Littlewood’s multiple formula for spin characters of symmetric groups

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LITTLEWOOD’S MULTIPLE FORMULA FOR SPIN CHARACTERS OF SYMMETRIC GROUPS

HIROSHI MIZUKAWA AND HIRO-FUMI YAMADA

Introduction

This paper deals with some character values of the symmetric group $S_n$ as well as its double cover $\tilde{S}_n$. Let $\chi_{\lambda}(\rho)$ be the irreducible character of $S_n$, indexed by the partition $\lambda$ and evaluated at the conjugacy class $\rho$. Comparing the character tables of $S_2$ and $S_4$, one observes that

\[ \chi_{\lambda}(2\rho) = \chi_{\lambda}(\rho) \]
\[ \chi_{\lambda}(2^2\rho) = \chi_{\lambda}(\rho) + \chi_{\lambda}(12\rho) \]

for $\rho = (2)$, $2\rho = (4)$ and $\rho = (1^2)$, $2\rho = (2^2)$. A number of such observations lead to what we call Littlewood’s multiple formula (Theorem 1.1). This formula appears in Littlewood’s book [2]. We include a proof that is based on an ‘inflation’ of the variables in a Schur function. This is different from one given in [2], and we claim that it is more complete than the one given there.

Our main objective is to obtain the spin character version of Littlewood’s multiple formula (Theorem 2.3). Let $\zeta_{\lambda}(\rho)$ be the irreducible negative character of $\tilde{S}_n$ (cf. [1]), indexed by the strict partition $\lambda$ and evaluated at the conjugacy class $\rho$. One finds character tables ($\zeta_{\lambda}(\rho)$) in [1] for $n \leq 14$. This time we evidently see that

\[ \zeta_{\lambda}(3\rho) = \zeta_{\lambda}(\rho) \]

for $\lambda = (4),(3,1)$ and $\rho = (3,1),(1^4)$. The proof of Theorem 2.3 is achieved in a way that is similar to the case of ordinary characters. Instead of a Schur function, we deal with Schur’s P-function, which is defined as a ratio of Pfaffians.

1. Littlewood’s multiple formula

We first recall the multiple formula for ordinary characters of the symmetric groups that is due to Littlewood [2, Chapter 8].

Throughout this section, we fix a positive integer $r$. Let $\lambda = (\lambda_1,\ldots,\lambda_N)$ be a partition. We always assume that $N$ is at least the length $l(\lambda)$ of $\lambda$. An $(r+1)$-tuple of partitions $(\lambda^c,\lambda^{[0]},\ldots,\lambda^{[r-1]})$ is attached to $\lambda$ : $\lambda^c$ is the $r$-core of $\lambda$ and the collection $\lambda^* = (\lambda^{[0]},\ldots,\lambda^{[r-1]})$ is the $r$-quotient of $\lambda$ (cf. [4, p. 12]). The effect of changing $N$ is to permute the $\lambda^{[k]}$ cyclically. Since we will only consider the Littlewood–Richardson coefficients $LR_{\lambda^{[0]},\ldots,\lambda^{[r-1]}}^{\mu\lambda}$, the ambiguity of choosing $N$ can be ignored. Put $r\lambda = (r\lambda_1,\ldots,r\lambda_N)$. We can easily verify that $(r\lambda)^c = \emptyset$ and
\( (r \lambda)[0] = (\lambda_r, r \lambda_{2r}, r \lambda_{3r}, \ldots), \) \( (r \lambda)[k] = (\lambda_{r-k}, r \lambda_{2r-k}, r \lambda_{3r-k}, \ldots) \) \((1 \leq k \leq r - 1).\) As is well known, the ordinary irreducible characters of the symmetric group \( S_n \) are parametrized by \( P(n) \), the set of all partitions of \( n \). Let \( \chi^\lambda(\rho) \) denote the irreducible character of \( S_n \) indexed by \( \lambda \), evaluated at the conjugacy class determined by the partition \( \rho \).

