X_0} \leqslant \varepsilon$, $\widetilde{\alpha} \leqslant \varepsilon$ with $g = f$, $\widetilde{\beta} \leqslant \varepsilon$ and $\widetilde{\gamma} \leqslant \varepsilon$, then a step function defined by (3.4) is called a *regular $\varepsilon$-approximate solution to* $(\mathrm{CP}; x, f)^u$ *on* $[0,\tau]$.

(b) A function $w \in C([0,\tau]; X)$ is said to be a *regular mild solution to* $(\mathrm{CP}; x, f)^u$ *on* $[0,\tau]$ provided that for each sufficiently small $\varepsilon > 0$ there exists a regular $\varepsilon$-approximate solution $w^\varepsilon$ to $(\mathrm{CP}; x, f)^u$ on $[0,\tau]$ such that $\|w^\varepsilon(t)\|_Y$ is uniformly bounded as $\varepsilon \downarrow 0$ and $\|w^\varepsilon(t) - w(t)\|_X \leqslant \varepsilon$ for $t$ in the domain of $w^\varepsilon$.

(c) A function $v \in C([0,\tau]; X)$ is called a *mild solution to* $(\mathrm{CP}; x, f)^u$ *on* $[0,\tau]$, if, for each sufficiently small $\varepsilon > 0$, there exists a regular mild solution $w^\varepsilon$ to $(\mathrm{CP}; y, g)^u$ on $[0,\tau]$ such that $\|x-y\|_{X_0} \leqslant \varepsilon$, $\|f-g\|_{L^p(0,\tau;E)} \leqslant \varepsilon$ and $\sup_{t\in[0,\tau]} \|w^\varepsilon(t) - v(t)\|_X \leqslant \varepsilon$.

The uniqueness of mild solutions is given by the following result.

Proposition 3.3. (i) *Let $u_0, \widehat{u}_0 \in D_0$, and let $u$ and $\widehat{u}$ be two regular mild solutions to* (QE; $u_0$) *and* (QE; $\widehat{u}_0$) *on* $[0,\tau]$ *respectively. Then we have*

$$\|u(t) - \widehat{u}(t)\|_X \leqslant C\|u_0 - \widehat{u}_0\|_{X_0} \quad \text{for } t \in [0,\tau].$$

(ii) *Let $u_0 \in D_0$ and $u$ be a regular mild solution to* (QE; $u_0$) *on* $[0,\tau]$. *Let $x, \widehat{x} \in X_0$ and $f, \widehat{f} \in L^p(0,\tau;E)$. If $v$ and $\widehat{v}$ are mild solutions to* $(\mathrm{CP}; x, f)^u$ *and* $(\mathrm{CP}; \widehat{x}, \widehat{f})^u$ *on* $[0,\tau]$ *respectively, then*

$$\|v(t) - \widehat{v}(t)\|_X \leqslant C(\|x - \widehat{x}\|_{X_0} + \|f - \widehat{f}\|_{L^p(0,\tau;E)})$$

*for $t \in [0,\tau]$. In particular, the* $(\mathrm{CP}; x, f)^u$ *has at most one mild solution on* $[0,\tau]$ *for each $x \in X_0$ and $f \in L^p(0,\tau;E)$.*

*Proof.* We begin by proving that assertion (ii) is true. Let $y, \widehat{y} \in X_0$ and $g, \widehat{g} \in L^p(0,\tau;E)$, and let $w$ and $\widehat{w}$ be regular mild solutions to $(\mathrm{CP}; y, g)^u$ and $(\mathrm{CP}; \widehat{y}, \widehat{g})^u$ on $[0,\tau]$ respectively. By the definition of regular mild solutions to $(\mathrm{CP}; y, g)^u$ on $[0,\tau]$, for each sufficiently small $\varepsilon > 0$ there exists a regular $\varepsilon$-approximate solution $w^\varepsilon$ to $(\mathrm{CP}; y, g)^u$ on $[0,\tau]$ such that $\|w^\varepsilon(t)\|_Y$ is uniformly bounded as $\varepsilon \downarrow 0$ and $\|w^\varepsilon(t) - w(t)\|_X \leqslant \varepsilon$ for $t$ in the domain of $w^\varepsilon$. Let $\{(t_i^\varepsilon, u_i^\varepsilon, \varepsilon_i^\varepsilon, w_i^\varepsilon, g_i^\varepsilon)\}_{i=1}^{N_\varepsilon}$ be a sequence in $[0,\tau] \times D \times E \times Y \times E$ satisfying (3.1) and (3.2), and let the regular $\varepsilon$-approximate solution $w^\varepsilon$ be defined like (3.4). Similarly, for each $\varepsilon > 0$ there exists the corresponding approximate solution $\widehat{w}^\varepsilon$ with $w$ replaced by $\widehat{w}$, satisfying the desired properties.

Now, let $\varepsilon \in (0, \lambda_0]$ and let $\{s_l\}_{l=1}^K$ be the minimal refinement of $\{t_i^\varepsilon\}_{i=1}^{N_\varepsilon}$ and $\{\widehat{t}_j^{\,\varepsilon}\}_{j=1}^{\widehat{N}^\varepsilon}$. Then by conditions (H3) and (H4) there exists a sequence $\{(v_l, z_l)\}_{l=1}^K$ in $D \times X$ such that

$$(v_l - v_{l-1})/(s_l - s_{l-1}) = A(s_{l-1}, v_{l-1})v_l$$

and

$$(z_l - z_{l-1})/(s_l - s_{l-1}) = A(s_{l-1}, v_{l-1})z_l + h(s_{l-1})$$

for $1 \leqslant l \leqslant K$, where $h \in C([0,\tau]; E)$, $v_0 = u_0$, $z_0 = \widehat{w}_0^\varepsilon$ and $\widehat{w}_0^\varepsilon$ is an element in $Y \cap X_0$ such that $\|\widehat{y} - \widehat{w}_0^\varepsilon\|_{X_0} \leqslant \varepsilon$. By Lemma 3.1 we have

$$\begin{aligned}\|w^\varepsilon(t) - \widehat{w}^\varepsilon(t)\|_X \leqslant C(&\|w_0^\varepsilon - \widehat{w}_0^\varepsilon\|_{X_0} + \|g - h\|_{L^p(0,\tau;E)} + \|\widehat{g} - h\|_{L^p(0,\tau;E)}) \\ &+ 2\delta(3\varepsilon + \rho_h(\varepsilon)) + C\widetilde{M}_Y^\varepsilon \delta(3\varepsilon)\end{aligned}$$

for $t \in [0, t_{N_\varepsilon}^\varepsilon] \cap [0, \widehat{t}_{\widehat{N}_\varepsilon}^{\,\varepsilon}]$, where

$$\widetilde{M}_Y^\varepsilon = \sup\{\|w^\varepsilon(t)\|_Y : t \in [0, t_{N_\varepsilon}^\varepsilon]\} + \sup\{\|\widehat{w}^\varepsilon(t)\|_Y : t \in [0, \widehat{t}_{\widehat{N}_\varepsilon}^{\,\varepsilon}]\}.$$

Since $\widetilde{M}_Y^\varepsilon$ is bounded as $\varepsilon \downarrow 0$, we have

$$\|w(t) - \widehat{w}(t)\|_X \leqslant C(\|y - \widehat{y}\|_{X_0} + \|g - \widehat{g}\|_{L^p(0,\tau;E)}) \quad \text{for } t \in [0,\tau].$$

Here we have taken the limit as $h \to \widehat{g}$ in $L^p(0,\tau;E)$. The desired claim follows from this inequality together with the definition of mild solutions to $(\mathrm{CP};x,f)^u$ on $[0,\tau]$.

Assertion (i) is verified in a way similar to the derivation of (ii). □

PROPOSITION 3.4. *Let $u_0 \in D_0$ and $u$ be a regular mild solution to* (QE; $u_0$) *on $[0,\tau]$. Let $x \in X_0$ and $f \in L^p(0,\tau;E)$. If $v$ is a mild solution to* $(\mathrm{CP};x,f)^u$ *on $[0,\tau]$ then we have*

$$\|v(t)-y\|_X \leqslant C(\|x-y\|_{X_0} + \|f\|_{L^p(0,\tau;E)} + \tau^{1/p} M_A \|y\|_Y)$$

*for $t \in [0,\tau]$ and $y \in X_0 \cap S^{-1}(X_0)$.*

*Proof.* It suffices to prove the proposition under the assumption that $v$ is a regular mild solution to $(\mathrm{CP};x,f)^u$ on $[0,\tau]$. Let $y \in X_0 \cap S^{-1}(X_0)$. Once the function $w(t) = y$ for $t \in [0,\tau]$ is shown to be a regular mild solution to $(\mathrm{CP};y,g)^u$ where $g(t) = -\overline{A}(t,u(t))y$ for $t \in [0,\tau]$, the desired inequality follows from (ii) of Proposition 3.3.

Since $v$ is a regular mild solution to $(\mathrm{CP};x,f)^u$ on $[0,\tau]$, there exists a sequence $\{(t_i,u_i,\varepsilon_i)\}_{i=1}^N$ in $[0,T] \times D \times E$ satisfying (3.1) such that $Su_i \in D(A(t_{i-1},u_{i-1}))$,

$$\xi_i := (Su_i - Su_{i-1})/(t_i - t_{i-1}) - (A(t_{i-1},u_{i-1}) + B(t_{i-1},u_{i-1}))Su_i \in E$$

and $\|\xi_i\|_E \leqslant \varepsilon$ for $1 \leqslant i \leqslant N$ and such that a step function defined by (3.3) is a regular $\varepsilon$-approximate solution to (QE; $u_0$) on $[0,\tau]$ whose difference from $u$ is less than or equal to $\varepsilon$. If we define $w_i = y$ for $0 \leqslant i \leqslant N$ and $g_i = -A(t_{i-1},u_{i-1})y$ for $1 \leqslant i \leqslant N$, then equation (3.2) is clearly satisfied. Since the family $\{\overline{A}(t,w) : (t,w) \in [0,T] \times \widetilde{D}\}$ is strongly continuous in $B(Y,E)$ on $[0,T]$ (see Remark 2.2(3)) and

$$\|A(s,u_{i-1}) - \overline{A}(s,u(t))\|_{Y,E} \leqslant L_A\varepsilon$$

for $s \in [0,T], t \in [t_{i-1},t_i]$ and $1 \leqslant i \leqslant N$, we see that the step function $w^\varepsilon$ defined like (3.4) is a regular $\varepsilon$-approximate solution to $(\mathrm{CP};y,g)^u$ on $[0,\tau]$, and $w^\varepsilon(t) = w(t)$ for $t \in [0,\tau]$. This means that $w$ is a regular mild solution to $(\mathrm{CP};y,g)^u$. □

*Proof of Lemma* 3.1. We use an auxiliary function $\widetilde{w} : [0,t_N] \to Y$ defined by

$$\widetilde{w}(t) = w_{i-1} + (t - t_{i-1})(w_i - w_{i-1})/(t_i - t_{i-1})$$

for $t \in [t_{i-1},t_i]$ and $i = 1,2,\dots,N$. By (1.1) and condition (H2), we have

$$h_j := (\widetilde{w}(s_j) - \widetilde{w}(s_{j-1}))/(s_j - s_{j-1}) - A(s_{j-1},v_{j-1})\widetilde{w}(s_j) \in E$$

for $j = 1,2,\dots,K$. We use the stability condition (H4) to find the estimate

$$\|z_j - \widetilde{w}(s_j)\|_X^p \leqslant M\left(\|z_0 - w_0\|_{X_0}^p + \sum_{l=1}^{j}(s_l - s_{l-1})\|h(s_{l-1}) - h_l\|_E^p\right) \tag{3.8}$$

for $j = 1,2,\dots,K$.

To estimate $\sum_{l=1}^{j}(s_l - s_{l-1})\|h(s_{l-1}) - h_l\|_E^p$, let $r \in (s_{l-1},s_l)$ for some $l \in \{1,2,\dots,K\}$. Since $P$ is a refinement of $\Delta$, there exists $k \in \{1,2,\dots,N\}$

such that $t_{k-1} \leqslant s_{l-1} < r < s_l \leqslant t_k$. By the definition of $\widetilde{w}$, we have

$$\begin{aligned} & A(s_{l-1}, u_{k-1})(\widetilde{w}(s_l) - w_{k-1}) \\ & \qquad = ((s_l - t_{k-1})/(t_k - t_{k-1}))A(s_{l-1}, u_{k-1})(w_k - w_{k-1}). \end{aligned} \tag{3.9}$$

Since

$$h(s_{l-1}) - h_l = (h(s_{l-1}) - g_k) + (A(s_{l-1}, v_{l-1})\widetilde{w}(s_l) - A(t_{k-1}, u_{k-1})w_k)$$

and $A(s_{l-1}, v_{l-1})\widetilde{w}(s_l) - A(t_{k-1}, u_{k-1})w_k$ is written as

$$\begin{aligned} & (A(s_{l-1}, v_{l-1}) - A(s_{l-1}, u_{k-1}))\widetilde{w}(s_l) + A(s_{l-1}, u_{k-1})(\widetilde{w}(s_l) - w_{k-1}) \\ & \quad + (A(s_{l-1}, u_{k-1}) - A(t_{k-1}, u_{k-1}))w_{k-1} + A(t_{k-1}, u_{k-1})(w_{k-1} - w_k), \end{aligned}$$

we have, by (3.9),

$$\begin{aligned} \|h(s_{l-1}) - h_l\|_E & \leqslant \|g(r) - g_k\|_E + \rho_h(|\Delta|) + \|h(r) - g(r)\|_E \\ & \qquad + L_A\widetilde{M}_Y\|v(r) - u(r)\|_X + 2M_A\widetilde{\gamma} + \widetilde{\beta}. \end{aligned}$$

Substituting this estimate into (3.8) we find that

$$\begin{aligned} \|z_j - \widetilde{w}(s_j)\|_X^p & \leqslant C(\|w_0 - z_0\|_{X_0}^p + \|g - h\|_{L^p(0,\tau;E)}^p) + \delta(\widetilde{\alpha} + \widetilde{\beta} + \widetilde{\gamma} + \rho_h(|\Delta|)) \\ & \qquad + C\widetilde{M}_Y^p\int_0^{s_j}\|u(r) - v(r)\|_X^p\, dr \end{aligned} \tag{3.10}$$

for $j = 1, 2, \ldots, K$.

Now, let $t \in [0, t_N)$. Since $P$ is a refinement of $\Delta$, there exist integers $i$ and $j$ such that $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant K$ and $t_{i-1} \leqslant s_{j-1} \leqslant t < s_j \leqslant t_i$, and then $w(t) = w_{i-1}$ and $z(t) = z_{j-1}$. By the definition of $\widetilde{w}$ we have

$$\widetilde{w}(s_{j-1}) - w_{i-1} = ((s_{j-1} - t_{i-1})/(t_i - t_{i-1}))(w_i - w_{i-1});$$

hence $\|\widetilde{w}(s_{j-1}) - w_{i-1}\|_X \leqslant c_X\widetilde{\gamma}$, where $c_X > 0$ is a constant such that $\|u\|_X \leqslant c_X\|u\|_Y$ for $u \in Y$. Combining this inequality and (3.10) we find that

$$\begin{aligned} \|w(t) - z(t)\|_X^p & \leqslant C(\|w_0 - z_0\|_{X_0}^p + \|g - h\|_{L^p(0,\tau;E)}^p) \\ & \qquad + \delta(\widetilde{\alpha} + \widetilde{\beta} + \widetilde{\gamma} + \rho_h(|\Delta|)) + C\widetilde{M}_Y^p\int_0^t\|u(r) - v(r)\|_X^p\, dr. \end{aligned} \tag{3.11}$$

Here we have used the fact that $s_{j-1} \leqslant t$. The inequality (3.10) with $j = K$ implies that (3.11) is also valid for $t = t_N$. If $w_0 = u_0$, $z_0 = v_0$, $g = h = 0$ and $\varepsilon_i = g_i$ for $1 \leqslant i \leqslant N$, then we see that $w(t) = u(t)$ and $z(t) = v(t)$ for $t \in [0, t_N]$ and that $\widetilde{\alpha} \leqslant T^{1/p}\alpha$ and $\widetilde{M}_Y = \max_{0 \leqslant i \leqslant N}\|u_i\|_Y \leqslant M_W$, where $M_W = \sup\{\|u\|_Y : u \in W\} < \infty$ because $W$ is bounded in $Y$; hence (3.11) implies (3.5) by Gronwall's inequality. The desired inequality (3.6) is obtained by substituting (3.5) into (3.11). □

## 4. *Convergence of regular approximate solutions*

The convergence of regular approximate solutions for (QE) is given as follows.

PROPOSITION 4.1. *Let $u_0 \in D_0$. Assume that for each sufficiently small $\varepsilon > 0$ there exists a regular $\varepsilon$-approximate solution $u^\varepsilon$ to* (QE; $u_0$) *on $[0, \tau]$. Then there* *exists a regular mild solution* $u$ *to* (QE; $u_0$) *on* $[0,\tau]$ *such that*

$$\lim_{\varepsilon\downarrow 0}(\sup\{\|u^\varepsilon(t)-u(t)\|_X : t\in[0,t^\varepsilon_{N_\varepsilon}]\})=0,$$

*where* $[0,t^\varepsilon_{N_\varepsilon}]$ *is the domain of* $u^\varepsilon$.

*Proof.* Let $u_0\in D_0$. By the definition of regular $\varepsilon$-approximate solutions to (QE; $u_0$), for each sufficiently small $\varepsilon>0$ there exists a sequence $\{(t^\varepsilon_i,u^\varepsilon_i,\varepsilon^\varepsilon_i)\}_{i=1}^{N_\varepsilon}$ in $[0,\tau]\times D\times E$, satisfying (3.1), such that a step function defined like (3.3) is the regular $\varepsilon$-approximate solution $u^\varepsilon$.