**Theorem 1.1** (Littlewood's multiple formula).

\[
\chi^r_{\lambda}(r \rho) = \sum_{\nu \in P(n)} \mathcal{L}_\nu^{r \lambda}(0) \mathcal{L}_\nu^{r \lambda}(1) \ldots \mathcal{L}_\nu^{r \lambda}(r-1) \chi^\nu(\rho)
\]

for any partitions \( \lambda \) and \( \rho \) of \( n \).

Littlewood proved a more general formula in [3, pp. 340–342]. However, in order to contrast the formula with the spin character case, we here deal only with this form. In the rest of this section, we give a simple proof of the theorem by using Schur functions.

Put \( x_N = (x_1, x_2, \ldots, x_N) \), \( x_{N,r} = (x_{r1}, x_{r2}, \ldots, x_{rN}) \) and \( x_{N,r} = (x_{r1}, x_{r2}, \ldots, x_{rN}) \) with \( x_{kN+i} = \omega^{k}x_i \) \((0 \leq k \leq r - 1, 1 \leq i \leq N)\), where \( \omega = \exp(2\pi \sqrt{-1}/r) \). We call \( x_{N,r} \) the \( r \)-inflation of \( x_N \). Let \( \delta_N = (N - 1, N - 2, \ldots, 1, 0) \). Following [4, p. 40], we define the Schur function of variables \( x_N \), corresponding to the partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) \), by

\[
s_{\lambda}(x_N) = \det A_{\lambda+\delta_N}(x_N) / \det A_{\delta_N}(x_N),
\]

where

\[
A_{\alpha}(x_N) = (x_{\alpha j}^{i})_{1 \leq i,j \leq N}
\]

for a sequence \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \) of non-negative integers.

**Theorem 1.2.**

\[
s_{r \lambda}(x_{N,r}) = \prod_{k=0}^{r-1} \mathcal{L}_{r \lambda[k]}(x_N)
\]

for any partition \( \lambda = (\lambda_1, \ldots, \lambda_r) \).

**Proof.** For a sequence \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \) of non-negative integers, put

\[
X_{N}^{x_{N,r}+N-j} = \begin{pmatrix} x_1^{x_{N,r}+N-j} \\ x_2^{x_{N,r}+N-j} \\ \vdots \\ x_N^{x_{N,r}+N-j} \end{pmatrix},
\]

the \( j \)th column of the matrix \( A_{r \lambda+\delta_N}(x_{N,r}) \). We first compute the numerator of \( s_{r \lambda}(x_{N,r}) \):

\[
A_{r \lambda+\delta_N}(x_{N,r}) = (\omega^{(i-1)(r-j)}x_{N,\lambda+rN-j})_{1 \leq i,0 \leq j \leq rN}.
\]

Let \( \tau \in S_N \) be defined by

\[
\tau(ir+j) = N(j-1)+(i+1) \quad 0 \leq i \leq N-1, \; 1 \leq j \leq r.
\]