To prove the convergence of $\{u^\varepsilon\}$ in $X$, let $\lambda,\mu\in(0,\lambda_0]$ be sufficiently small. We want to estimate the difference between $u^\lambda$ and $u^\mu$ in $X$. For this purpose, let $\{s_l\}_{l=1}^K$ be the minimal refinement of $\{t^\lambda_i\}_{i=1}^{N_\lambda}$ and $\{t^\mu_j\}_{j=1}^{N_\mu}$. Then by condition (H3) there exists a sequence $\{v_l\}_{l=1}^K$ in $D$ such that $(v_l-v_{l-1})/(s_l-s_{l-1})=A(s_{l-1},v_{l-1})v_l$ for $1\leqslant l\leqslant K$, where $v_0=u_0$. By Lemma 3.1 we have

$$\|u^\lambda(t)-u^\mu(t)\|_X\leqslant C(\delta(3\lambda)+\delta(3\mu))$$

for $t\in[0,t^\lambda_{N_\lambda}]\cap[0,t^\mu_{N_\mu}]$, which shows that the sequence $\{u^\varepsilon\}$ converges in $X$ uniformly on every subinterval of $[0,\tau)$. Once the limit function of the sequence $\{u^\varepsilon\}$ is uniformly continuous on $[0,\tau)$, the desired assertion follows from the principle of extension by continuity.

It remains to show that the limit function of the sequence $\{u^\varepsilon\}$ is uniformly continuous on $[0,\tau)$ in $X$. By (3.1) we have $\|u^\varepsilon_j-u^\varepsilon_i\|_E\leqslant(t^\varepsilon_j-t^\varepsilon_i)(M_AM_W+\varepsilon)$ for $1\leqslant i\leqslant j\leqslant N_\varepsilon$. Since $E$ is continuously embedded in $X$, it is shown that the limit function is Lipschitz continuous on $[0,\tau)$ in $X$. □

The following proposition establishes the convergence of regular approximate solutions to $(\mathrm{CP};x,f)^u$.

PROPOSITION 4.2. *Let* $u_0\in D_0$ *and let* $u$ *be a regular mild solution to* (QE; $u_0$) *on* $[0,\tau]$.

(i) *Let* $y\in X_0$ *and* $g\in L^p(0,\tau;E)$. *Assume that for each sufficiently small* $\varepsilon>0$ *there exists a regular* $\varepsilon$*-approximate solution* $w^\varepsilon$ *to* $(\mathrm{CP};y,g)^u$ *on* $[0,\tau]$ *such that* $\|w^\varepsilon(t)\|_Y$ *is uniformly bounded as* $\varepsilon\downarrow 0$. *Then there exists a regular mild solution to* $(\mathrm{CP};y,g)^u$ *on* $[0,\tau]$.

(ii) *Assume that there exists a regular mild solution to* $(\mathrm{CP};y,g)^u$ *on* $[0,\tau]$ *for each* $y$ *in a dense subset of* $X_0$ *and* $g$ *in a dense subset of* $L^p(0,\tau;E)$. *Then there exists a mild solution to* $(\mathrm{CP};x,f)^u$ *on* $[0,\tau]$ *for every* $x\in X_0$ *and* $f\in L^p(0,\tau;E)$.

*Proof.* Similarly to the proof of Proposition 4.1, the desired assertion (i) follows from Lemma 3.1. (See also the proof of (ii) of Proposition 3.3.) To prove (ii), let $x\in X_0$ and $f\in L^p(0,\tau;E)$. Then there exist a sequence $\{y_n\}$ in $X_0$ and a sequence $\{g_n\}$ in $L^p(0,\tau;E)$ such that $\lim_{n\to\infty}\|y_n-x\|_{X_0}=\lim_{n\to\infty}\|g_n-f\|_{L^p(0,\tau;E)}=0$ and such that the $(\mathrm{CP};y_n,g_n)^u$ has a regular mild solution $w_n$ for each $n\geqslant 1$. Since a regular mild solution to $(\mathrm{CP};y_n,g_n)^u$ on $[0,\tau]$ is clearly a mild solution to $(\mathrm{CP};y_n,g_n)^u$ on $[0,\tau]$, we deduce from (ii) of Proposition 3.3 that the sequence $\{w_n\}$ converges in $X$ uniformly on $[0,\tau]$. The limit function is the desired mild solution to $(\mathrm{CP};x,f)^u$ on $[0,\tau]$. □

LEMMA 4.3. *Let $0 < \tau \leqslant T$. Let $u_0 \in D_0$ and $u$ be a regular mild solution to* (QE; $u_0$) *on* $[0,\tau]$. *Assume that there exists $v \in C([0,\tau];X)$ such that $v(t) \in S(W)$ for $t \in [0,\tau]$ and $v$ is a mild solution to* (CP; $Su_0, f^v)^u$ *on* $[0,\tau]$, *where $f^v(t) = B(t, S^{-1}v(t))v(t)$ for $t \in [0,\tau]$. Then the following assertions hold:*

(i) *there exists a sequence $\{u^\varepsilon\}$ of regular $\varepsilon$-approximate solutions to* (QE; $u_0$) *on $[0,\tau]$ such that it converges to $u$ in $Y$ uniformly on every compact subinterval of $[0,\tau)$, as $\varepsilon \downarrow 0$.*

(ii) *$u$ is a classical solution to* (QE; $u_0$) *on $[0,\tau]$, and $Su(t) = v(t)$ for $t \in [0,\tau]$.*

*Proof.* Let $\eta > 0$ be fixed arbitrarily. By the definition of mild solutions to (CP; $Su_0, f^v)^u$ on $[0,\tau]$, there exists a regular mild solution $w$ to (CP; $y, g)^u$ on $[0,\tau]$ such that $\|y - Su_0\|_{X_0} \leqslant \eta$, $\|f^v - g\|_{L^p(0,\tau;E)} \leqslant \eta$ and $\sup_{t\in[0,\tau]} \|w(t) - v(t)\|_X \leqslant \eta$. By the definition of regular mild solutions to (CP; $y, g)^u$ on $[0,\tau]$, for each sufficiently small $\varepsilon > 0$ there exists a regular $\varepsilon$-approximate solution $w^\varepsilon$ to (CP; $y, g)^u$ on $[0,\tau]$ such that $\|w^\varepsilon(t)\|_Y$ is uniformly bounded as $\varepsilon \downarrow 0$ and $\|w^\varepsilon(t) - w(t)\|_X \leqslant \varepsilon$ for $t$ in the domain of $w^\varepsilon$. By the definition of regular $\varepsilon$-approximate solutions to (CP; $y, g)^u$ on $[0,\tau]$ there exist $w_0^\varepsilon \in Y \cap X_0$ and a sequence $\{(t_i^\varepsilon, u_i^\varepsilon, \varepsilon_i^\varepsilon, w_i^\varepsilon, g_i^\varepsilon)\}_{i=1}^{N_\varepsilon}$ in $[0,\tau] \times D \times E \times Y \times E$ satisfying (3.1) and (3.2) such that

$$Su_i^\varepsilon \in D(A(t_{i-1}^\varepsilon, u_{i-1}^\varepsilon)),$$
$$\xi_i^\varepsilon := (Su_i^\varepsilon - Su_{i-1}^\varepsilon)/(t_i^\varepsilon - t_{i-1}^\varepsilon) - (A(t_{i-1}^\varepsilon, u_{i-1}^\varepsilon) + B(t_{i-1}^\varepsilon, u_{i-1}^\varepsilon))Su_i^\varepsilon \in E$$

and $\|\xi_i^\varepsilon\|_E \leqslant \varepsilon$ for $1 \leqslant i \leqslant N_\varepsilon$, such that a step function $u^\varepsilon$ defined by (3.3) is a regular $\varepsilon$-approximate solution to (QE; $u_0$) on $[0,\tau]$ whose difference from $u$ is less than or equal to $\varepsilon$ and such that $\|w_0^\varepsilon - y\|_{X_0} \leqslant \varepsilon$, $\widetilde{\alpha} \leqslant \varepsilon$, $\widetilde{\beta} \leqslant \varepsilon$ and $\widetilde{\gamma} \leqslant \varepsilon$. Notice that $t_i^\varepsilon - t_{i-1}^\varepsilon \leqslant \varepsilon$ for $1 \leqslant i \leqslant N_\varepsilon$, $\alpha \leqslant \varepsilon$, $\beta \leqslant \varepsilon$ and $\gamma \leqslant \varepsilon$ because $u^\varepsilon$ is a regular $\varepsilon$-approximate solution to (QE; $u_0$) on $[0,\tau]$. Here $\alpha$, $\beta$, $\gamma$, $\widetilde{\alpha}$, $\widetilde{\beta}$ and $\widetilde{\gamma}$ are the symbols defined as in Lemma 3.1.

We apply condition (H4) to the sequence $\{Su_i^\varepsilon - w_i^\varepsilon\}_{i=0}^{N_\varepsilon}$, so that

$$\|Su^\varepsilon(t) - w^\varepsilon(t)\|_X^p \leqslant M\left(\|Su_0 - w_0^\varepsilon\|_{X_0}^p + \sum_{k=1}^{i-1}(t_k^\varepsilon - t_{k-1}^\varepsilon)\|B(t_{k-1}^\varepsilon, u_{k-1}^\varepsilon)Su_k^\varepsilon + \xi_k^\varepsilon - g_k^\varepsilon\|_E^p\right) \tag{4.1}$$

for $t \in [t_{i-1}^\varepsilon, t_i^\varepsilon)$ and $i = 1, 2, \ldots, N_\varepsilon$. We write

$$\begin{aligned} B(t_{k-1}^\varepsilon, u_{k-1}^\varepsilon)Su_k^\varepsilon - g_k^\varepsilon &= (B(t_{k-1}^\varepsilon, u_{k-1}^\varepsilon) - B(t_{k-1}^\varepsilon, S^{-1}v(r)))Su_k^\varepsilon \\ &\quad + B(t_{k-1}^\varepsilon, S^{-1}v(r))S(u_k^\varepsilon - u_{k-1}^\varepsilon) \\ &\quad + B(t_{k-1}^\varepsilon, S^{-1}v(r))(Su_{k-1}^\varepsilon - v(r)) \\ &\quad + (B(t_{k-1}^\varepsilon, S^{-1}v(r)) - B(r, S^{-1}v(r)))v(r) \\ &\quad + (f^v(r) - g(r)) + (g(r) - g_k^\varepsilon) \end{aligned}$$

for $r \in (t_{k-1}^\varepsilon, t_k^\varepsilon)$ and $k = 1, 2, \ldots, N_\varepsilon$. By condition (B) in (H6) and Remark 2.2(2) we find that

$$\begin{aligned} &\|B(t_{k-1}^\varepsilon, u_{k-1}^\varepsilon)Su_k^\varepsilon - g_k^\varepsilon\|_E \\ &\quad \leqslant L_B\|S^{-1}\|_{X,Y}\|Su^\varepsilon(r) - v(r)\|_X\|S\|_{Y,X}M_W + M_B\|S\|_{Y,X}\gamma \\ &\qquad + M_B\|Su^\varepsilon(r) - v(r)\|_X + \|(B(t_{k-1}^\varepsilon, S^{-1}v(r)) - B(r, S^{-1}v(r)))v(r)\|_E \\ &\qquad + \|f^v(r) - g(r)\|_E + \|g(r) - g_k^\varepsilon\|_E \end{aligned} \tag{4.2}$$

for $r \in (t^\varepsilon_{k-1}, t^\varepsilon_k)$ and $k = 1, 2, \ldots, N_\varepsilon$. Here we have used the fact that

$$\|Su^\varepsilon_k\|_X \leqslant \|S\|_{Y,X} M_W \quad \text{for } 1 \leqslant k \leqslant N_\varepsilon.$$

Since $v$ is continuous on $[0,\tau]$ in $X$, the set $K = \{v(r) : r \in [0,\tau]\}$ is compact in $X$. This fact, together with the strong continuity of $B$ on $[0,\tau] \times W$ in $B(X,E)$, implies that the function $(t,z) \to B(t, S^{-1}z)z$ is uniformly continuous on $[0,\tau] \times K$ in $E$, so that the fourth term on the right-hand side of (4.2) is bounded by $\delta(\varepsilon)$. Integrating the $p$th power of both sides of (4.2) over $(t^\varepsilon_{k-1}, t^\varepsilon_k)$ and summing up the resultant inequality from $k = 1$ to $k = i-1$, we have

$$\begin{aligned}&\sum_{k=1}^{i-1} (t^\varepsilon_k - t^\varepsilon_{k-1}) \|B(t^\varepsilon_{k-1}, u^\varepsilon_{k-1}) Su^\varepsilon_k - g^\varepsilon_k\|^p_E \\ &\qquad \leqslant C \int_0^{t^\varepsilon_{i-1}} \|Su^\varepsilon(r) - v(r)\|^p_X \, dr + \delta(\gamma) + \delta(\varepsilon) + \|f^v - g\|^p_{L^p(0,\tau;E)} + \widetilde{\alpha}^p\end{aligned}$$

for $i = 1, 2, \ldots, N_\varepsilon$. Combining (4.1) and the above inequality gives

$$\|Su^\varepsilon(t) - v(t)\|^p_X \leqslant \delta(\varepsilon) + \delta(\eta) + C \int_0^t \|Su^\varepsilon(r) - v(r)\|^p_X \, dr$$

for $t \in [0, t^\varepsilon_{N_\varepsilon})$. By Gronwall's inequality we have

$$\limsup_{\varepsilon \downarrow 0} \left( \sup_{t \in [0, t^\varepsilon_{N_\varepsilon})} \|Su^\varepsilon(t) - v(t)\|^p_X \right) \leqslant \exp(C\tau)\delta(\eta),$$

and the right-hand side tends to zero as $\eta \downarrow 0$. This means that the sequence $\{Su^\varepsilon(t)\}$ is uniformly convergent to $v(t)$ in $X$ on every compact subinterval of $[0,\tau)$, as $\varepsilon \downarrow 0$. Since $\|u^\varepsilon(t) - u(t)\|_X \leqslant \varepsilon$ for $t$ in the domain of $u^\varepsilon$, we have by (H1), $u(t) \in Y$ and $Su(t) = v(t)$ for $t \in [0,\tau)$. This is also valid for $t = \tau$, since $u$ and $v$ are both continuous in $X$ on $[0,\tau]$. Assertion (i) is thus shown to be true. By (3.1) we have

$$\begin{aligned}&u^\varepsilon(t) - u_0 - \int_0^{t^\varepsilon_{i-1}} A(r, u(r))u(r)\, dr \\ &\quad = \sum_{k=1}^{i-1} \int_{t^\varepsilon_{k-1}}^{t^\varepsilon_k} (A(t^\varepsilon_{k-1}, u^\varepsilon_{k-1})u^\varepsilon_k - A(r, u(r))u(r))\, dr + \sum_{k=1}^{i-1} (t^\varepsilon_k - t^\varepsilon_{k-1})\varepsilon^\varepsilon_k\end{aligned}$$

for $t \in [t^\varepsilon_{i-1}, t^\varepsilon_i)$ and $i = 1, 2, \ldots, N_\varepsilon$. A passage to the limit implies (ii), by the strong continuity of $A$ in $B(Y,E)$ and assertion (i). □

Proposition 4.4. *Let $u_0 \in D_0$ and $u$ be a regular mild solution to* (QE; $u_0$) *on* $[0,T]$. *If for each $x \in X_0$ and $f \in L^p(0,T;E)$ there exists a mild solution to* (CP; $x, f)^u$ *on* $[0,T]$ *then $u$ is a classical solution to* (QE; $u_0$) *on* $[0,T]$.

*Proof.* Since $S(W)$ is open in $X$ there exists $r > 0$ such that

$$\{x \in X : \|x - Su_0\|_X \leqslant r\} \subset S(W).$$

Let us define a space $G$ by the set of all functions $v \in C([0,\tau]; X)$ such that $v(0) = Su_0$ and $\|v(t) - Su_0\|_X \leqslant r$ for $t \in [0,\tau]$, where $\tau > 0$ is yet to be determined. Clearly, the space $G$ is a complete metric space with the usual distance. Since the function $v(t) = Su_0$ for $t \in [0,\tau]$ belongs to $G$, the space $G$ is non-empty.

Let $v \in G$, and define $f^v(t) = B(t, S^{-1}v(t))v(t)$ for $t \in [0,\tau]$. By assumption, there exists a unique mild solution $z^v$ to (CP; $Su_0, f^v)^u$ on $[0,\tau]$. The uniqueness of mild solutions follows from (ii) of Proposition 3.3. Now, consider the mapping $\Phi$ from $G$ into $C([0,\tau];X)$ defined by $\Phi v = z^v$. Then we have, by Proposition 3.4,

$$\|(\Phi v)(t) - y\|_X \leqslant C(\|Su_0 - y\|_{X_0} + \tau^{1/p}(M_B\|S\|_{Y,X}M_W + M_A\|y\|_Y)) \tag{4.3}$$

for $t \in [0,\tau]$ and $y \in X_0 \cap S^{-1}(X_0)$. Notice that $\|\cdot\|_X \leqslant c_{X_0}\|\cdot\|_{X_0}$ for some $c_{X_0} > 0$ because $X_0$ is continuously embedded in $X$. Since $Su_0 \in S(D_0) \subset X_0$ by (1.3), the above fact and (1.2) together imply that $\|Su_0 - y_0\|_X + C\|Su_0 - y_0\|_{X_0} \leqslant \frac{1}{2}r$ for some $y_0 \in X_0 \cap S^{-1}(X_0)$ $(\subset Y)$. By (4.3) with $y = y_0$ there exists $\tau > 0$ such that the operator $\Phi$ maps $G$ into itself. By (ii) of Proposition 3.3 we have

$$\|(\Phi v)(t) - (\Phi\widehat{v})(t)\|_X^p \leqslant C\int_0^\tau \|B(t, S^{-1}v(t))v(t) - B(t, S^{-1}\widehat{v}(t))\widehat{v}(t)\|_E^p\,dt,$$

and the integrand on the right-hand side is bounded by

$$(L_B\|S^{-1}\|_{X,Y}\|S\|_{Y,X}M_W + M_B)^p\|v(t) - \widehat{v}(t)\|_X^p \quad \text{for } t \in [0,\tau],$$

by virtue of condition (B) and Remark 2.2(2). This implies that the operator $\Phi$ is strictly contractive on $G$ for sufficiently small $\tau > 0$. By the Picard–Banach fixed-point theorem, the operator $\Phi$ has a fixed point $v$ in $G$; namely there exists $v \in C([0,\tau];X)$ such that $v(t) \in S(W)$ for $t \in [0,\tau]$ and $v$ is a mild solution to (CP; $Su_0, f^v)^u$ on $[0,\tau]$. Such a function $v$ is unique by Proposition 3.3 and an application of Gronwall's inequality.