Permuting columns of the above matrix according to \( \tau \), and setting \( p = \)
diag \((x_1, x_2, \ldots, x_N)\), we obtain
\[
\det A_{r, \beta_\lambda}(x_N) = sgn(\tau) \det(a_j^{i-r-j}) \det(A_{(r\beta_\lambda)})(1 \leq i, j \leq r)
\]
Thus we have
\[
\det A_{r, \beta_\lambda}(x_N) = sgn(\tau)(\det p)^c \prod_{j=0}^{r-1} \det(A_{(r\beta_\lambda)})(x_N^j)
\]
where
\[
\begin{cases}
  \alpha = \sum_{j=1}^r (r - j) = \binom{r}{2} \\
  c = det(a_j^{i-r-j})E_N.
\end{cases}
\]
Putting \(\lambda = (0, \ldots, 0)\), we get
\[
\det A_{\beta_\lambda}(x_N) = sgn(\tau)p^c \left(\det A_{\beta_\lambda}(x_N^r)\right)^r.
\]
From (1) and (2), we see that
\[
s_{r, \lambda}(x_N) = \frac{\det A_{r, \beta_\lambda}(x_N)}{\det A_{\beta_\lambda}(x_N)}
\]
\[
= \prod_{k=0}^{r-1} \frac{\det A_{(r, \beta_\lambda)}(x_N^k)}{\det A_{\beta_\lambda}(x_N^k)}
\]
\[
= \prod_{k=0}^{r-1} s_{(r, \beta_\lambda)}(x_N^k).
\]
We remark here that the right-hand side of the expression in Theorem 1.2 can also be expressed as
\[
\prod_{k=0}^{r-1} s_{(r, \beta_\lambda)}(x_N^k) = \sum_{\nu \in \mathcal{P}(n)} LR_{\nu(\lambda)}(x_N^0, x_N^1, \ldots, x_N^{r-1}) s_{\nu}(x_N).
\]
In order to translate the above identity of Schur functions into that of irreducible characters of \(S_n\), we need the power sum symmetric functions
\[
p_m(x_N) = \sum_{i=1}^N x_{ni}^m \quad m = 1, 2, \ldots.
\]
For a partition \(\rho = (\rho_1, \rho_2, \ldots)\), define
\[
p_\rho = p_{\rho_1}p_{\rho_2} \ldots.
\]
The well known Frobenius formula is
\[
s_{\lambda}(x_N) = \sum_{\rho \in \mathcal{P}(n)} z_{\rho}^{-1} \chi^\rho(\rho)p_{\rho}(x_N),
\]
where
\[
z_{\rho} = \prod_{i=1}^n m_i! \ i^m
\]
for the partition $\rho = (1^{m_1} 2^{m_2} \ldots n^{m_n})$ of $n$. As for the $r$-inflation of $x_N$, we have the following simple fact.

**Lemma 1.3.**

$$p_m(x_{N,r}) = \begin{cases} \quad rp_m(x_N) & m \equiv 0 \pmod{r} \\ 0 & m \not\equiv 0 \pmod{r}. \end{cases}$$

**Proof.** We easily see that

$$p_m(x_{N,r}) = \sum_{i=1}^{rN} x_i^m = \sum_{i=1}^{N} \left( \sum_{k=0}^{r-1} (\omega^k x_i)^m \right) = \begin{cases} \quad r \sum_{i=1}^{N} x_i^m & m \equiv 0 \pmod{r} \\ N \sum_{i=1}^{N} 1 - (\omega^m)^i x_i^m = 0 & m \not\equiv 0 \pmod{r}. \end{cases}$$

Combining this lemma with the Frobenius formula, we see that

$$s_{r\lambda}(x_{N,r}) = \sum_{\rho \in P(n)} z^{-1} \chi^{r\lambda}(\rho) p_\rho(x_{N,r})$$

$$s_{r\lambda}(x_{N,r}) = \sum_{\rho \in P(n)} z^{-1} \chi^{r\lambda}(\rho) p_\rho(x_N)$$

$$s_{r\lambda}(x_{N,r}) = \sum_{\rho \in P(n)} z^{-1} \chi^{r\lambda}(\rho) p_\rho(x_N).$$

Therefore we have

$$\sum_{\rho \in P(n)} z^{-1} \chi^{r\lambda}(r\rho) p_\rho(x_N)$$

$$= \sum_{\nu \in P(n)} L_R^{r\lambda}(x_{[0],[r],[r],[r],[r],\ldots}) \left( \sum_{\rho \in P(n)} z^{-1} \chi^{r\lambda}(\rho) p_\rho(x_N) \right)$$

$$= \sum_{\rho \in P(n)} \sum_{\nu \in P(n)} L_R^{r\lambda}(x_{[0],[r],[r],[r],[r],\ldots}) \chi^{r\lambda}(\rho) z^{-1} p_\rho(x_N).$$

Since the $p_\rho$ are linearly independent, we obtain

$$\chi^{r\lambda}(r\rho) = \sum_{\nu \in P(n)} L_R^{r\lambda}(x_{[0],[r],[r],[r],[r],\ldots}) \chi^{r\lambda}(\rho),$$

as desired.

The proof of Lemma 1.3 given here can immediately be extended to the case of partition $\mu \in P(n)$ whose $r$-core is empty. Littlewood’s multiple formula is

$$s_\mu(x_{N,r}) = e \prod_{k=0}^{r-1} s_{\mu(k)}(x_N),$$
and
\[ \chi^\mu(r \rho) = \varepsilon \sum_{\nu \in P(\mu)} L R^r_{\mu[0][\nu[1][2][...][r-1][\mu]} \chi^\nu(\rho), \]

where \( \varepsilon \) is the sign depending only on \( \mu \).

2. Spin characters

We consider Littlewood’s multiple formula for spin characters of symmetric groups. Since the theory of spin characters is Pfaffian by nature, Schur’s P-functions play an important role. Here we adopt the definition of P-functions due to Nimmo [6, Appendix] (see also [4, p. 267]).

A partition \( \lambda = (\lambda_1, \ldots, \lambda_l) \) is said to be strict if \( \lambda_1 > \ldots > \lambda_l > 0 \).

The set of strict partitions of \( n \) is denoted by \( \text{SP}(n) \).

Let \( A(x_N) = \begin{pmatrix} x_i - x_j \\ x_i + x_j \end{pmatrix} \) for a strict partition \( \lambda = (\lambda_1, \ldots, \lambda_l) \). Put
\[ A_{\lambda}(x_N) = \begin{pmatrix} A(x_N) & D_{\lambda}(x_N) \\ -D_{\lambda}^T(x_N) & 0 \end{pmatrix}, \]

which is a skew-symmetric matrix of \( N + l \) rows and columns. Define \( \text{Pf}_{\phi}(x_N) \) to be the Pfaffian of \( A_{\lambda}(x_N) \) if \( N + l \) is even, and to be the Pfaffian of \( A_{\lambda}(x_{N+1}) \) if \( N + l \) is odd, agreeing \( x_{N+1} = 0 \). When \( \lambda = \emptyset \), we have
\[ \text{Pf}_{\phi}(x_N) = \begin{cases} \text{Pf}_{\phi}(x_{N+1}) & \text{if } N \text{ is odd} \\ \text{Pf}_{\phi}(x_{N+1}) & \text{if } N \text{ is even} \end{cases}. \]

It is a good exercise to verify that
\[ \text{Pf}_{\phi}(x_N) = \prod_{1 \leq i < j \leq N} \frac{x_i - x_j}{x_i + x_j}. \tag{3} \]

Now define Schur’s P-function corresponding to the strict partition \( \lambda \) by
\[ P_{\lambda}(x_N) = \frac{\text{Pf}_{\phi}(x_N)}{\text{Pf}_{\phi}(x_{N+1})}. \]

Throughout this section we fix a positive odd integer \( r \). The \( r \)-inflation of \( x_N \) is defined, as in Section 1, by \( x_{N,r} = (x_1, x_2, \ldots, x_{rN}) \) with \( x_{kN+i} = \omega^k x_i \) \((0 \leq k \leq r-1, 1 \leq i \leq N)\), where \( \omega = \exp(2\pi \sqrt{-1}/r) \).

**Lemma 2.1.**
\[ \text{Pf}_{\phi}(x_{N,r}) = c_r \left( \text{Pf}_{\phi}(x_N') \right)^r, \]
where \( c_r \) is a non-zero complex number.

**Proof.** Put
\[ B_k = \begin{pmatrix} x_i - \omega^k x_j \\ x_i + \omega^k x_j \end{pmatrix} \] \(0 \leq k \leq r-1, 1 \leq i \leq j \leq N\),
\[ B = (B_{k,j})_{0 \leq k \leq r-1, 0 \leq j \leq r-1}. \]
Then we see that $\mathcal{B}$ is a skew-symmetric matrix. We also see that

$$\text{Pf}_\varnothing(x_{N,r}) = \text{Pf}(\mathcal{B})$$

if $N$ is even, and

$$\text{Pf}_\varnothing(x_{N,r}) = \text{Pf}\left( \begin{array}{cc} \mathcal{B} & \theta \\ -\theta & 0 \end{array} \right)$$

if $N$ is odd. Here $\theta = (1, \ldots, 1)$ is a row vector of length $r$. Put

$$b_k^+ = \prod_{1 \leq i < j \leq N} \frac{x_i - \omega^k x_j}{x_i + \omega^k x_j}, \quad b_k^- = \prod_{1 \leq i < j \leq N} \frac{x_i - \omega^k x_j}{x_i + \omega^k x_j},$$