Now, let us define $t_{\max}$ by the supremum of $\tau \in [0,T]$ such that there exists $v \in C([0,\tau];X)$ such that $v(t) \in S(W)$ for $t \in [0,\tau]$ and $v$ is a mild solution to (CP; $Su_0, f^v)^u$ on $[0,\tau]$. By the preceding argument we have $0 < t_{\max} \leqslant T$. By the definition of $t_{\max}$ and uniqueness, there exists $\overline{v} \in C([0,t_{\max});X)$ such that $\overline{v}(t) \in S(W)$ for $t \in [0,t_{\max})$ and such that for each $\tau \in (0,t_{\max})$, the restriction of $\overline{v}$ to $[0,\tau]$ is a mild solution to (CP; $Su_0, f^{\overline{v}})^u$ on $[0,\tau]$. By Lemma 4.3 we see that $u$ is a classical solution on $[0,t_{\max})$ and $Su(t) = \overline{v}(t)$ for $t \in [0,t_{\max})$. To prove that the limit $\lim_{t\uparrow t_{\max}} u(t)$ exists in $Y$, we have only to show that the limit $\lim_{t\uparrow t_{\max}} \overline{v}(t)$ exists in $X$. Since $\overline{v}(t) \in S(W)$ for $t \in [0,t_{\max})$ and $W$ is bounded in $Y$, we have $f^{\overline{v}} \in L^\infty(0,t_{\max};E)$. By assumption there exists a mild solution $w$ to (CP; $Su_0, f^{\overline{v}})^u$ on $[0,t_{\max}]$. By (ii) of Proposition 3.3 we have $\overline{v}(t) = w(t)$ for $t \in [0,t_{\max})$. Since $w$ is continuous in $X$ on $[0,t_{\max}]$, the above fact implies that the limit $\lim_{t\uparrow t_{\max}} \overline{v}(t) = w(t_{\max})$ exists in $X$.

Once $t_{\max} = T$ is proved, the fact shown above implies that the desired claim is true. Assume to the contrary that $t_{\max} < T$, and set $v^* = \lim_{t\uparrow t_{\max}} \overline{v}(t)$. Then we have $S^{-1}v^* = \lim_{t\uparrow t_{\max}} S^{-1}\overline{v}(t) = \lim_{t\uparrow t_{\max}} u(t)$ in $Y$. Since $u(t) \in D$ for $t \in [0,t_{\max})$ and $D$ is closed in $Y$, we have $v^* \in S(W)$. Let us define $R > 0$ such that $\{x \in X : \|x - v^*\|_X \leqslant R\} \subset S(W)$, and define another complete metric space $\widetilde{G}$ by the set of all functions $z \in C([0,\tau];X)$ such that $z(t) = \overline{v}(t)$ for $t \in [0,t_{\max})$ and $\|z(t) - v^*\|_X \leqslant R$ for $t \in [t_{\max},\tau]$ where $\tau$ is a number such that $t_{\max} < \tau \leqslant T$, which will be determined in later arguments. The space $\widetilde{G}$ is clearly non-empty.

Let $z \in \widetilde{G}$. By assumption there exists a mild solution $w^z$ to (CP; $Su_0, f^z)^u$ on $[0,\tau]$. Consider the operator $\Psi$ from $\widetilde{G}$ into $C([0,\tau];X)$ defined by $\Psi z = w^z$. Since

$f^z(t) = f^{\overline{v}}(t)$ for $t \in [0, t_{\max})$, we have $(\Psi z)(t) = w^z(t) = \overline{v}(t)$ for $t \in [0, t_{\max})$, by uniqueness. Let us define $g \in L^p(0, T; E)$ by $g(t) = f^{\overline{v}}(t)$ for $t \in [0, t_{\max})$, and 0 for $t \in [t_{\max}, T]$. By assumption there exists a mild solution $\widetilde{w}$ to (CP; $Su_0, g)^u$ on $[0, T]$. By the definition of $g$ we have $\widetilde{w}(t) = \overline{v}(t)$ for $t \in [0, t_{\max})$; hence $\widetilde{w}(t_{\max}) = v^*$. By (ii) of Proposition 3.3 we find that

$$\|(\Psi z)(t) - \widetilde{w}(t)\|_X^p \leqslant C\int_{t_{\max}}^{\tau} \|f^z(t)\|_E^p\,dt \leqslant C(\tau - t_{\max})(M_B\|S\|_{Y,X}M_W)^p$$

for $t \in [0, \tau]$. It follows that

$$\|(\Psi z)(t) - v^*\|_X \leqslant C(\tau - t_{\max})^{1/p} M_B\|S\|_{Y,X}M_W + \|\widetilde{w}(t) - v^*\|_X$$

for $t \in [t_{\max}, \tau]$. Since $\widetilde{w}$ is independent of $z \in \widetilde{G}$ and $\lim_{t\downarrow t_{\max}} \widetilde{w}(t) = \widetilde{w}(t_{\max}) = v^*$, the above inequality implies that $\|(\Psi z)(t) - v^*\|_X \leqslant R$ for $t \in [t_{\max}, \tau]$ if $\tau$ ($> t_{\max}$) is chosen close to $t_{\max}$. It is thus shown that the operator $\Psi$ maps $\widetilde{G}$ into itself for such numbers $\tau$. Let $z, \widehat{z} \in \widetilde{G}$. Since $f^z(t) = f^{\widehat{z}}(t)$ for $t \in [0, t_{\max})$, we find, by (ii) of Proposition 3.3,

$$\|(\Psi z)(t) - (\Psi\widehat{z})(t)\|_X^p \leqslant C\int_{t_{\max}}^{\tau} \|f^z(t) - f^{\widehat{z}}(t)\|_E^p\,dt$$

for $t \in [0, \tau]$. In a way similar to the derivation of strict contractivity of $\Phi$, we see that the operator $\Psi$ is strictly contractive on $\widetilde{G}$ for some $\tau$ ($> t_{\max}$). By the Picard–Banach fixed-point theorem, there exist $\tau$ ($> t_{\max}$) and $z \in C([0, \tau]; X)$ such that $z(t) \in S(W)$ for $t \in [0, \tau]$ and $z$ is a mild solution to (CP; $Su_0, f^z)^u$ on $[0, \tau]$. This is a contradiction to the definition of $t_{\max}$. □

REMARK 4.1. In the proof, the limit $u^* = \lim_{t\uparrow t_{\max}} u(t)$ was shown to exist in $Y$. If $u^*$ were in $D_0$, a routine argument would imply the extension of the classical solution $u$ on $[0, t_{\max})$. However, the fact that $u^*$ is in $D_0$ is in general false. This is a reason why we have introduced another space $\widetilde{G}$ and had a continuation argument of new type.

## 5. *Existence of classical solutions to the Cauchy problem for* (QE)

We begin with a sequence of lemmas, which are necessary for the construction of approximate solutions to the Cauchy problem for (QE).

LEMMA 5.1. *Let $\eta_0 \in (0, \varepsilon_0]$ and $i \geqslant 1$. Let $\{t_l\}_{l=0}^{i-1}$ be a sequence such that $0 = t_0 < t_1 < \ldots < t_{i-1} < t_{i-1} + \eta_0 \leqslant T$ and $t_l - t_{l-1} \leqslant \varepsilon_0$ for $1 \leqslant l \leqslant i-1$. Let $u_0 \in D_0$, $w_0 \in X_0$ and $\{f_l\}_{l=1}^{i}$ be a sequence in $E$. Let $\{(u_l, w_l)\}_{l=1}^{i-1}$ be a sequence in $D \times X$ such that*

$$(u_l - u_{l-1})/(t_l - t_{l-1}) = A(t_{l-1}, u_{l-1})u_l \quad \text{for } 1 \leqslant l \leqslant i-1,$$
$$(w_l - w_{l-1})/(t_l - t_{l-1}) = A(t_{l-1}, u_{l-1})w_l + f_l \quad \text{for } 1 \leqslant l \leqslant i-1.$$

*Then the following assertions hold:*

(i) *for each $h \in (0, \eta_0]$ there exists a unique $w^h$ in $X$ such that*

$$(w^h - w_{i-1})/h = A(t_{i-1}, u_{i-1})w^h + f_i;$$

(ii) *the function $w$ on $[0, \eta_0]$ defined by $w(0) = w_{i-1}$ and $w(h) = w^h$ for $h \in (0, \eta_0]$ is continuous on $[0, \eta_0]$ in $X$.*

*Proof.* To prove (i), let $h \in (0, \eta_0]$. Let $s_l = t_l$ for $0 \leqslant l \leqslant i-1$ and $s_i = t_{i-1} + h$. Since $0 = s_0 < s_1 < \ldots < s_{i-1} < s_i \leqslant T$, there exists a sequence $\{v_l\}_{l=0}^i$ in $X$ such that $v_0 = u_0$ and $(v_l - v_{l-1})/(s_l - s_{l-1}) = A(s_{l-1}, v_{l-1})v_l$ for $1 \leqslant l \leqslant i$, by condition (H3). Such a sequence is unique by Remark 2.2(4); hence $v_l = u_l$ for $0 \leqslant l \leqslant i-1$. For the sequences $\{v_l\}_{l=0}^i$ and $\{f_l\}_{l=1}^i$, condition (H4) implies the existence of a unique sequence $\{z_l\}_{l=0}^i$ in $X$ such that $z_0 = w_0$ and $(z_l - z_{l-1})/(s_l - s_{l-1}) = A(s_{l-1}, v_{l-1})z_l + f_l$ for $1 \leqslant l \leqslant i$. By uniqueness we have $z_l = w_l$ for $0 \leqslant l \leqslant i-1$. This means that $z_i$ is a desired solution.

To prove (ii), let $z_0 \in X_0 \cap S^{-1}(X_0)$ and $\{g_l\}_{l=1}^i$ be a sequence in $S^{-1}(E)$ (which is a dense subspace of $E$ by (H1) and (1.1) with the inclusions continuous and dense). Then by an argument similar to the derivation of (i), there exist a sequence $\{z_l\}_{l=1}^{i-1}$ in $X$ and a function $z$ on $[0, \eta_0]$ such that $z(0) = z_{i-1}$, $(z_l - z_{l-1})/(t_l - t_{l-1}) = A(t_{l-1}, u_{l-1})z_l + g_l$ for $1 \leqslant l \leqslant i-1$, and

$$(z(h) - z_{i-1})/h = A(t_{i-1}, u_{i-1})z(h) + g_i \tag{5.1}$$

for $h \in (0, \eta_0]$. By Proposition 2.3 we notice that $z$ is bounded in $Y$ on $[0, \eta_0]$. In view of this fact, the continuity of $z$ in $X$ at $h = 0$ follows from (5.1). Now, let $h_0 \in (0, \eta_0]$ and $h \in [0, \eta_0]$. Since

$$z(h) - z(h_0) = h_0 A(t_{i-1}, u_{i-1})(z(h) - z(h_0)) + (h - h_0)(A(t_{i-1}, u_{i-1})z(h) + g_i),$$

the sequence $\{v_l\}_{l=0}^i$ defined by $v_l = 0$ for $0 \leqslant l \leqslant i-1$ and $v_i = z(h) - z(h_0)$ is a solution of the difference equation $(v_l - v_{l-1})/(t_l - t_{l-1}) = A(t_{l-1}, u_{l-1})v_l + \xi_l$ for $1 \leqslant l \leqslant i$, where $t_i = t_{i-1} + h_0$, $\xi_l = 0$ for $1 \leqslant l \leqslant i-1$ and

$$\xi_i = (h - h_0)(A(t_{i-1}, u_{i-1})z(h) + g_i)/h_0.$$

Since $\xi_i \in E$ by condition (H2), we have by condition (H4),

$$\begin{aligned}\|z(h) - z(h_0)\|_X^p &\leqslant Mh_0 \|(h - h_0)(A(t_{i-1}, u_{i-1})z(h) + g_i)/h_0\|_E^p \\ &\leqslant Mh_0 |h - h_0|^p (M_A \|z(h)\|_Y + \|g_i\|_E)^p (1/h_0)^p,\end{aligned}$$

which tends to zero as $h \to h_0$. The continuity of $z$ in $X$ on $[0, \eta_0]$ is thus proved.

We use condition (H4) again to find that

$$\|w(h) - z(h)\|_X^p \leqslant M\bigg(\|w_0 - z_0\|_{X_0}^p + \sum_{l=1}^{i-1}(t_l - t_{l-1})\|f_l - g_l\|_E^p + h\|f_i - g_i\|_E^p\bigg)$$

for $h \in [0, \eta_0]$. Since $X_0 \cap S^{-1}(X_0)$ and $S^{-1}(E)$ are dense in $X_0$ and $E$ respectively, the above inequality, together with the continuity of $z$ in $X$ on $[0, \eta_0]$, implies that assertion (ii) is true. □

LEMMA 5.2. *Let $\eta_0 \in (0, \varepsilon_0]$ and $i \geqslant 1$. Let $\{t_l\}_{l=0}^{i-1}$, $\{(u_l, w_l)\}_{l=0}^{i-1}$ and $\{f_l\}_{l=1}^i$ be the sequences in Lemma* 5.1, *and $w$ the function defined in* (ii) *of Lemma* 5.1. *If $w_0 \in S^{-1}(X_0)$ and $f_l \in S^{-1}(E)$ for $1 \leqslant l \leqslant i$, then $w$ is continuous in $Y$ on $[0, \eta_0]$.*

*Proof.* By Proposition 2.3, we notice that $w_l \in Y$ for $1 \leqslant l \leqslant i-1$, $w(h) \in Y$ for $h \in (0, \eta_0]$, $(Sw_l - Sw_{l-1})/(t_l - t_{l-1}) = (A(t_{l-1}, u_{l-1}) + B(t_{l-1}, u_{l-1}))Sw_l + Sf_l$ for $1 \leqslant l \leqslant i-1$, and $(Sw(h) - Sw_{i-1})/h = (A(t_{i-1}, u_{i-1}) + B(t_{i-1}, u_{i-1}))Sw(h) + Sf_i$ for $h \in (0, \eta_0]$.

Let $h_0 \in [0, \eta_0]$. Let $z_0 = Sw_0 \in X_0$, $g_l = B(t_{l-1}, u_{l-1})Sw_l + Sf_l$ for $1 \leqslant l \leqslant i-1$ and $g_i = B(t_{i-1}, u_{i-1})Sw(h_0) + Sf_i$. Since $g_l \in E$ for $1 \leqslant l \leqslant i$, there exists a sequence $\{z_l\}_{l=1}^{i-1}$ in $X$ such that $(z_l - z_{l-1})/(t_l - t_{l-1}) = A(t_{l-1}, u_{l-1})z_l + g_l$ for $1 \leqslant l \leqslant i-1$, by condition (H4). By Lemma 5.1, there exists a continuous function $z$ in $X$ on $[0, \eta_0]$ such that $z(0) = z_{i-1}$ and $(z(h) - z_{i-1})/h = A(t_{i-1}, u_{i-1})z(h) + g_i$ for $h \in (0, \eta_0]$. By condition (H4) we have $Sw_l = z_l$ for $1 \leqslant l \leqslant i-1$, $Sw(h_0) = z(h_0)$ and $\|Sw(h) - z(h)\|_X^p \leqslant Mh\|B(t_{i-1}, u_{i-1})(Sw(h) - Sw(h_0))\|_E^p$, which gives

$$\|S(w(h) - w(h_0))\|_X \leqslant (1 - (Mh)^{1/p}M_B)^{-1}\|z(h) - z(h_0)\|_X$$

for $h \in [0, \eta_0]$. Since $z$ is continuous in $X$ on $[0, \eta_0]$, the desired claim is shown by the choice of $\varepsilon_0$. □

LEMMA 5.3. *Let $\{t_i\}_{i=1}^\infty$ be a sequence such that $0 = t_0 < t_1 < \ldots < t_i < \ldots < T$ and $t_i - t_{i-1} \leqslant \varepsilon_0$ for $i = 1, 2, \ldots$ . Let $u_0 \in D_0$, $w_0 \in X_0$ and $\{f_i\}_{i=1}^\infty$ be a sequence in $E$. Let $\{(u_i, w_i)\}_{i=1}^\infty$ be a sequence in $D \times X$ such that*

$$(u_i - u_{i-1})/(t_i - t_{i-1}) = A(t_{i-1}, u_{i-1})u_i \quad \text{for } i = 1, 2, \ldots,$$
$$(w_i - w_{i-1})/(t_i - t_{i-1}) = A(t_{i-1}, u_{i-1})w_i + f_i \quad \text{for } i = 1, 2, \ldots .$$

*Let $\{\gamma_i\}_{i=1}^\infty$ be a null sequence such that $0 < \gamma_i \leqslant \varepsilon_0$ and $t_{i-1} + \gamma_i \leqslant T$ for $i = 1, 2, \ldots$ . Let $\widetilde{w}_0 = w_0$ and $\{\widetilde{w}_i\}_{i=1}^\infty$ be a sequence such that*

$$(\widetilde{w}_i - w_{i-1})/\gamma_i = A(t_{i-1}, u_{i-1})\widetilde{w}_i + f_i$$

*for $i = 1, 2, \ldots$ . If the sequence $\{f_i\}_{i=1}^\infty$ converges in $E$ as $i \to \infty$, then the sequence $\{\widetilde{w}_i\}_{i=1}^\infty$ converges in $X$ as $i \to \infty$.*

*Proof.* Let $\varepsilon > 0$ and $z_0 \in X_0 \cap S^{-1}(X_0)$. Since $\{f_i\}_{i=1}^\infty$ is a convergent sequence in $E$ and $S^{-1}(E)$ is dense in $E$, there exists a sequence $\{g_i\}_{i=1}^\infty$ in $S^{-1}(E)$ such that $\|f_i - g_i\|_E \leqslant \varepsilon$ for $i \geqslant 1$ and $\sup_{i\geqslant 1}(\|g_i\|_X + \|Sg_i\|_E) < \infty$. Indeed, the set $K = \{f_1, f_2, \ldots\}$ is relatively compact in $E$ and $E = \bigcup_{e\in S^{-1}(E)} B_\varepsilon(e)$ where $B_\varepsilon(e)$ is an open ball in $E$ with center $e$ and radius $\varepsilon$. Hence $K \subset \bigcup_{k=1}^N B_\varepsilon(e_k)$ for some sequence $\{e_k\}_{k=1}^N$ in $S^{-1}(E)$. This implies that to each $i \geqslant 1$ there corresponds $k(i) \in \{1, 2, \ldots, N\}$ such that $\|f_i - e_{k(i)}\|_E < \varepsilon$. The desired sequence is obtained by defining $g_i = e_{k(i)}$ for $i \geqslant 1$.

By condition (H4) there exists a sequence $\{z_i\}_{i=1}^\infty$ such that

$$(z_i - z_{i-1})/(t_i - t_{i-1}) = A(t_{i-1}, u_{i-1})z_i + g_i \quad \text{for } i = 1, 2, \ldots .$$

Assertion (i) of Lemma 5.1 enables us to define a sequence $\{\widetilde{z}_i\}_{i=1}^\infty$ by $(\widetilde{z}_i - z_{i-1})/\gamma_i = A(t_{i-1}, u_{i-1})\widetilde{z}_i + g_i$ for $i = 1, 2, \ldots$ . Since

$$\sup_{i\geqslant 1}(\|g_i\|_X + \|Sg_i\|_E) < \infty,$$

we see by Proposition 2.3 that the sequence $\{z_i\}_{i=1}^\infty$ is bounded in $Y$ as $i \to \infty$, and so is the sequence $\{\widetilde{z}_i\}_{i=1}^\infty$. It follows that

$$\begin{aligned}\|\widetilde{z}_j - \widetilde{z}_i\|_X &\leqslant \|\widetilde{z}_j - z_{j-1}\|_X + \|z_{j-1} - z_{i-1}\|_X + \|z_{i-1} - \widetilde{z}_i\|_X \\ &\leqslant \widetilde{C}((\gamma_j + \gamma_i) + (t_{j-1} - t_{i-1})) \quad \text{for } j \geqslant i \geqslant 1,\end{aligned}$$

where $\widetilde{C}$ is a positive constant. This implies that the sequence $\{\widetilde{z}_i\}_{i=1}^\infty$ is a Cauchy sequence in $X$ because $\{t_i\}$ is a convergent sequence and $\lim_{i\to\infty}\gamma_i = 0$. By condition (H4) we have $\|\widetilde{w}_i - \widetilde{z}_i\|_X^p \leqslant M(\|w_0 - z_0\|_{X_0}^p + T\varepsilon^p)$ for $i \geqslant 1$, by the choice of the sequence $\{g_i\}$. Since $\{\widetilde{z}_i\}$ converges in $X$ as $i \to \infty$, the desired claim is shown by a density argument. □

LEMMA 5.4. *Let* $\{t_i\}_{i=0}^{\infty}$, $\{(u_i, w_i)\}_{i=0}^{\infty}$, $\{f_i\}_{i=1}^{\infty}$, $\{\gamma_i\}_{i=1}^{\infty}$ *and* $\{\widetilde{w}_i\}_{i=1}^{\infty}$ *be the sequences in Lemma* 5.3. *If* $w_0 \in S^{-1}(X_0)$, $f_i \in S^{-1}(E)$ *for* $i = 1, 2, \ldots$, *and* $\|Sf_i\|_E$ *is bounded as* $i \to \infty$, *then the sequence* $\{\widetilde{w}_i\}_{i=1}^{\infty}$ *is convergent in* $Y$ *as* $i \to \infty$.

*Proof.* By Proposition 2.3 we first notice that $w_i \in Y$ and $\widetilde{w}_i \in Y$ for $i \geqslant 1$. For each $k \geqslant 1$, let $\{v_l^{(k)}\}_{l=1}^{\infty}$ be a sequence in $X$ such that $v_0^{(k)} = Sw_0$ and

$$(v_l^{(k)} - v_{l-1}^{(k)})/(t_l - t_{l-1}) = A(t_{l-1}, u_{l-1})v_l^{(k)} + g_l^{(k)} \quad \text{for } l = 1, 2, \ldots,$$

where $g_l^{(k)} = B(t_{l-1}, u_{l-1})Sw_l + Sf_l \in E$ for $1 \leqslant l \leqslant k$ and $g_l^{(k)} = 0$ for $l \geqslant k+1$. By (i) of Lemma 5.1 we see that for each $k \geqslant 1$, there exists a sequence $\{\widetilde{v}_i^{(k)}\}_{i=0}^{\infty}$ in $X$ such that $\widetilde{v}_0^{(k)} = Sw_0$ and $(\widetilde{v}_i^{(k)} - v_{i-1}^{(k)})/\gamma_i = A(t_{i-1}, u_{i-1})\widetilde{v}_i^{(k)} + g_i^{(k)}$ for $i = 1, 2, \ldots$. Since $\{g_i^{(k)}\}_{i=1}^{\infty}$ is a convergent sequence in $E$ as $i \to \infty$, we deduce from Lemma 5.3 that the sequence $\{\widetilde{v}_i^{(k)}\}_{i=1}^{\infty}$ converges in $X$ as $i \to \infty$, for each $k \geqslant 1$.

Now, let $k \geqslant 1$ and $i, j \geqslant k+1$. Then we apply condition (H4) to the sequence $\{v_0^{(k)} - Sw_0, \ldots, v_{i-1}^{(k)} - Sw_{i-1}, \widetilde{v}_i^{(k)} - S\widetilde{w}_i\}$, so that

$$\begin{aligned}&\|\widetilde{v}_i^{(k)} - S\widetilde{w}_i\|_X^p \\ &\quad \leqslant M\biggl(\sum_{l=k+1}^{i-1}(t_l - t_{l-1})\|B(t_{l-1}, u_{l-1})Sw_l + Sf_l\|_E^p + \gamma_i\|B(t_{i-1}, u_{i-1})S\widetilde{w}_i + Sf_i\|_E^p\biggr).\end{aligned}$$

Here we have used the relation (2.8). Since the sequences $\{Sw_i\}$ and $\{S\widetilde{w}_i\}$ are both bounded in $X$ as $i \to \infty$ (by (2.9)), the right-hand side is estimated by $M\widetilde{C}(t_{i-1} + \gamma_i - t_k)$ for some constant $\widetilde{C} > 0$. This fact, together with the convergence of each sequence $\{\widetilde{v}_i^{(k)}\}_{i=1}^{\infty}$ in $X$ as $i \to \infty$, implies that

$$\limsup_{i,j\to\infty} \|S(\widetilde{w}_i - \widetilde{w}_j)\|_X \leqslant 2(M\widetilde{C}(\bar{t} - t_k))^{1/p}$$

for all integers $k \geqslant 1$, where $\bar{t} = \lim_{i\to\infty} t_i$. The desired claim is proved by taking the limit as $k \to \infty$. □

The following establishes the existence of regular approximate solutions.

PROPOSITION 5.5. *Let* $0 < \tau \leqslant T$ *and* $\varepsilon \in (0, \varepsilon_0]$, *and let* $u_0 \in D_0$, $w_0 \in X_0 \cap S^{-1}(X_0)$, $g \in C([0,\tau]; Y)$ *and* $Sg \in C([0,\tau]; E)$. *Then there exists a sequence* $\{(t_i, u_i, w_i)\}_{i=1}^{N}$ *in* $[0,\tau] \times D \times Y$ *such that the following conditions are satisfied:*

(i) $0 = t_0 < t_1 < \ldots < t_N = \tau$;
(ii) $t_i - t_{i-1} \leqslant \varepsilon$ *for* $i = 1, 2, \ldots, N$;
(iii) $(u_i - u_{i-1})/(t_i - t_{i-1}) = A(t_{i-1}, u_{i-1})u_i$ *for* $i = 1, 2, \ldots, N$;
(iv) $\|u_i - u_{i-1}\|_Y \leqslant \varepsilon$ *for* $i = 1, 2, \ldots, N$;
(v) $\|(A(t, u_{i-1}) - A(t_{i-1}, u_{i-1}))u_{i-1}\|_E \leqslant \varepsilon$ *for* $t \in [t_{i-1}, t_i]$ *and* $i = 1, 2, \ldots, N$;
(vi) $(w_i - w_{i-1})/(t_i - t_{i-1}) = A(t_{i-1}, u_{i-1})w_i + g(t_{i-1})$ *for* $i = 1, 2, \ldots, N$;
(vii) $\|w_i - w_{i-1}\|_Y \leqslant \varepsilon$ *for* $i = 1, 2, \ldots, N$;
(viii) $\|(A(t, u_{i-1}) - A(t_{i-1}, u_{i-1}))w_{i-1}\|_E \leqslant \varepsilon$ *for* $t \in [t_{i-1}, t_i]$ *and* $i = 1, 2, \ldots, N$.

*Proof.* Let $k \geqslant 1$ and assume that a sequence $\{(t_i, u_i, w_i)\}_{i=1}^{k-1}$ in $[0, \tau] \times D \times Y$ is chosen such that conditions (i) through (viii) are satisfied. If $t_{k-1} = \tau$ then the proof is complete. Now, we may assume that $t_{k-1} < \tau$ and define $h_k$ by the largest number satisfying the following conditions:

$$
\begin{gathered}
0 \leqslant h_k \leqslant \varepsilon, \quad t_{k-1} + h_k \leqslant \tau, \\
\|u_k(h_k) - u_{k-1}\|_Y \leqslant \varepsilon, \quad \|w_k(h_k) - w_{k-1}\|_Y \leqslant \varepsilon, \\
\|(A(t, u_{k-1}) - A(t_{k-1}, u_{k-1}))u_{k-1}\|_E \leqslant \varepsilon \quad \text{for } t \in [t_{k-1}, t_{k-1} + h_k], \\
\|(A(t, u_{k-1}) - A(t_{k-1}, u_{k-1}))w_{k-1}\|_E \leqslant \varepsilon \quad \text{for } t \in [t_{k-1}, t_{k-1} + h_k].
\end{gathered}
$$

Here $u_k(h)$ and $w_k(h)$ are two functions on $[0, (\tau - t_{k-1}) \wedge \varepsilon_0]$ such that $u_k(0) = u_{k-1}$, $w_k(0) = w_{k-1}$,

$$(u_k(h) - u_{k-1})/h = A(t_{k-1}, u_{k-1})u_k(h)$$

for $h \in (0, (\tau - t_{k-1}) \wedge \varepsilon_0]$, and

$$(w_k(h) - w_{k-1})/h = A(t_{k-1}, u_{k-1})w_k(h) + g(t_{k-1})$$

for $h \in (0, (\tau - t_{k-1}) \wedge \varepsilon_0]$. The existence of such functions is ensured by Lemma 5.1. Since $A(t, u_{k-1})$ is strongly continuous in $B(Y, E)$ on $[0, \tau]$ and the functions $u_k(h)$ and $w_k(h)$ are both continuous in $Y$ on $[0, (\tau - t_{k-1}) \wedge \varepsilon_0]$ (by Lemma 5.2), we have $h_k > 0$. If we define $u_k = u_k(h_k)$, $w_k = w_k(h_k)$ and $t_k = t_{k-1} + h_k$, then conditions (i) through (viii) are satisfied with $i = k$.

It remains to show that there exists an integer $N \geqslant 1$ such that $t_N = \tau$. Suppose to the contrary that $t_i < \tau$ for all integers $i \geqslant 1$, and set $\bar{t} = \lim_{i\to\infty} t_i$. For each $i \geqslant 1$, let us define $\gamma_i = \bar{t} - t_{i-1}$. Clearly, $\gamma_i > h_i$ and $t_{i-1} + \gamma_i \leqslant \tau$ for $i \geqslant 1$, and $\lim_{i\to\infty} \gamma_i = 0$. Lemma 5.4 then asserts that the sequence $\{w_i(\gamma_i)\}_{i=1}^{\infty}$ converges in $Y$ as $i \to \infty$, and so does the sequence $\{w_i\}_{i=1}^{\infty}$ because $\lim_{i\to\infty} h_i = 0$. Since $\|w_i(\gamma_i) - w_{i-1}\|_E \leqslant (M_A\|w_i(\gamma_i)\|_Y + \|g(t_{i-1})\|_E)\gamma_i$ for $i \geqslant 1$ and since the right-hand side tends to zero as $i \to \infty$, we deduce that the two sequences $\{w_i(\gamma_i)\}$ and $\{w_i\}$ have the same limit in $Y$, and so do the two sequences $\{u_i(\gamma_i)\}$ and $\{u_i\}$ by an argument similar to the above one. Hence there exists an integer $i_0 \geqslant 1$ such that $\gamma_i \leqslant \varepsilon$, $\|u_i(\gamma_i) - u_{i-1}\|_Y \leqslant \varepsilon$ and $\|w_i(\gamma_i) - w_{i-1}\|_Y \leqslant \varepsilon$ for $i \geqslant i_0$. Since $\gamma_i > h_i$, we see by the definition of $h_i$ that for each $i \geqslant i_0$, either

(i) there exists $\widehat{t}_i \in [t_{i-1}, \bar{t}]$ such that $\|(A(\widehat{t}_i, u_{i-1}) - A(t_{i-1}, u_{i-1}))u_{i-1}\|_E > \varepsilon$ or

(ii) there exists $\widetilde{t}_i \in [t_{i-1}, \bar{t}]$ such that $\|(A(\widetilde{t}_i, u_{i-1}) - A(t_{i-1}, u_{i-1}))w_{i-1}\|_E > \varepsilon$.

Since $A$ is strongly continuous in $B(Y, E)$ on $[0, \tau] \times D$ and since $\lim_{i\to\infty} t_i = \bar{t}$ and the sequences $\{u_i\}$ and $\{w_i\}$ are convergent in $Y$, the left-hand side tends to zero as $i \to \infty$. This implies that $\varepsilon \leqslant 0$, which is a contradiction to the fact that $\varepsilon > 0$. □

The main result of this paper is given by the following.

THEOREM 5.6. *Let $Y \subset E \subset X$ be three real Banach spaces such that the inclusions are continuous and dense and that condition* (H1) *is satisfied. Let $D$ be a closed, bounded subset of $Y$. Let $X_0$ be another real Banach space satisfying* (1.2) *and continuously embedded in $X$, and let $D_0$ be a subset of $X_0$ satisfying* (1.3). *If a family $\{A(t, w) : (t, w) \in [0, T] \times D\}$ of closed linear operators in $X$ satisfies conditions* (H2) *through* (H6), *then the abstract Cauchy problem for the* *quasi-linear evolution equation*

$$u'(t) = A(t, u(t))u(t) \quad \text{for } t \in [0, T] \tag{QE}$$

*is well-posed in the sense of Hadamard. Namely, for each initial datum $u_0$ in the set $D_0$ there exists a unique function $u(t; u_0)$ in the class $C([0,T]; D) \cap C^1([0,T]; E)$ satisfying* (QE) *and the initial condition $u(0; u_0) = u_0$, and there exists $C > 0$ such that*

$$\|u(t; u_0) - u(t; \widehat{u}_0)\|_X \leqslant C\|u_0 - \widehat{u}_0\|_{X_0}$$

*for $t \in [0, T]$ and $u_0, \widehat{u}_0 \in D_0$.*

*Proof.* Let $u_0 \in D_0$. By Proposition 5.5 with $\tau = T$, $w_0 = 0$ and $g = 0$, we see that for each $\varepsilon \in (0, \varepsilon_0]$, there exists a regular $\varepsilon$-approximate solution to (QE; $u_0$) on $[0, T]$. Proposition 4.1 asserts that the (QE; $u_0$) has a regular mild solution $u$ on $[0, T]$.

Let $w_0 \in X_0 \cap S^{-1}(X_0)$ and let $g \in C([0,T]; Y)$ such that $Sg \in C([0,T]; E)$. Let $\varepsilon \in (0, \varepsilon_0]$. Then we deduce from Proposition 5.5 again that there exists a sequence $\{(t_i, u_i, w_i)\}_{i=1}^N$ in $[0, T] \times D \times Y$ such that (3.1) and (3.2) are satisfied with $\varepsilon_i = 0$ and $g_i = g(t_{i-1})$ respectively, $\widetilde{\alpha} \leqslant T^{1/p}\rho_g(\varepsilon)$, $\widetilde{\beta} \leqslant \varepsilon$, $\widetilde{\gamma} \leqslant \varepsilon$, and a step function defined like (3.3) is a regular $\varepsilon$-approximate solution $u^\varepsilon$ to (QE; $u_0$) on $[0, T]$. Proposition 4.1, together with the uniqueness of regular mild solutions to (QE; $u_0$) on $[0, T]$ (by (i) of Proposition 3.3), implies that $u^\varepsilon$ converges to $u$ in $X$, uniformly on $[0, T]$. Moreover, since (3.1) is satisfied with $\varepsilon_i = 0$, we have $u_i \in Y$, $Su_i \in D(A(t_{i-1}, u_{i-1}))$ and

$$\xi_i := (Su_i - Su_{i-1})/(t_i - t_{i-1}) - (A(t_{i-1}, u_{i-1}) + B(t_{i-1}, u_{i-1}))Su_i = 0$$

for $1 \leqslant i \leqslant N$, by Proposition 2.3. It follows that (CP; $w_0, g)^u$ has a regular $\varepsilon$-approximate solution $w^\varepsilon$ on $[0, T]$. By Proposition 2.3 the sequence $\{w^\varepsilon(t)\}$ is uniformly bounded in $Y$ as $\varepsilon \downarrow 0$. We deduce from (i) of Proposition 4.2 that there exists a regular mild solution to (CP; $w_0, g)^u$ on $[0, T]$.

Since $X_0 \cap S^{-1}(X_0)$ is dense in $X_0$ and the set $\{g \in C([0,T]; Y); Sg \in C([0,T]; E)\}$ is dense in $L^p(0, T; E)$, there exists a mild solution to (CP; $x, f)^u$ on $[0, T]$ for every $x \in X_0$ and $f \in L^p(0, T; E)$, by (ii) of Proposition 4.2. We conclude from Proposition 4.4 that the (QE; $u_0$) has a classical solution on $[0, T]$. The continuous dependence of classical solutions on initial data follows from Propositions 3.2 and 3.3. □

## 6. *Relations with Kato's theory*

Let $X$ and $Y$ be two real Banach spaces such that $Y$ is continuously and densely embedded in $X$. Let $W$ be an open, bounded subset of $Y$ and $\{A(t, w) : (t, w) \in [0, T_0] \times W\}$ a family of closed linear operators in $X$. We discuss the local well-posedness of the abstract Cauchy problem for the quasi-linear evolution equation

$$\begin{cases} u'(t) = A(t, u(t))u(t) & \text{for } t \in [0, T_0], \\ u(0) = u_0, \end{cases}$$

under the assumptions described below.

(A1) There exists an isomorphism $S$ of $Y$ onto $X$.

(A2) The domain $D(A(t,w)) \supset Y$ for $(t,w) \in [0,T_0] \times W$, and $A$ is strongly continuous in $B(Y,X)$ on $[0,T_0] \times W$.

(A3) There exists $h_0 > 0$ such that $R(I - hA(t,w)) = X$ for $(t,w) \in [0,T_0] \times W$ and $h \in (0,h_0]$.