and

$$d_k = \left( \frac{1 - \omega^k}{1 + \omega^k} \right)^N$$

for $0 \leq k \leq r - 1$. Then we check that

$$b_k^- = (-1)^{N(N-1)/2} b_k^+,$$

and

$$\prod_{k=0}^{r-1} b_k^+ = \prod_{1 \leq i < j \leq N} \left( \prod_{k=0}^{r-1} \frac{x_i - \omega^k x_j}{x_i + \omega^k x_j} \right) = \prod_{1 \leq i < j \leq N} \frac{x_i' - x_j'}{x_i' + x_j'},$$

$$= \text{Pf}_\varnothing(x_N').$$

By (3), we have

$$\text{Pf}_\varnothing(x_{N,r}) = c_r \left( \prod_{k=0}^{r-1} b_k^+ \right)^r = c_r \left( \text{Pf}_\varnothing(x_N') \right)^r,$$

where

$$c_r = \prod_{k=1}^{r-1} d_k^{-(r-k)},$$

which is a non-zero constant. \qed

**Theorem 2.2.**

$$P_{r\lambda}(x_{N,r}) = P_{\lambda}(x_N')$$

for any strict partition $\lambda = (\lambda_1, \ldots, \lambda_l)$.

**Proof.** We give a proof for the case where $N + l$ is even, since the other case can be proved with only a slight modification. Set

$$D = D_{r}(x_N').$$

Since $rN + l$ is even, we have

$$\text{Pf}_{r\lambda}(x_{N,r}) = \text{Pf}\left( \begin{array}{cc} \mathcal{B} & \mathcal{D} \\ \mathcal{D} & 0 \end{array} \right).$$

Here $\mathcal{D} = ('D, \ldots, 'D)$ is an $N \times rN$ matrix. Apply the following elementary row-block and column-block operations successively to the above matrix:
(1) Subtract the \( r \)th row-block from the first, second, \ldots, \((r-1)\)th row-blocks.
(2) Subtract the \( r \)th column-block from the first, second, \ldots, \((r-1)\)th column-blocks.
(3) Add the \( r \)th row-block, multiplied by \( \frac{1}{r} \), to the first, second, \ldots, \((r-1)\)th row-blocks.
(4) Add the \( r \)th column-block, multiplied by \( \frac{1}{r} \), to the first, second, \ldots, \((r-1)\)th column-blocks.

This series of operations preserves the skew-symmetricity of the matrix and does not change its Pfaffian. Therefore we have, setting \( B = \frac{1}{r} \sum_{k=0}^{r-1} B_k \),

\[
Pf_{r\lambda}(x_{N,r}) = Pf \begin{pmatrix} A' & 0 & 0 \\ 0 & B & D \\ 0 & -D & 0 \end{pmatrix}
= Pf(A')Pf \begin{pmatrix} B & D \\ -D & 0 \end{pmatrix}
\]
for some skew-symmetric matrix \( A' \) of \( r(N-1) \) rows and columns. By a simple calculation, we can show that the \((i,j)\)-entry of \( B \) is equal to

\[
\frac{1}{r} \sum_{k=0}^{r-1} x_i - \omega^k x_j = x_i - x_j
\]
and thus we see that \( Pf(B) = Pf(x_{N}'_r) \). Hence we have

\[
Pf \begin{pmatrix} B & D \\ -D & 0 \end{pmatrix} = Pf_{r\lambda}(x_{N,r}).
\]
By the same elementary operations as above, we see that

\[
Pf_{r\lambda}(x_{N,r}) = Pf \begin{pmatrix} A' & 0 \\ 0 & B \end{pmatrix}
= Pf(A')Pf_{r\lambda}(x_{N,r}).
\]

Since the left-hand side is equal to \( c_r \{Pf_{r\lambda}(x_{N,r}^r)\}^r \), we have

\[
Pf(A') = c_r \{Pf_{r\lambda}(x_{N,r}^r)\}^{r-1}.
\]
Consequently we see that

\[
P_{r\lambda}(x_{N,r}) = \frac{Pf_{r\lambda}(x_{N,r})}{Pf_{r\lambda}(x_{N,r})} = c_r \{Pf_{r\lambda}(x_{N,r}^r)\}^{r-1} \frac{Pf_{r\lambda}(x_{N,r})}{c_r \{Pf_{r\lambda}(x_{N,r}^r)\}^r} = P_{r\lambda}(x_{N,r}).\]