(A4) There exists a family $\{\|\cdot\|_{(t,w)} : (t,w) \in [0,T_0] \times W\}$ of norms in $X$ such that the following three conditions are satisfied:

(i) there exist $M_X \geqslant m_X > 0$ such that $m_X\|u\|_X \leqslant \|u\|_{(t,w)} \leqslant M_X\|u\|_X$ for $u \in X$ and $(t,w) \in [0,T_0] \times W$;

(ii) there exists $\omega \geqslant 0$ such that $\|u\|_{(t,w)} \leqslant \|u - hA(t,w)u\|_{(t,w)} + h\omega\|u\|_{(t,w)}$ for $(t,w) \in [0,T_0] \times W$, $u \in D(A(t,w))$ and $h \in (0,h_0]$;

(iii) there exists $L_X > 0$ such that $\|u\|_{(t+h,w_h)} \leqslant \|u\|_{(t,w)} + hL_X\|u\|_{(t,w)}$ for $(t,w) \in [0,T_0] \times W$, $u \in X$, $h \in (0,h_0]$ with $t+h \leqslant T_0$ and $w_h \in W$ with $w = w_h - hA(t,w)w_h$.

(A5) There exists $L_A > 0$ such that $\|A(t,w) - A(t,z)\|_{Y,X} \leqslant L_A\|w-z\|_X$ for $(t,w),(t,z) \in [0,T_0] \times W$.

(A6) There exists a strongly continuous family $\{B(t,w) : (t,w) \in [0,T_0] \times W\}$ in $B(X)$ such that $SA(t,w)S^{-1} = A(t,w) + B(t,w)$ for $(t,w) \in [0,T_0] \times W$. There exists $L_B > 0$ such that

$$\|B(t,w) - B(t,z)\|_X \leqslant L_B\|w-z\|_Y \quad \text{for } (t,w),(t,z) \in [0,T_0] \times W.$$

Remark 6.1. In Kato's theory [**8**, **9**] (see also [**13**]), conditions (K1) through (K3) below are assumed instead of conditions (A2) through (A4). A condition similar to condition (A4) was proposed by Hughes *et al.* [**7**], but their smoothness assumption of norms in $(t,w)$ is stronger than (iii) of condition (A4). It is not necessary for us to use another Banach space $\widetilde{X}$ in condition (K1).

(K1) There exists another Banach space $\widetilde{X}$ such that $Y \subset \widetilde{X} \subset X$ with all the inclusions continuous and dense.

(K2) For each $(t,w) \in [0,T_0] \times W$, $A(t,w)$ is the infinitesimal generator of a semigroup on $X$ of class $(C_0)$. If $v \in C([0,T_0];W) \cap \mathrm{Lip}([0,T_0];\widetilde{X})$, then there exist $M \geqslant 1$ and $\beta \geqslant 0$ depending only on $\mathrm{Lip}_{\widetilde{X}}(v)$ such that for each $t \in [0,T_0]$, the set $(\beta,\infty)$ is contained in the resolvent set $\rho(A(t,v(t)))$ of $A(t,v(t))$ and such that $\|\prod_{k=1}^{i}(I - \lambda_k A(t_k,v(t_k)))^{-1}\|_X \leqslant M\prod_{k=1}^{i}(1-\lambda_k\beta)^{-1}$ for every finite sequence $\{t_k\}_{k=1}^{i}$ with $0 \leqslant t_1 \leqslant \ldots \leqslant t_i \leqslant T_0$ and every finite sequence $\{\lambda_k\}_{k=1}^{i}$ with $\lambda_k > 0$ and $\lambda_k\beta < 1$ for $1 \leqslant k \leqslant i$.

(K3) The domain $D(A(t,w)) \supset Y$ and $A(t,w)(Y) \subset \widetilde{X}$ for $(t,w) \in [0,T_0] \times W$. The family $\{A(t,w) : (t,w) \in [0,T_0] \times W\}$ is strongly continuous in $B(Y,\widetilde{X})$. There exists $\lambda_A > 0$ such that $\|A(t,w)\|_{Y,\widetilde{X}} \leqslant \lambda_A$ for $(t,w) \in [0,T_0] \times W$.

Lemma 6.1. *Let $W_0$ be a convex set in $W$. If conditions* (K1) *through* (K3) *are assumed, then conditions* (A2) *through* (A4) *are satisfied with $W = W_0$.*

*Proof.* Condition (A2) follows from condition (K3), since $\widetilde{X}$ is continuously embedded in $X$ by condition (K1).

Let $L = \lambda_A M_W$ where $M_W = \sup\{\|w\|_Y : w \in W\} < \infty$. For each $t \in [0,T_0)$, let us define $E_t = \{v \in C([t,T_0];W_0) : \|v(s) - v(r)\|_{\widetilde{X}} \leqslant L|s-r| \quad \text{for } s,r \in [t,T_0]\}$. By condition (K2) there exist $M \geqslant 1$ and $\beta \geqslant 0$ depending only on $L$ such that

$$(\beta, \infty) \subset \rho(A(t, v(t))) \quad \text{for } t \in [0, T_0], \tag{6.1}$$

$$\Big\| \prod_{k=1}^{i} (I - \lambda_k A(t_k, v(t_k)))^{-1} \Big\|_X \leqslant M \prod_{k=1}^{i} (1 - \lambda_k \beta)^{-1} \tag{6.2}$$

for $v \in E_0$, $0 \leqslant t_1 \leqslant \ldots \leqslant t_i \leqslant T_0$ and $\lambda_k > 0$ with $\lambda_k \beta < 1$ for $1 \leqslant k \leqslant i$. For each $w \in W_0$, the function $v(s) = w$ for $s \in [0, T_0]$ is clearly an element of $E_0$, so that we have by (6.1), $(\beta, \infty) \subset \rho(A(t, w))$ for $(t, w) \in [0, T_0] \times W_0$. This implies that condition (A3) is satisfied with $h_0 > 0$ such that $h_0 \beta < 1$.

For each $(t, w) \in [0, T_0] \times W_0$, let us define a norm $\| \cdot \|_{(t,w)}$ in $X$ by

$$\|x\|_{(t,w)} = \sup \left\{ \Big\| \prod_{k=1}^{i} (1 - \lambda_k \beta)(I - \lambda_k A(t_k, v(t_k)))^{-1} x \Big\|_X \right\},$$

where the supremum is taken over all finite sequences $\{t_k\}_{k=1}^{i}$ with $t \leqslant t_1 \leqslant \ldots \leqslant t_i \leqslant T_0$, all finite sequences $\{\lambda_k\}_{k=1}^{i}$ with $\lambda_k > 0$ and $\lambda_k \beta < 1$ for $1 \leqslant k \leqslant i$, and all functions $v \in E_t$ with $v(t) = w$. By (6.2) we have $\|x\|_X \leqslant \|x\|_{(t,w)} \leqslant M \|x\|_X$ for $x \in X$ and $(t, w) \in [0, T_0] \times W_0$. Here we have used the fact that if $v \in E_t$ and $v(t) = w$ then the function $v$ extended to $[0, T_0]$ by $v(s) = w$ for $s \in [0, t]$ belongs to the set $E_0$.

To check (ii) of condition (A4), let $(t, w) \in [0, T_0] \times W_0$, $u \in D(A(t, w))$ and $h \in (0, h_0]$. Let $\{t_k\}_{k=1}^{i}$ be a finite sequence with $t \leqslant t_1 \leqslant \ldots \leqslant t_i \leqslant T_0$, $\{\lambda_k\}_{k=1}^{i}$ be a finite sequence with $\lambda_k > 0$ and $\lambda_k \beta < 1$ for $1 \leqslant k \leqslant i$, and $v \in E_t$ be such that $v(t) = w$. If we set $t_0 = t$ and $\lambda_0 = h$, then we have

$$\begin{aligned} (1 - h\beta) & \Big\| \prod_{k=1}^{i} (1 - \lambda_k \beta)(I - \lambda_k A(t_k, v(t_k)))^{-1} u \Big\|_X \\ & = \Big\| \prod_{k=0}^{i} (1 - \lambda_k \beta)(I - \lambda_k A(t_k, v(t_k)))^{-1} (u - hA(t, w)u) \Big\|_X \\ & \leqslant \|u - hA(t, w)u\|_{(t,w)}. \end{aligned}$$

This implies that condition (ii) is satisfied with $\omega = \beta$.

Finally, we shall prove that (iii) of condition (A4) is satisfied. Let $(t, w) \in [0, T_0] \times W_0$ and $h \in (0, h_0]$ with $t + h \leqslant T_0$ and $w_h \in W_0$ such that $w = w_h - hA(t, w)w_h$. Since $\|w_h - w\|_{\widetilde{X}} \leqslant h \lambda_A \|w_h\|_Y \leqslant Lh$ (by (K3)) and $W_0$ is convex, we see that a function $v \in E_{t+h}$ with $v(t + h) = w_h$ can be extended to $[t, T_0]$ so that $v \in E_t$ with $v(t) = w$, by defining $v(s) = (w_h - w)(s - t)/h + w$ for $s \in [t, t + h]$. This implies that $\|x\|_{(t+h, w_h)} \leqslant \|x\|_{(t,w)}$ for $x \in X$, by definition. □

By Lemma 6.1, the following is an improvement of the results in [**8**] and [**13**].

Theorem 6.2. *Let $Y \subset X$ be two real Banach spaces with the inclusion dense and continuous, and let $W$ be an open, bounded set in $Y$. If a family $\{A(t, w) : (t, w) \in [0, T_0] \times W\}$ of closed linear operators in $X$ satisfies conditions* (A1) *through* (A6)*, then for each $u_0 \in W$ there exist $T \in (0, T_0]$ and a unique function $u$ in the class $C([0, T]; W) \cap C^1([0, T]; X)$ such that $u'(t) = A(t, u(t))u(t)$ for $t \in [0, T]$, and $u(0) = u_0$.*

*Proof.* Let $u_0 \in W$. Since $W$ is open in $Y$, there exists $r_0 > 0$ such that

$$\{w \in Y : \|S(w - u_0)\|_X \leqslant r_0\} \subset W. \tag{6.3}$$

Choose $R > 0$ and $R_0 > 0$ such that $(M_X/m_X)R_0 < R$ and $(M_X/m_X)R < r_0$, and define $D = \{w \in Y : \|S(w - u_0)\|_X \leqslant R\}$ and $D_0 = \{w \in Y : \|S(w - u_0)\|_X \leqslant R_0\}$. Clearly, we have $D_0 \subset D$. Since $Y$ is dense in $X$, a number $T > 0$ and an element $z \in Y$ can be chosen so that

$$(M_X/m_X)\exp(2\widetilde{\omega}T)(R_0 + \|Su_0 - z\|_X + T(\widetilde{M}_A\|z\|_Y + \widetilde{M}_B\|z\|_X)) \\ + \|z - Su_0\|_X \leqslant R,$$

where

$$\widetilde{\omega} = (\omega + L_X + \widetilde{M}_B)(M_X/m_X),$$

$$\widetilde{M}_A = \sup\{\|A(t, w)\|_{Y,X} : (t, w) \in [0, T_0] \times W\}$$

and

$$\widetilde{M}_B = \sup\{\|B(t, w)\|_X : (t, w) \in [0, T_0] \times W\}.$$

We shall prove that conditions (H1) through (H6) are satisfied with $X_0 = E = X$. Since $D \subset W$, conditions (H1), (H2), (H5) and (H6) follow from (A1), (A2), (A5) and (A6) respectively.

Let $\lambda_0 \in (0, h_0]$ be a number such that $2\widetilde{\omega}\lambda_0 \leqslant 1$, but a smaller one will be chosen again in a later argument. To check condition (H3), let $x_0 \in D_0$ and $\{t_k\}_{k=1}^{i}$ be a sequence such that $0 = t_0 < t_1 < \ldots < t_i \leqslant T$ and $t_k - t_{k-1} \leqslant \lambda_0$ for $1 \leqslant k \leqslant i$. Let $1 \leqslant l \leqslant i$ and assume that there exists a sequence $\{x_k\}_{k=1}^{l-1}$ in $D$ such that (2.1) is satisfied for $1 \leqslant k \leqslant l-1$. By condition (A3), there exists $x_l \in D(A(t_{l-1}, x_{l-1}))$ such that (2.1) is satisfied for $k = l$. Such an element $x_l$ is unique by (ii) of condition (A4). We want to show that $x_l \in D$. For this purpose, we notice that $x_k \in Y$, $Sx_k \in D(A(t_{k-1}, x_{k-1}))$ and

$$Sx_{k-1} = Sx_k - (t_k - t_{k-1})(A(t_{k-1}, x_{k-1}) + B(t_{k-1}, x_{k-1}))Sx_k \tag{6.4}$$

for $1 \leqslant k \leqslant l$. Indeed, by conditions (A3) and (A4), $A(t, w) - \omega I$ is $m$-dissipative in $X$ equipped with the norm $\|\cdot\|_{(t,w)}$, for each $(t, w) \in [0, T_0] \times W$. By the Lumer–Phillips theorem and a perturbation theorem due to Phillips [**12**], the operator $A(t, w) + B(t, w)$ is the infinitesimal generator of a semigroup on $X$ of class $(C_0)$ and the type of the semigroup is smaller than $\widetilde{\omega}$. Notice that $x_{k-1} \in D \subset Y$ for $1 \leqslant k \leqslant l$ (by the hypothesis of induction). Then there exists $z_k \in X$ such that $Sx_{k-1} = z_k - (t_k - t_{k-1})(A(t_{k-1}, x_{k-1}) + B(t_{k-1}, x_{k-1}))z_k$ for $1 \leqslant k \leqslant l$. By condition (A6) we have $x_{k-1} = S^{-1}z_k - (t_k - t_{k-1})A(t_{k-1}, x_{k-1})S^{-1}z_k$; hence $x_k = S^{-1}z_k \in Y$ for $1 \leqslant k \leqslant l$, by uniqueness.

We begin by proving that $x_l \in W$. To do this, let $y \in Y$ $(\subset D(A(t_{l-1}, x_{l-1})))$ be fixed arbitrarily. By (ii) of condition (A4) with $u = Sx_l - y$, $(t, w) = (t_{l-1}, x_{l-1})$ and $h = t_l - t_{l-1}$ we use (6.4) with $k = l$ and (i) of condition (A4) to find that

$$m_X\|Sx_l - y\|_X \leqslant M_X\|Sx_{l-1} - y\|_X \\ + (t_l - t_{l-1})M_X(\widetilde{M}_A\|y\|_Y + \widetilde{M}_B\|Sx_l - y\|_X + \widetilde{M}_B\|y\|_X) \\ + (t_l - t_{l-1})\omega M_X\|Sx_l - y\|_X.$$

This, together with the hypothesis of induction that $x_{l-1} \in D$, implies that

$$\|S(x_l - u_0)\|_X \leqslant (M_X/m_X)(1-\lambda_0\widetilde{\omega})^{-1}(R + \|Su_0 - y\|_X + \lambda_0(\widetilde{M}_A\|y\|_Y + \widetilde{M}_B\|y\|_X)) + \|Su_0 - y\|_X$$

for any $y \in Y$. Since $Y$ is dense in $X$ there exists $\lambda_0 \in (0, h_0]$ such that $x_l \in W$ by (6.3) and the choice of $R$.

Now, we turn to the proof of the fact that $x_l \in D$. Notice that $Sx_k \in D(A(t_{k-1}, x_{k-1}))$, $z \in Y \subset D(A(t_{k-1}, x_{k-1}))$ and $x_k \in W$ for $1 \leqslant k \leqslant l$. By (ii) combined with (iii) of condition (A4) we have

$$\|u\|_{(t+h,w_h)} \leqslant \|u - hA(t,w)u\|_{(t,w)} + h(L_X + \omega)\|u\|_{(t,w)} \tag{6.5}$$

for $(t,w) \in [0,T_0] \times W$, $u \in D(A(t,w))$, $h \in (0,h_0]$ with $t + h \leqslant T_0$ and $w_h \in W$ with $w = w_h - hA(t,w)w_h$. This inequality is applied to the case where $u = Sx_k - z$, $(t,w) = (t_{k-1}, x_{k-1})$, $h = t_k - t_{k-1}$ and $w_h = x_k$ and then (6.4) is used, so that

$$\begin{aligned}
&\|Sx_k - z\|_{(t_k,x_k)} \\
&\quad\leqslant \|Sx_{k-1} - z\|_{(t_{k-1},x_{k-1})} + (t_k - t_{k-1})(M_X/m_X)(\omega + L_X)\|Sx_k - z\|_{(t_k,x_k)} \\
&\qquad + (t_k - t_{k-1})M_X(\widetilde{M}_A\|z\|_Y + (\widetilde{M}_B/m_X)\|Sx_k - z\|_{(t_k,x_k)} + \widetilde{M}_B\|z\|_X)
\end{aligned}$$

for $1 \leqslant k \leqslant l$. Here we have used the inequality

$$\|u\|_{(t,w)} \leqslant (M_X/m_X)\|u\|_{(s,z)} \tag{6.6}$$

for $(t,w),(s,z) \in [0,T_0] \times W$ and $u \in X$. This inequality follows from (i) of condition (A4). The above recursive estimate implies that

$$\|Sx_l - z\|_{(t_l,x_l)} \leqslant \exp(2\widetilde{\omega}T)(\|Sx_0 - z\|_{(t_0,x_0)} + TM_X(\widetilde{M}_A\|z\|_Y + \widetilde{M}_B\|z\|_X)).$$

By (i) of condition (A4) we have

$$\begin{aligned}
\|S(x_l - u_0)\|_X \leqslant (M_X/m_X)\exp(2\widetilde{\omega}T)(\|S(x_0 - u_0)\|_X + \|Su_0 - z\|_X \\
+ T(\widetilde{M}_A\|z\|_Y + \widetilde{M}_B\|z\|_X)) + \|z - Su_0\|_X \leqslant R,
\end{aligned}$$

by the choice of $T > 0$ and $z \in Y$. This means that $x_l \in D$, and so the desired claim follows by induction. Condition (H3) is thus checked. The combination of (6.5) and (6.6) gives (2.4) with $(p,\omega) = (1,\widetilde{\omega})$. Condition (H4) follows from Proposition 2.1.

Since $u_0 \in D_0$, the theorem is a direct consequence of Theorem 5.6. □

## 7. *Application to global solvability*

In the previous section, it was proved that Theorem 5.6 is applicable to the local well-posedness in the usual sense for quasi-linear equations. To show that Theorem 5.6 can be also applied to the global solvability of the Cauchy problems for quasi-linear hyperbolic equations, we study the Cauchy problem for the abstract inhomogeneous quasi-linear equation of Kirchhoff type,

$$u''(t) + \sigma(\|A^{1/2}u(t)\|^2)Au(t) + \gamma u'(t) = f(t) \quad \text{for } t \geqslant 0, \tag{7.1}$$

where $A$ is a positive self-adjoint operator in a real Hilbert space $H$ with inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$, $\sigma \in C^1([0,\infty);\mathbb{R})$ satisfies $\sigma(r) > 0$ for $r \geqslant 0$, $f \in C([0,\infty); D(A^{1/2}))$ and $\gamma > 0$.

We start with the following theorem which is a special case of Theorem 5.6.

THEOREM 7.1. *Let $Y \subset X$ be two real Banach spaces with the inclusion continuous and dense. Let $D$ be a closed, bounded set in $Y$, and $D_0 \subset D$. If a family $\{A(t,w) : (t,w) \in [0,T] \times D\}$ of closed linear operators in $X$ satisfies conditions* (H1) *through* (H3), (H5) *and* (H6) *with $X = E = X_0$, and the two conditions* (i) *and* (ii) *of Proposition* 2.1, *then for each $u_0 \in D_0$ the equation* (QE) *has a unique solution $u$ in the class $C([0,T];Y) \cap C^1([0,T];X)$ satisfying the initial condition $u(0) = u_0$. Moreover, $u(t)$ depends Lipschitz continuously in $X$ on initial data $u_0$.*

The Cauchy problem for (7.1) was studied by Yamazaki [**18**] and the following theorem was obtained. This section presents an operator-theoretic approach to the Cauchy problem for (7.1).

THEOREM 7.2. *Let $(\phi,\psi) \in D(A) \times D(A^{1/2})$, and assume that*

$$\|A\phi\| + \|A^{1/2}\psi\| + \sup_{t \geqslant 0} \|A^{1/2} f(t)\|$$

*is sufficiently small. Then there exists a unique solution $u$ in the class*

$$C([0,\infty); D(A)) \cap C^1([0,\infty); D(A^{1/2})) \cap C^2([0,\infty); H)$$

*satisfying* (7.1) *and the initial condition $u(0) = \phi$ and $u'(0) = \psi$. Moreover, $(u(t), u'(t))$ depends Lipschitz continuously on the initial data $(\phi,\psi)$ in $D(A^{1/2}) \times H$.*

To prove the theorem, we need the following lemma which will be used to obtain energy inequalities for a difference equation associated with (7.1).

LEMMA 7.3. *Let $L \geqslant 0$, $\beta > 0$ and $\lambda_0 > 0$. Let $\{h_k\}_{k=1}^i$ be a sequence such that $0 < h_k \leqslant \lambda_0$ for $1 \leqslant k \leqslant i$. If $\{a_k\}_{k=0}^i$ is a sequence of non-negative numbers such that*

$$(1+\beta h_k)a_k - a_{k-1} \leqslant L h_k a_k^{1/2} \tag{7.2}$$

*for $1 \leqslant k \leqslant i$, then we have $a_i^{1/2} \leqslant a_0^{1/2} + (2L/\beta)(1+\beta\lambda_0)^{1/2}$.*

*Proof.* Set $\alpha_0 = 1$ and $\alpha_k = \prod_{l=1}^k (1+\beta h_l)$ for $1 \leqslant k \leqslant i$. Let $1 \leqslant k \leqslant i$. Then we multiply both sides of (7.2) by $\alpha_{k-1}$, and divide both sides of the resultant inequality by $(\alpha_k a_k)^{1/2} + (\alpha_{k-1} a_{k-1})^{1/2}$. This yields

$$(\alpha_k a_k)^{1/2} - (\alpha_{k-1} a_{k-1})^{1/2} \leqslant L \alpha_{k-1}^{1/2} h_k. \tag{7.3}$$

Since the function $x \to (1+x)^{1/2}$ is concave for $x \geqslant 0$, we have

$$(1+\beta h_k)^{1/2} - 1 \geqslant \tfrac{1}{2}\beta h_k (1+\beta h_k)^{-1/2}.$$

Substituting this inequality into the right-hand side of (7.3), we see that the right-hand side of (7.3) is bounded by $(2L/\beta)(1+\beta\lambda_0)^{1/2}(\alpha_k^{1/2} - \alpha_{k-1}^{1/2})$. We add the inequalities obtained above from $k=1$ and $k=i$, so that

$$(\alpha_i a_i)^{1/2} \leqslant a_0^{1/2} + (2L/\beta)(1+\beta\lambda_0)^{1/2}\alpha_i^{1/2}.$$

This implies the desired inequality. □

*Proof of Theorem* 7.2. To prove the theorem by Theorem 7.1, let

$$X = D(A^{1/2}) \times H \times \mathbb{R}$$

and

$$\|(u,v,\xi)\|_X = (\|A^{1/2}u\|^2 + \|v\|^2 + |\xi|^2)^{1/2} \quad \text{for } (u,v,\xi) \in X.$$

Let $Y = D(A) \times D(A^{1/2}) \times \mathbb{R}$ and $\|(u,v,\xi)\|_Y = (\|Au\|^2 + \|A^{1/2}v\|^2 + |\xi|^2)^{1/2}$ for $(u,v,\xi) \in Y$, and use the operator $S$ defined by $S(u,v,\xi) = (A^{1/2}u, A^{1/2}v, \xi)$ for $(u,v,\xi) \in Y$, as an isomorphism of $Y$ onto $X$.

We introduce two sets $D$, $D_0$ by employing the functional $V$ on $D(A) \times D(A^{1/2})$ defined by $V(u,v) = 2\sigma(\|A^{1/2}u\|^2)\|Au\|^2 + \|A^{1/2}v\|^2 + \|A^{1/2}(\gamma u + v)\|^2$. Since $\gamma > 0$, we choose $R > 0$ and $\beta > 0$ such that

$$\frac{2M_{\sigma'}(R^2)R^2(1+\gamma)}{m_\sigma(R^2)} + \left(1 + \left(\frac{\gamma^2}{m_\sigma(R^2)c_A} \vee 1\right)\right)\beta \leqslant \gamma, \tag{7.4}$$

where $c_A$ is a positive constant such that $\|A^{1/2}u\|^2 \geqslant c_A\|u\|^2$ for $u \in D(A^{1/2})$ (by the positivity of $A$) and symbols $M_g$ and $m_g$ are defined by

$$M_g(R) = \sup\{|g(r)| : r \in [0,R]\} \quad \text{and} \quad m_g(R) = \inf\{|g(r)| : r \in [0,R]\}$$

for $g \in C([0,\infty);\mathbb{R})$. Assume that $C_f := \sup_{t\geqslant 0}\|A^{1/2}f(t)\|$ is finite, and choose $0 < \lambda_0 \leqslant 1$ and $R_0 > 0$ so small that

$$((2/\gamma) \vee 1)(4C_f\lambda_0 + R_0) \leqslant R. \tag{7.5}$$

Let $D = \{(u,v,\xi) \in Y : V(u,v) \leqslant R_0^2 \text{ and } \xi = 1\}$. Since $V$ is continuous on $D(A) \times D(A^{1/2})$, the set $D$ is closed in $Y$. Let $(u,v) \in D(A) \times D(A^{1/2})$ satisfy $V(u,v) \leqslant R_0^2$. Since $\|A^{1/2}u\| \leqslant \gamma^{-1}(\|A^{1/2}(\gamma u+v)\| + \|A^{1/2}v\|) \leqslant 2\gamma^{-1}R_0$ it follows that $2m_\sigma((2\gamma^{-1}R_0)^2)\|Au\|^2 + \|A^{1/2}v\|^2 \leqslant V(u,v) \leqslant R_0^2$, which implies that the set $D$ is bounded in $Y$.

Let $T > 0$ be fixed arbitrarily. We use the homogeneous reduction technique. For each $t \in [0,T]$ and $(w,z,\eta) \in D$, let us define

$$A(t,(w,z,\eta))(u,v,\xi) = (v, -\sigma(\|A^{1/2}w\|^2)Au - \gamma v + \xi f(t), 0)$$

for $(u,v,\xi) \in Y$. Notice that the problem (7.1) on $[0,T]$ with initial condition $u(0) = \phi$ and $u'(0) = \psi$ is equivalent to the quasi-linear problem

$$\begin{cases} (u(t),v(t),\xi(t))' = A(t,(u(t),v(t),\xi(t)))(u(t),v(t),\xi(t)) & \text{for } t \in [0,T], \\ (u(0),v(0),\xi(0)) = (\phi,\psi,1). \end{cases}$$

By Theorem 7.1 we only need to verify that the family

$$\{A(t,(w,z,\eta)) : (t,(w,z,\eta)) \in [0,T] \times D\}$$

satisfies the hypotheses in Theorem 7.1.

It is easily seen that for each $t \in [0,T]$ and $(w,z,\eta) \in D$, $A(t,(w,z,\eta))$ is a closed linear operator in $X$ satisfying condition (H2) with $D(A(t,(w,z,\eta))) = Y$

and condition (H5) with $L_A = 2M_{\sigma'}(R^2)R$. Here we have used the fact that $\|A^{1/2}w\| \leqslant (1/\gamma)(\|A^{1/2}(\gamma w + z)\| + \|A^{1/2}z\|) \leqslant (2/\gamma)R_0 \leqslant R$ for $(w, z, \eta) \in D$. This fact and the inequality $\|A^{1/2}z\| \leqslant R_0 \leqslant R$ for $(w, z, \eta) \in D$ will be used in later arguments. By defining

$$W = \{(u, v, \xi) \in Y : V(u, v) < 2R_0^2 \text{ and } |\xi| < 2\}$$

and

$$B(t, (w, z, \eta))(u, v, \xi) = (0, \xi(A^{1/2}f(t) - f(t)), 0)$$

for $t \in [0, T]$, $(w, z, \eta) \in W$ and $(u, v, \xi) \in X$, condition (H6) is seen to be satisfied.

To check condition (H3), we consider the set

$$D_0 := \{(u, v, \xi) \in Y : V(u, v) \leqslant r_0^2 \text{ and } \xi = 1\},$$

where $r_0 > 0$ will be determined in later arguments. If $R_0 \geqslant r_0$ then the set inclusion $D_0 \subset D$ is true. Let $(u_0, v_0, \xi_0) \in D_0$ and $\{t_k\}_{k=1}^i$ be a sequence such that $0 = t_0 < t_1 < \ldots < t_i \leqslant T$ and $t_k - t_{k-1} \leqslant \lambda_0$ for $1 \leqslant k \leqslant i$. Then there exists a sequence $\{(u_k, v_k)\}_{k=1}^i$ in $D(A^{3/2}) \times D(A)$ such that

$$(u_k - u_{k-1})/(t_k - t_{k-1}) = v_k, \tag{7.6}$$

$$(v_k - v_{k-1})/(t_k - t_{k-1}) + \sigma(\|A^{1/2}u_{k-1}\|^2)Au_k + \gamma v_k = f(t_{k-1}) \tag{7.7}$$

for $1 \leqslant k \leqslant i$. This means that the sequence $\{(u_k, v_k, 1)\}_{k=1}^i$ satisfies (2.1) with $x_k = (u_k, v_k, 1)$ for $1 \leqslant k \leqslant i$.

We need to show that $(u_k, v_k, 1) \in D$ for $0 \leqslant k \leqslant i$. We take the inner products of the equation obtained by (7.7) plus $\gamma$ times (7.6) and the equation (7.7) with $A(\gamma u_k + v_k)$ and $Av_k$ respectively. This yields

$$\begin{aligned}&\langle A^{1/2}(\gamma u_k + v_k), A^{1/2}(\gamma u_k + v_k) - A^{1/2}(\gamma u_{k-1} + v_{k-1})\rangle/(t_k - t_{k-1})\\ &\quad + \sigma(\|A^{1/2}u_{k-1}\|^2)(\gamma\|Au_k\|^2 + \langle Au_k, Av_k\rangle) = \langle A^{1/2}(\gamma u_k + v_k), A^{1/2}f(t_{k-1})\rangle\end{aligned}$$

and

$$\begin{aligned}&\langle A^{1/2}v_k, A^{1/2}v_k - A^{1/2}v_{k-1}\rangle/(t_k - t_{k-1}) + \sigma(\|A^{1/2}u_{k-1}\|^2)\langle Au_k, Av_k\rangle\\ &\qquad + \gamma\|A^{1/2}v_k\|^2 = \langle A^{1/2}v_k, A^{1/2}f(t_{k-1})\rangle\end{aligned}$$

for $1 \leqslant k \leqslant i$. Adding these equalities and substituting the equality

$$\langle Au_k, Av_k\rangle = \langle Au_k, Au_k - Au_{k-1}\rangle/(t_k - t_{k-1})$$

(by (7.6)), we see by the inequality $\|u\|^2 - \|v\|^2 \leqslant 2\langle u, u - v\rangle$ for $u, v \in H$ that

$$\begin{aligned}&2\sigma(\|A^{1/2}u_{k-1}\|^2)\|Au_k\|^2 + \|A^{1/2}v_k\|^2 + \|A^{1/2}(\gamma u_k + v_k)\|^2\\ &\quad - V(u_{k-1}, v_{k-1}) + 2\gamma(\sigma(\|A^{1/2}u_{k-1}\|^2)\|Au_k\|^2 + \|A^{1/2}v_k\|^2)(t_k - t_{k-1})\\ &\quad \leqslant 2\langle A^{1/2}(\gamma u_k + v_k) + A^{1/2}v_k, A^{1/2}f(t_{k-1})\rangle(t_k - t_{k-1})\end{aligned} \tag{7.8}$$

for $1 \leqslant k \leqslant i$.

We prove by induction that $(u_k, v_k, 1) \in D$ for $0 \leqslant k \leqslant i$. To do this, we employ another functional $\widetilde{V}$ defined by $\widetilde{V}(u, v) = \sigma(\|A^{1/2}u\|^2)\|Au\|^2 + \|A^{1/2}v\|^2$ for $(u, v) \in D(A) \times D(A^{1/2})$. Since $D_0 \subset D$, the desired claim is true for $k = 0$. Let $1 \leqslant k \leqslant i$ and assume that $(u_p, v_p, 1) \in D$ for $0 \leqslant p \leqslant k - 1$. To show that $(u_k, v_k, 1) \in D$, let $1 \leqslant l \leqslant k$. Since $(u_{l-1}, v_{l-1}, 1) \in D$ by the hypothesis of induction, we have $\|A^{1/2}u_{l-1}\| \leqslant R$ and $\|A^{1/2}v_{l-1}\| \leqslant R$. By (7.8) we have

$$\begin{aligned}&\|A^{1/2}v_l\|^2 + \|A^{1/2}(\gamma u_l + v_l)\|^2 \\ &\qquad \leqslant V(u_{l-1}, v_{l-1}) + 4(\|A^{1/2}v_l\|^2 + \|A^{1/2}(\gamma u_l + v_l)\|^2)^{1/2} C_f (t_l - t_{l-1}).\end{aligned}$$

This inequality implies that

$$\begin{aligned}(\|A^{1/2}v_l\|^2 + \|A^{1/2}(\gamma u_l + v_l)\|^2)^{1/2} &\leqslant 2C_f\lambda_0 + (R_0^2 + (2C_f\lambda_0)^2)^{1/2} \\ &\leqslant 4C_f\lambda_0 + R_0;\end{aligned}$$

hence $\|A^{1/2}v_l\| \leqslant R$ and $\|A^{1/2}u_l\| \leqslant R$ by (7.5), because

$$A^{1/2}u_l = (1/\gamma)(A^{1/2}(\gamma u_l + v_l) - A^{1/2}v_l).$$

By this fact we use (7.6) to obtain

$$|\|A^{1/2}u_l\| - \|A^{1/2}u_{l-1}\|| \leqslant \|A^{1/2}v_l\|(t_l - t_{l-1}) \leqslant R(t_l - t_{l-1}).$$

Since $m_\sigma(R^2)\|Au_l\|^2 \leqslant \widetilde{V}(u_l, v_l)$ it follows that

$$|\sigma(\|A^{1/2}u_l\|^2) - \sigma(\|A^{1/2}u_{l-1}\|^2)|\|Au_l\|^2 \leqslant (2M_{\sigma'}(R^2)R^2/m_\sigma(R^2))\widetilde{V}(u_l, v_l)(t_l - t_{l-1}).$$

By this inequality and (7.8) we have

$$\begin{aligned}&(1 + \beta(t_l - t_{l-1}))V(u_l, v_l) - V(u_{l-1}, v_{l-1}) \\ &\qquad + (2\widetilde{V}(u_l, v_l)(\gamma - 2M_{\sigma'}(R^2)R^2(1+\gamma)/m_\sigma(R^2)) - \beta V(u_l, v_l))(t_l - t_{l-1}) \\ &\quad \leqslant 4C_f V(u_l, v_l)^{1/2}(t_l - t_{l-1}).\end{aligned}$$

Since

$$\begin{aligned}V(u_l, v_l) &\leqslant 2\widetilde{V}(u_l, v_l) + 2((\gamma^2/c_A)\|Au_l\|^2 + \|A^{1/2}v_l\|^2) \\ &\leqslant 2\widetilde{V}(u_l, v_l)\big(1 + ((\gamma^2/(m_\sigma(R^2)c_A)) \vee 1)\big),\end{aligned}$$

we have, by (7.4),

$$(1 + \beta(t_l - t_{l-1}))V(u_l, v_l) - V(u_{l-1}, v_{l-1}) \leqslant 4C_f V(u_l, v_l)^{1/2}(t_l - t_{l-1})$$

for $1 \leqslant l \leqslant k$. We apply Lemma 7.3 to the sequence $\{V(u_l, v_l)\}_{l=0}^k$, so that

$$V(u_k, v_k)^{1/2} \leqslant r_0 + (8C_f/\beta)(1 + \beta\lambda_0)^{1/2}.$$

If $r_0 > 0$ and $C_f$ are chosen so that

$$r_0 + (8C_f/\beta)(1 + \beta\lambda_0)^{1/2} \leqslant R_0,$$

then we have $V(u_k, v_k) \leqslant R_0^2$, namely $(u_k, v_k, 1) \in D$. It is thus shown inductively that condition (H3) is satisfied.

Finally, we check the two conditions (i) and (ii) of Proposition 2.1. Let $t \in [0, T]$ and $(w, z, \eta) \in D$. Since $A$ is maximal monotone in $H$, it is easily shown that for each $(u_0, v_0, \xi_0) \in X$ and $h > 0$, the system

$$(u - u_0)/h = v, \tag{7.9}$$

$$(v - v_0)/h + \sigma(\|A^{1/2}w\|^2)Au + \gamma v = \xi f(t), \tag{7.10}$$

$$(\xi - \xi_0)/h = 0 \tag{7.11}$$

has a solution $(u,v,\xi) \in Y$. This means that $R(I - hA(t,(w,z,\eta))) = X$ for every $h > 0$. We check (ii) of Proposition 2.1. For each $t \in [0,T]$ and $(w,z,\eta) \in D$, let us define $\|(u,v,\xi)\|_{(t,(w,z,\eta))} = (\sigma(\|A^{1/2}w\|^2)\|A^{1/2}u\|^2 + \|v\|^2 + |\xi|^2)^{1/2}$ for $(u,v,\xi) \in X$. Condition (N1) of Proposition 2.1 is satisfied with $m_X = (m_\sigma(R^2) \wedge 1)^{1/2}$ and $M_X = (M_\sigma(R^2) \vee 1)^{1/2}$. To verify condition (N2), let $t \in [0,T]$, $(w,z,\eta) \in D$, $(u,v,\xi) \in D(A(t,(w,z,\eta)))$ and $h > 0$, and let $(w_h, z_h, \eta_h) \in D$ satisfy the equation

$$(w_h, z_h, \eta_h) - hA(t,(w,z,\eta))(w_h, z_h, \eta_h) = (w,z,\eta).$$

If we set $(u_0, v_0, \xi_0) = (u,v,\xi) - hA(t,(w,z,\eta))(u,v,\xi)$, then equations (7.9) through (7.11) are satisfied. Taking the inner products of (7.9) and (7.10) with $\sigma(\|A^{1/2}w\|^2)Au$ and $v$ respectively, and adding the two resultant equalities, we find that

$$\sigma(\|A^{1/2}w\|^2)(\|A^{1/2}u\|^2 - \|A^{1/2}u_0\|^2) + (\|v\|^2 - \|v_0\|^2) + 2\gamma h\|v\|^2 \leqslant 2\xi\langle v, f(t)\rangle h;$$

hence

$$\begin{aligned}
&\|(u,v,\xi)\|^2_{(t+h,(w_h,z_h,\eta_h))} - \|(u_0,v_0,\xi_0)\|^2_{(t,(w,z,\eta))} \\
&\quad \leqslant (\sigma(\|A^{1/2}w_h\|^2) - \sigma(\|A^{1/2}w\|^2))\|A^{1/2}u\|^2 + (C_f/c_A^{1/2})(|\xi|^2 + \|v\|^2)h.
\end{aligned}$$

Since $(w_h - w)/h = z_h$, $\|A^{1/2}w_h\| \leqslant R$ and $\|A^{1/2}z_h\| \leqslant R$, the first term on the right-hand side is bounded by $(2M_{\sigma'}(R^2)R^2/m_\sigma(R^2))\sigma(\|A^{1/2}w_h\|^2)\|A^{1/2}u\|^2 h$. Condition (N2) is thus shown to be satisfied with

$$\omega = (2M_{\sigma'}(R^2)R^2/m_\sigma(R^2)) \vee (C_f/c_A^{1/2}) \quad \text{and} \quad p = 2. \qquad \square$$

## 8. *Local well-posedness for Kirchhoff equations of degenerate type*

This section is devoted to an application of Theorem 5.6 to the local well-posedness for the degenerate abstract Kirchhoff equation

$$\begin{cases} u''(t) + \sigma(\|A^{1/2}u(t)\|^2)Au(t) = F(t,u(t)) & \text{for } t \in [0,T_0], \\ u(0) = \phi \quad \text{and} \quad u'(0) = \psi, \end{cases} \tag{8.1}$$

where $\sigma \in C^1([0,\infty);\mathbb{R})$ satisfies $\sigma(0) = 0$ and $\sigma'(r) \geqslant 0$ for $r \in [0,\infty)$, $A$ is a non-negative self-adjoint operator in a real Hilbert space $H$ and $F$ is a continuous mapping from $[0,T_0] \times D(A^{3/2})$ into $D(A^{3/2})$ satisfying the following condition.

(F) For each $R > 0$ there exists $L_F(R) > 0$ such that

$$\begin{aligned}
\|F(t,w) - F(t,z)\|_{D(A^{1/2})} &\leqslant L_F(R)\|w - z\|_{D(A^{1/2})}, \\
\|F(t,w) - F(t,z)\|_{D(A^{3/2})} &\leqslant L_F(R)\|w - z\|_{D(A^{3/2})}
\end{aligned}$$

for $t \in [0,T_0]$, $w, z \in D(A^{3/2})$ with $\|w\|_{D(A^{3/2})} \leqslant R$ and $\|z\|_{D(A^{3/2})} \leqslant R$.

Here for each non-negative integer $k$, we define $\|w\|_{D(A^{k/2})} = \max_{0 \leqslant l \leqslant k} \|A^{l/2}w\|$ for $w \in D(A^{k/2})$.

In case of $F = 0$, the Cauchy problem (8.1) was studied in [**16**], by the regularized semigroup-theoretic method. In the case where $\sigma(r) = r^\alpha$ and $F = 0$, another approach is found in a paper by Yamada [**17**]. The following is a generalization of [**16**, Theorem 4.2].

THEOREM 8.1. *If $\phi \in D(A^2)$ and $\psi \in D(A^{3/2})$ satisfy $A^{1/2}\phi = 0$ and $A^{1/2}\psi \neq 0$ respectively, then there exists $T \in (0, T_0]$ such that (8.1) has a unique solution $u$ in the class*

$$C([0,T]; D(A^{3/2})) \cap C^1([0,T]; D(A)) \cap C^2([0,T]; D(A^{1/2})).$$

*Moreover, $(u(t), u'(t))$ in $D(A^{1/2}) \times H$ depends Lipschitz continuously on initial data $(\phi, \psi)$ if the initial condition is measured in $D(A) \times D(A^{1/2})$.*

We need the following lemma to check a stability condition.

LEMMA 8.2. *Let $T > 0$, $M \geqslant 0$, $L \geqslant 0$ and $M_F \geqslant 0$. Let $j \geqslant 1$. Let $\{h_i\}_{i=1}^j$ be a sequence of positive numbers such that $4(1+T)h_i \leqslant 1$ for $1 \leqslant i \leqslant j$ and $\sum_{i=1}^j h_i \leqslant T$, $\{a_i\}_{i=0}^{j-1}$ a sequence such that $0 \leqslant a_i \leqslant M$ and $0 \leqslant a_i - a_{i-1} \leqslant Lh_i$ for $0 \leqslant i \leqslant j-1$, where $a_{-1} = a_0$ and $h_0 = 0$, and $\{F_i\}_{i=0}^{j-1}$ a sequence in $D(A^{1/2})$ such that $\|F_i\|^2 \leqslant M_F$ and $\|A^{1/2}F_i\|^2 \leqslant M_F$ for $0 \leqslant i \leqslant j-1$. Let $\{(f_i, g_i, e_i)\}_{i=1}^j$ be a sequence in $D(A^{1/2}) \times D(A^{1/2}) \times \mathbb{R}$ and $(w_0, z_0, \eta_0) \in D(A^{1/2}) \times D(A^{1/2}) \times \mathbb{R}$. Then there exists a unique sequence $\{(w_i, z_i, \eta_i)\}_{i=1}^j$ such that $w_i \in D(A^{1/2})$, $a_{i-1}w_i \in D(A)$, $z_i \in D(A^{1/2})$ and $\eta_i \in \mathbb{R}$ for $1 \leqslant i \leqslant j$, satisfying the system*

$$\begin{cases} (w_i - w_{i-1})/h_i = z_i + f_i, \\ (z_i - z_{i-1})/h_i + A(a_{i-1}w_i) = \eta_i F_{i-1} + g_i, \\ (\eta_i - \eta_{i-1})/h_i = e_i \end{cases}$$

*for $1 \leqslant i \leqslant j$. Moreover, we have*

$$\begin{aligned}
\|A^{1/2}w_l\|^2 &\leqslant (C(T)+1)M_2 + C(T)M_4 + M_F C(T)M_5, \\
\|z_l\|^2 &\leqslant (L+M)(C(T)+1)M_2 + (C(T)+1)M_3 \\
&\quad + (L+M)C(T)M_4 + (L+M+1)M_F C(T)M_5, \\
\eta_l^2 &\leqslant (C(T)+1)M_5, \\
\|w_l\|^2 &\leqslant (C(T)+1)M_1 + (L+M)C(T)M_2 + C(T)M_3 \\
&\quad + (L+M)C(T)M_4 + (L+M+1)M_F C(T)M_5
\end{aligned}$$

*for $1 \leqslant l \leqslant j$, where $C(\delta)$ is a positive function on $(0,\infty)$ such that $\lim_{\delta\downarrow 0} C(\delta) = 0$ and the symbols $M_i$ for $i = 1,2,3,4,5$ are defined by*

$$M_1 = \|w_0\|^2 + \sum_{i=1}^j h_i\|f_i\|^2, \quad M_2 = \|A^{1/2}w_0\|^2 + \sum_{i=1}^j h_i\|A^{1/2}f_i\|^2,$$

$$M_3 = \|z_0\|^2 + \sum_{i=1}^j h_i\|g_i\|^2, \quad M_4 = \|A^{1/2}z_0\|^2 + \sum_{i=1}^j h_i\|A^{1/2}g_i\|^2,$$

$$M_5 = \eta_0^2 + \sum_{i=1}^j h_i e_i^2.$$

*Proof.* We begin by proving the uniqueness of sequences. To do this, let $(\widehat{w}_0, \widehat{z}_0, \widehat{\eta}_0) = (w_0, z_0, \eta_0)$, and let $\{(\widehat{w}_i, \widehat{z}_i, \widehat{\eta}_i)\}_{i=1}^j$ be another sequence which has the same properties as $\{(w_i, z_i, \eta_i)\}_{i=1}^j$ has. Since $\eta_0 = \widehat{\eta}_0$ we have $\eta_i = \widehat{\eta}_i$ for $1 \leqslant i \leqslant j$. By this fact we have $(z_i - \widehat{z}_i) - (z_{i-1} - \widehat{z}_{i-1}) + h_i A(a_{i-1}(w_i - \widehat{w}_i)) = 0$ for

$1 \leqslant i \leqslant j$. Let $1 \leqslant i \leqslant j$ and assume that $w_k = \widehat{w}_k$ and $z_k = \widehat{z}_k$ for $0 \leqslant k \leqslant i-1$. Taking the inner product of the above equality with $z_i - \widehat{z}_i$ $(= (w_i - \widehat{w}_i)/h_i)$ we find that $\|z_i - \widehat{z}_i\|^2 + a_{i-1}\|A^{1/2}(w_i - \widehat{w}_i)\|^2 = 0$. This, together with the non-negativity of $a_{i-1}$, implies that $z_i = \widehat{z}_i$, and then $w_i = \widehat{w}_i$. The uniqueness is thus shown by induction.

To prove the existence of a sequence $\{(w_i, z_i, \eta_i)\}_{i=1}^j$ satisfying the desired properties, let $\varepsilon \in (0,1]$ and consider the system of three equations

$$(w_i^\varepsilon - w_{i-1}^\varepsilon)/h_i = z_i^\varepsilon + f_i, \tag{8.2}$$

$$(z_i^\varepsilon - z_{i-1}^\varepsilon)/h_i + a_{i-1}^\varepsilon A w_i^\varepsilon = \eta_i^\varepsilon F_{i-1} + g_i, \tag{8.3}$$

$$(\eta_i^\varepsilon - \eta_{i-1}^\varepsilon)/h_i = e_i \tag{8.4}$$

for $1 \leqslant i \leqslant j$, where $a_i^\varepsilon = a_i + \varepsilon$ for $-1 \leqslant i \leqslant j-1$, and $(w_0^\varepsilon, z_0^\varepsilon, \eta_0^\varepsilon) = (w_0, z_0, \eta_0)$. Since $a_{i-1}^\varepsilon \geqslant \varepsilon > 0$ for $1 \leqslant i \leqslant j$, the above system has a solution $\{(w_i^\varepsilon, z_i^\varepsilon, \eta_i^\varepsilon)\}_{i=1}^j$ in $D(A) \times D(A^{1/2}) \times \mathbb{R}$. We prove that

$$\|A^{1/2}w_l^\varepsilon\|^2 \leqslant (C(T)+1)M_2 + C(T)M_4 + M_F C(T) M_5, \tag{8.5}$$

$$\begin{aligned} \|z_l^\varepsilon\|^2 \leqslant{} & (L+M+\varepsilon)(C(T)+1)M_2 + (C(T)+1)M_3 \\ & + (L+M+\varepsilon)C(T)M_4 + (L+M+\varepsilon+1)M_F C(T) M_5, \end{aligned} \tag{8.6}$$

$$(\eta_l^\varepsilon)^2 \leqslant (C(T)+1)M_5, \tag{8.7}$$

$$\begin{aligned} \|w_l^\varepsilon\|^2 \leqslant{} & (C(T)+1)M_1 + (L+M+\varepsilon)C(T)M_2 + C(T)M_3 \\ & + (L+M+\varepsilon)C(T)M_4 + (L+M+\varepsilon+1)M_F C(T) M_5, \end{aligned} \tag{8.8}$$

for $1 \leqslant l \leqslant j$. By (8.4) we have

$$(\eta_i^\varepsilon)^2 - (\eta_{i-1}^\varepsilon)^2 \leqslant 2\eta_i^\varepsilon(\eta_i^\varepsilon - \eta_{i-1}^\varepsilon) = 2h_i\eta_i^\varepsilon e_i \leqslant h_i((\eta_i^\varepsilon)^2 + e_i^2) \quad \text{for } 1 \leqslant i \leqslant l,$$

which implies that $(\eta_l^\varepsilon)^2 \leqslant M_5 + \sum_{i=1}^l h_i(\eta_i^\varepsilon)^2$ for $1 \leqslant l \leqslant j$. Since $0 < h_i \leqslant \frac{1}{2}$ for $1 \leqslant i \leqslant l$, we apply Lemma 2.4 to obtain $(\eta_l^\varepsilon)^2 \leqslant \exp(2T)M_5$ for $1 \leqslant l \leqslant j$. The desired inequality (8.7) is thus proved. By (8.2) we have

$$\|w_i^\varepsilon\|^2 - \|w_{i-1}^\varepsilon\|^2 \leqslant 2\langle w_i^\varepsilon, h_i(z_i^\varepsilon + f_i)\rangle \leqslant 2h_i\|w_i^\varepsilon\|^2 + h_i\|z_i^\varepsilon\|^2 + h_i\|f_i\|^2$$

for $1 \leqslant i \leqslant l$. Once (8.6) is proved, the desired inequality (8.8) is obtained similarly to the derivation of (8.7).

Let $1 \leqslant l \leqslant j$ and set $x_i^\varepsilon = \sum_{k=i+1}^l h_k A w_k^\varepsilon$ for $0 \leqslant i \leqslant l-1$ and $x_l^\varepsilon = 0$. Let $1 \leqslant i \leqslant l$. Then we have, by (8.2) and (8.3),

$$\begin{aligned} \|A^{1/2}w_i^\varepsilon\|^2 - \|A^{1/2}w_{i-1}^\varepsilon\|^2 &\leqslant 2\langle A^{1/2}w_i^\varepsilon, A^{1/2}w_i^\varepsilon - A^{1/2}w_{i-1}^\varepsilon\rangle \\ &= 2\langle A w_i^\varepsilon, h_i(z_i^\varepsilon + f_i)\rangle \\ &= 2\langle x_{i-1}^\varepsilon - x_i^\varepsilon, z_i^\varepsilon\rangle + 2h_i\langle A^{1/2}w_i^\varepsilon, A^{1/2}f_i\rangle \end{aligned}$$

and

$$\begin{aligned} 2\langle z_i^\varepsilon - z_{i-1}^\varepsilon, x_{i-1}^\varepsilon\rangle &= -2a_{i-1}^\varepsilon\langle x_{i-1}^\varepsilon - x_i^\varepsilon, x_{i-1}^\varepsilon\rangle + 2h_i\langle \eta_i^\varepsilon F_{i-1}, x_{i-1}^\varepsilon\rangle + 2h_i\langle g_i, x_{i-1}^\varepsilon\rangle \\ &\leqslant -a_{i-1}^\varepsilon(\|x_{i-1}^\varepsilon\|^2 - \|x_i^\varepsilon\|^2) + 2h_i\langle \eta_i^\varepsilon F_{i-1}, x_{i-1}^\varepsilon\rangle + 2h_i\langle g_i, x_{i-1}^\varepsilon\rangle. \end{aligned}$$

Adding both sides of the two inequalities above, and using the fact that

$a^\varepsilon_{i-1} \geqslant a^\varepsilon_{i-2}$, we have

$$\|A^{1/2}w^\varepsilon_i\|^2 - \|A^{1/2}w^\varepsilon_{i-1}\|^2 + 2(\langle x^\varepsilon_i, z^\varepsilon_i\rangle - \langle x^\varepsilon_{i-1}, z^\varepsilon_{i-1}\rangle) - (a^\varepsilon_{i-1}\|x^\varepsilon_i\|^2 - a^\varepsilon_{i-2}\|x^\varepsilon_{i-1}\|^2)$$
$$\leqslant 2h_i\langle A^{1/2}w^\varepsilon_i, A^{1/2}f_i\rangle + 2h_i\sum_{k=i}^{l} h_k\langle \eta^\varepsilon_i A^{1/2}F_{i-1}, A^{1/2}w^\varepsilon_k\rangle$$
$$+ 2h_i\sum_{k=i}^{l} h_k\langle A^{1/2}g_i, A^{1/2}w^\varepsilon_k\rangle,$$

and the right-hand side is bounded by

$$h_i\|A^{1/2}w^\varepsilon_i\|^2 + h_i\|A^{1/2}f_i\|^2 + Th_i(\eta^\varepsilon_i)^2M_F + h_i\sum_{k=1}^{l} h_k\|A^{1/2}w^\varepsilon_k\|^2$$
$$+ Th_i\|A^{1/2}g_i\|^2 + h_i\sum_{k=1}^{l} h_k\|A^{1/2}w^\varepsilon_k\|^2.$$

We sum the inequalities obtained above from $i = 1$ to $i = l$, and use the fact that $x^\varepsilon_l = 0$ and the non-negativity of $a_{-1}$. This yields

$$\|A^{1/2}w^\varepsilon_l\|^2 \leqslant M_2 + 2\langle x^\varepsilon_0, z^\varepsilon_0\rangle + (1 + 2T)\sum_{k=1}^{l} h_k\|A^{1/2}w^\varepsilon_k\|^2$$
$$+ TM_F\sum_{i=1}^{l} h_i(\eta^\varepsilon_i)^2 + T\sum_{i=1}^{l} h_i\|A^{1/2}g_i\|^2. \tag{8.9}$$

By the definition of $x^\varepsilon_0$ we have $2\langle x^\varepsilon_0, z^\varepsilon_0\rangle = 2\sum_{k=1}^{l} h_k\langle A^{1/2}w^\varepsilon_k, A^{1/2}z^\varepsilon_0\rangle$, and Yang's inequality shows that the right-hand side is estimated by

$$\sum_{k=1}^{l} h_k\|A^{1/2}w^\varepsilon_k\|^2 + T\|A^{1/2}z^\varepsilon_0\|^2.$$

Substituting this inequality and (8.7) into (8.9), and applying Lemma 2.4 to the resultant inequality, we obtain the desired inequality (8.5).

To prove (8.6), let $1 \leqslant l \leqslant j$ and $1 \leqslant i \leqslant l$. We take the inner products of (8.2) and (8.3) with $a^\varepsilon_{i-1}Aw^\varepsilon_i$ and $z^\varepsilon_i$ respectively and add the two resultant equalities, so that we find, by Schwarz's inequality,

$$a^\varepsilon_{i-1}\|A^{1/2}w^\varepsilon_i\|^2 - a^\varepsilon_{i-2}\|A^{1/2}w^\varepsilon_{i-1}\|^2 + \|z^\varepsilon_i\|^2 - \|z^\varepsilon_{i-1}\|^2$$
$$\leqslant (a^\varepsilon_{i-1} - a^\varepsilon_{i-2})\|A^{1/2}w^\varepsilon_{i-1}\|^2$$
$$+ h_i(a^\varepsilon_{i-1}(\|A^{1/2}w^\varepsilon_i\|^2 + \|A^{1/2}f_i\|^2) + (\eta^\varepsilon_i)^2M_F + 2\|z^\varepsilon_i\|^2 + \|g_i\|^2).$$

By using the properties of $\{a_i\}$ and (8.5), we see that the right-hand side is bounded by

$$(Lh_{i-1} + (M + \varepsilon)h_i)((C(T) + 1)M_2 + C(T)M_4 + M_FC(T)M_5)$$
$$+ h_i(M + \varepsilon)\|A^{1/2}f_i\|^2 + M_Fh_i(\eta^\varepsilon_i)^2 + h_i\|g_i\|^2 + 2h_i\|z^\varepsilon_i\|^2.$$

The desired inequality (8.6) is obtained by adding the inequalities obtained above from $i = 1$ to $i = l$, and applying Lemma 2.4. It is thus shown that (8.5) through (8.8) are satisfied.

By (8.5) through (8.8), for each $1 \leqslant i \leqslant j$ the sequence $\{(w_i^\varepsilon, z_i^\varepsilon, \eta_i^\varepsilon)\}$ is bounded in $D(A^{1/2}) \times H \times \mathbb{R}$ as $\varepsilon \downarrow 0$. By reflexivity there exists a null sequence $\{\varepsilon(n)\}$ such that for each $1 \leqslant i \leqslant j$, $\{w_i^{\varepsilon(n)}\}$ and $\{z_i^{\varepsilon(n)}\}$ converge weakly to $w_i$ and $z_i$ in $D(A^{1/2})$ and $H$ respectively, and $\{\eta_i^{\varepsilon(n)}\}$ converges to $\eta_i$ in $\mathbb{R}$, as $n \to \infty$. The desired estimate for the sequence $\{(w_i, z_i, \eta_i)\}_{i=1}^j$ follows from the estimates (8.5) through (8.8). Let $\varphi \in D(A^{1/2})$. Taking the inner product of (8.3) with $\varphi$ and the limit as $n \to \infty$, we have $\langle \varphi, (z_i - z_{i-1})/h_i \rangle + a_{i-1} \langle A^{1/2}\varphi, A^{1/2} w_i \rangle = \langle \varphi, \eta_i F_{i-1} + g_i \rangle$, which implies that $a_{i-1} w_i \in D(A)$ and $-A(a_{i-1} w_i) = (z_i - z_{i-1})/h_i - (\eta_i F_{i-1} + g_i)$ for $1 \leqslant i \leqslant j$. We conclude that $\{(w_i, z_i, \eta_i)\}_{i=1}^j$ is a sequence satisfying the desired properties. □

*Proof of Theorem* 8.1. Let $X = D(A^{1/2}) \times H \times \mathbb{R}$, $E = D(A) \times D(A^{1/2}) \times \mathbb{R}$ and $Y = D(A^{3/2}) \times D(A) \times \mathbb{R}$, and define $S(u, v, \xi) = (u + Au, v + Av, \xi)$ for $(u, v, \xi) \in Y$. Then condition (H1) is clearly satisfied.

As in the previous section we use the homogeneous reduction technique. To define two sets $D$ and $D_0$ we use three real numbers $R > R_0 > r_0 > 0$. Let us define $D = \{(u, v, \xi) \in Y : \|u\|_{D(A^{3/2})} \leqslant R, \|v\|_{D(A)} \leqslant R, \xi = 1\}$. Then it is obvious that the set $D$ is closed and bounded in $Y$. We prove the theorem by applying Theorem 5.6 to the family $\{A(t, (w, z, \eta)) : (t, (w, z, \eta)) \in [0, T_0] \times D\}$ in $X$ defined by

$$A(t, (w, z, \eta))(u, v, \xi) = (v, -A(\sigma(\|A^{1/2} w\|^2) u) + \xi F(t, w), 0)$$

for $(u, v, \xi) \in D(A(t, (w, z, \eta)))$, where

$$D(A(t, (w, z, \eta))) := \{(u, v, \xi) \in X : v \in D(A^{1/2}), \sigma(\|A^{1/2} w\|^2) u \in D(A)\}.$$

It is easily seen that $\{A(t, (w, z, \eta)) : (t, (w, z, \eta)) \in [0, T_0] \times D\}$ is a family of closed linear operators in $X$ satisfying condition (H2).

Let $X_0 = D(A) \times D(A^{1/2}) \times \mathbb{R}$ $(= E)$. Then it is obvious that

$$X_0 \cap S^{-1}(X_0) = D(A^2) \times D(A^{3/2}) \times \mathbb{R}$$

and (1.2) is satisfied. If $D_0$ is defined by the set of all elements $(u, v, \xi)$ in $D(A^2) \times D(A^{3/2}) \times \mathbb{R}$ such that $A^{1/2} u = 0$, $\|u\| \leqslant R_0$, $r_0 \leqslant \|A^{1/2} v\| \leqslant R_0$, $\|v\| \leqslant R_0$, $\|Av\| \leqslant R_0$, $\|A^{3/2} v\| \leqslant R_0$ and $\xi = 1$, then condition (1.3) is clearly satisfied.

Condition (H5) follows from condition (F) and the inequality

$$|\sigma(\|A^{1/2} w\|^2) - \sigma(\|A^{1/2} \widehat{w}\|^2)| \leqslant 2 M_{\sigma'}(R^2) R \|A^{1/2}(w - \widehat{w})\|$$

for $w, \widehat{w} \in D(A^{1/2})$ with $\|A^{1/2} w\| \leqslant R$ and $\|A^{1/2} \widehat{w}\| \leqslant R$. Condition (H6) is satisfied with $B(t, (w, z, \eta))(u, v, \xi) = (0, \xi A F(t, w), 0)$ for $(u, v, \xi) \in X$, $t \in [0, T_0]$ and $(w, z, \eta) \in W$, where $W$ is the open, bounded set in $Y$ consisting of all elements $(u, v, \xi) \in Y$ such that $\|u\|_{D(A^{3/2})} < 2R$, $\|v\|_{D(A)} < 2R$ and $|\xi| < 2$.

We prove that the family $\{A(t, (w, z, \eta)) : (t, (w, z, \eta)) \in [0, T] \times D\}$ satisfies condition (H3), where $T > 0$ is yet to be determined. For this purpose, let $\lambda_0 > 0$ be chosen such that $4(1 + T_0)\lambda_0 \leqslant 1$, and let $(u_0, v_0, \xi_0) \in D_0$ and $\{t_k\}_{k=1}^i$ be a finite sequence such that $0 = t_0 < t_1 < \ldots < t_i \leqslant T$ and $t_k - t_{k-1} \leqslant \lambda_0$ for $1 \leqslant k \leqslant i$. Let $M = M_\sigma(R^2)$, $M_F = \sup\{\|F(t, w)\|^2_{D(A^{3/2})} : t \in [0, T_0], \|w\|_{D(A^{3/2})} \leqslant R\} < \infty$ (by condition (F)) and $L = 2 M_{\sigma'}(R^2) R^2$. We shall inductively show that there exists a solution $\{(u_k, v_k, \xi_k)\}_{k=1}^i$ in $D(A^{3/2}) \times D(A) \times \mathbb{R}$ of the system

$$(u_k - u_{k-1})/(t_k - t_{k-1}) = v_k, \tag{8.10}$$

$$(v_k - v_{k-1})/(t_k - t_{k-1}) + \sigma(\|A^{1/2}u_{k-1}\|^2)Au_k = \xi_k F(t_{k-1}, u_{k-1}), \tag{8.11}$$

$$(\xi_k - \xi_{k-1})/(t_k - t_{k-1}) = 0, \tag{8.12}$$

for $k = 1, 2, \dots, i$, satisfying the following properties:

$$0 \leqslant a_k \leqslant M \quad \text{for } 0 \leqslant k \leqslant i; \tag{8.13}$$

$$0 \leqslant a_k - a_{k-1} \leqslant L(t_k - t_{k-1}) \quad \text{for } 0 \leqslant k \leqslant i; \tag{8.14}$$

$$\|F(t_k, u_k)\|^2_{D(A^{3/2})} \leqslant M_F \quad \text{for } 0 \leqslant k \leqslant i; \tag{8.15}$$

$$(u_k, v_k, \xi_k) \in D \quad \text{for } 0 \leqslant k \leqslant i. \tag{8.16}$$

Here we set $t_{-1} = 0$, $a_{-1} = 0$ and $a_k = \sigma(\|A^{1/2}u_k\|^2)$ for $0 \leqslant k \leqslant i$.

Since $A^{1/2}u_0 = 0$, the desired properties (8.13) through (8.16) are true for $i = 0$. Let $1 \leqslant j \leqslant i$ and assume that there exists a solution $\{(u_k, v_k, \xi_k)\}_{k=1}^{j-1}$ in $D(A^{3/2}) \times D(A) \times \mathbb{R}$ of the equations (8.10) through (8.12) satisfying (8.13) through (8.16) for $0 \leqslant k \leqslant j-1$. For each $l = 0, 1, 2$, let

$$(w_0^l, z_0^l, \eta_0^l) = (A^{l/2}u_0, A^{l/2}v_0, \xi_0) \quad \text{and} \quad F_k^l = A^{l/2}F(t_k, u_k)$$

for $0 \leqslant k \leqslant j-1$. Let $l \in \{0, 1, 2\}$. Since $(u_0, v_0, \xi_0) \in D_0$ we have

$$(w_0^l, z_0^l, \eta_0^l) \in D(A^{1/2}) \times D(A^{1/2}) \times \mathbb{R}.$$

Since $u_k \in D(A^{3/2})$ for $0 \leqslant k \leqslant j-1$ (by the hypothesis of induction), we see by the property of $F$ that $\{F_k^l\}_{k=0}^{j-1}$ is the sequence in $D(A^{1/2})$. By (8.13) through (8.15) for $0 \leqslant k \leqslant j-1$ (the hypothesis of induction), we apply Lemma 8.2 with $h_k = t_k - t_{k-1}$ and $(f_k, g_k, e_k) = (0, 0, 0)$ for $1 \leqslant k \leqslant j$, so that there exists a unique sequence $\{(w_i^l, z_i^l, \eta_i^l)\}_{i=1}^j$ such that $w_i^l \in D(A^{1/2})$, $a_{i-1}w_i^l \in D(A)$, $z_i^l \in D(A^{1/2})$ and $\eta_i^l \in \mathbb{R}$ for $1 \leqslant i \leqslant j$, satisfying the system

$$\begin{cases} (w_i^l - w_{i-1}^l)/(t_i - t_{i-1}) = z_i^l, \\ (z_i^l - z_{i-1}^l)/(t_i - t_{i-1}) + A(a_{i-1}w_i^l) = \eta_i^l F_{i-1}^l, \\ (\eta_i^l - \eta_{i-1}^l)/(t_i - t_{i-1}) = 0, \end{cases}$$

for $1 \leqslant i \leqslant j$. Moreover, since $\eta_0^l = \xi_0 = 1$ we have

$$\begin{aligned} \|A^{1/2}w_j^l\|^2 &\leqslant (C(T)+1)\|A^{1/2}w_0^l\|^2 + C(T)\|A^{1/2}z_0^l\|^2 + M_F C(T), \\ \|z_j^l\|^2 &\leqslant (L+M)(C(T)+1)\|A^{1/2}w_0^l\|^2 + (C(T)+1)\|z_0^l\|^2 \\ &\quad + (L+M)C(T)\|A^{1/2}z_0^l\|^2 + (L+M+1)M_F C(T), \\ \|w_j^l\|^2 &\leqslant (C(T)+1)\|w_0^l\|^2 + (L+M)C(T)\|A^{1/2}w_0^l\|^2 + C(T)\|z_0^l\|^2 \\ &\quad + (L+M)C(T)\|A^{1/2}z_0^l\|^2 + (L+M+1)M_F C(T). \end{aligned}$$

Define $(u_j, v_j, \xi_j) = (w_j^0, z_j^0, \eta_j^0)$. By uniqueness we see that $(w_i^0, z_i^0, \eta_i^0) = (u_i, v_i, \xi_i)$ for $1 \leqslant i \leqslant j-1$ and that

$$(A^{1/2}w_j^l, A^{1/2}z_j^l, \eta_j^l) = (w_j^{l+1}, z_j^{l+1}, \eta_j^{l+1}) \in D(A^{1/2}) \times D(A^{1/2}) \times \mathbb{R}$$

for $l = 0, 1$. Since $\xi_0 = 1$ we have $\eta_j^0 = 1$. These facts together imply that (8.10) through (8.12) are satisfied with $k = j$ and that

$$(u_j, v_j, \xi_j) \in D(A^{3/2}) \times D(A) \times \{1\}, \quad w_j^l = A^{l/2} u_j \quad \text{and} \quad z_j^l = A^{l/2} v_j$$

for $l = 0, 1, 2$. Notice that $A^{1/2} w_0^l = A^{l/2}(A^{1/2} u_0) = 0$ for $l = 0, 1, 2$. Then the three estimates above show that

$$\|A^{(l+1)/2} u_j\|^2 \leqslant C(T)(R_0^2 + M_F),$$
$$\|A^{l/2} v_j\|^2 \leqslant (C(T) + 1)R_0^2 + (L + M)C(T)R_0^2 + (L + M + 1)M_F C(T),$$
$$\|A^{l/2} u_j\|^2 \leqslant (C(T) + 1)R_0^2 + (L + M + 1)C(T)(R_0^2 + M_F).$$

Since $\lim_{T\downarrow 0} C(T) = 0$, it is possible to choose $T > 0$ so small that $(u_j, v_j, \xi_j) \in D$. This means that (8.16) holds for $k = j$. From this fact we deduce that the desired properties (8.13) and (8.15) are valid for $k = j$.

It remains to prove that (8.14) is true for $k = j$. Since

$$\begin{aligned}(d/d\theta)\sigma(\|\theta A^{1/2} u_j + (1 - \theta)A^{1/2} u_{j-1}\|^2) \\ = 2\sigma'(\|\theta A^{1/2} u_j + (1 - \theta)A^{1/2} u_{j-1}\|^2) \\ \times \langle \theta A^{1/2} u_j + (1 - \theta)A^{1/2} u_{j-1}, A^{1/2} v_j\rangle (t_j - t_{j-1})\end{aligned}$$

and the right-hand side is estimated by $2M_{\sigma'}(R^2)R^2(t_j - t_{j-1})$, we have $a_j - a_{j-1} \leqslant L(t_j - t_{j-1})$. The derivative of $\sigma(\|\theta A^{1/2} u_j + (1 - \theta)A^{1/2} u_{j-1}\|^2)$ is rewritten as

$$\begin{aligned}2\sigma'(\|\theta A^{1/2} u_j + (1 - \theta)A^{1/2} u_{j-1}\|^2) \\ \times (\theta\|A^{1/2} v_j\|^2 (t_j - t_{j-1}) + \langle A^{1/2} u_{j-1}, A^{1/2} v_j\rangle)(t_j - t_{j-1}).\end{aligned}$$

If $w_k := F(t_{k-1}, u_{k-1}) - a_{k-1} A u_k$ for $1 \leqslant k \leqslant j$ then

$$u_k = u_0 + t_k v_0 + \sum_{p=1}^{k} (t_p - t_{p-1}) \sum_{q=1}^{p} (t_q - t_{q-1}) w_q \quad \text{and} \quad v_k = v_0 + \sum_{p=1}^{k} (t_p - t_{p-1}) w_p$$

by (8.10) and (8.11). Since $A^{1/2} u_0 = 0$, $r_0 \leqslant \|A^{1/2} v_0\| \leqslant R_0$ and $\|A^{1/2} w_p\| \leqslant M_F^{1/2} + MR$ for $1 \leqslant p \leqslant j$ we have

$$\langle A^{1/2} u_{j-1}, A^{1/2} v_j\rangle \geqslant t_{j-1}(r_0^2 - 2TR(M_F^{1/2} + MR) - T^2(M_F^{1/2} + MR)^2),$$

and the right-hand side is non-negative, if $T$ is chosen sufficiently small. It follows that the function $\sigma(\|\theta A^{1/2} u_j + (1 - \theta)A^{1/2} u_{j-1}\|^2)$ is non-decreasing in $\theta \in [0, 1]$; hence $a_j \geqslant a_{j-1}$. It is thus shown that condition (H3) is satisfied.

The argument above shows that (8.13) through (8.15) are true if $(u_0, v_0, \xi_0) \in D_0$ and (8.16) is satisfied. By virtue of this fact, condition (H4) is a direct consequence of Lemma 8.2. □

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