Let \( \tilde{S}_n \) \((n \geq 4)\) be the double cover of the symmetric group \( S_n \), which is generated by \( t_1, \ldots, t_{n-1} \) and \( z \) subject to the relations

\[
\begin{aligned}
z^2 &= 1 \\
z t_i &= t_i z, \quad t_i^2 = z & 1 \leq i \leq n - 1 \\
(t_i t_{i+1})^3 &= z & 1 \leq i \leq n - 2 \\
t_i t_j &= z t_j t_i & |i-j| \geq 2.
\end{aligned}
\]
The group $\hat{S}_n$ can be regarded as a central extension

$$1 \longrightarrow Q \longrightarrow \hat{S}_n \longrightarrow S_n \longrightarrow 1,$$

where $Q = \{1, z\} \cong S_2$. The central element $z$ acts on an irreducible representation by $\pm 1$. An irreducible representation of $\hat{S}_n$ is said to be negative if $z = -1$.

The irreducible negative representations of $\hat{S}_n$ are parametrized up to associativity by $\text{SP}(n)$ [1, Chapter 8]. We denote by $\zeta^z$ the character of the irreducible negative representation corresponding to $\lambda \in \text{SP}(n)$. These spin characters are related to the $P$-functions by the following Schur formula [5, p. 64]:

$$P_{\lambda}(x_N) = \sum_{\rho \in \text{OP}(n)} \frac{2^{l(\rho)} - l(\lambda) + \epsilon(\lambda)/2}{z^{|\lambda|/2}} \zeta^z(\rho)p_{\rho}(x_N),$$

where $\text{OP}(n)$ denotes the set of partitions of $n$ consisting of odd parts, $\zeta^z(\rho)$ is the value of $\zeta^z$ on the conjugacy class determined by $\rho \in \text{OP}(n)$, and

$$\epsilon(\lambda) = \begin{cases} 0 & n - l(\lambda) \text{ is even} \\ 1 & n - l(\lambda) \text{ is odd.} \end{cases}$$

Using the Schur formula, we can translate Theorem 2.2 into the following multiple formula for spin characters.

**Theorem 2.3.** If $r$ is an odd positive integer, we have

$$\zeta^{rz}(r\rho) = \zeta^z(\rho)$$

for any $\lambda \in \text{SP}(n)$ and $\rho \in \text{OP}(n)$.

**Proof.** We have

$$P_{rz}(x_{N,r}) = \sum_{\rho \in \text{OP}(n)} \frac{2^{l(\rho)} - l(\lambda) + \epsilon(\lambda)/2}{z^{|\lambda|/2}} \zeta^z(\rho)p_{\rho}(x_{N,r})$$

$$= \sum_{\rho \in \text{OP}(n)} \frac{2^{l(\rho)} - l(\lambda) + \epsilon(\lambda)/2}{z^{|\lambda|/2}} \zeta^z(\rho)p_{\rho}(x_{N,r})$$

$$= \sum_{\rho \in \text{OP}(n)} \frac{2^{l(\rho)} - l(\lambda) + \epsilon(\lambda)/2}{z^{|\lambda|/2}} \zeta^z(\rho)p_{\rho}(x_{N,r}).$$

The left-hand side is equal to

$$P_z(x_N) = \sum_{\rho \in \text{OP}(n)} \frac{2^{l(\rho)} - l(\lambda) + \epsilon(\lambda)/2}{z^{|\lambda|/2}} \zeta^z(\rho)p_{\rho}(x_N).$$

Since $l(\rho) = l(\rho)$, $l(\lambda) = l(\lambda)$ and $\epsilon(\lambda) = \epsilon(\lambda)$, we can conclude, comparing the coefficients of $p_{\rho}(x_N)$, that $\zeta^{rz}(r\rho) = \zeta^z(\rho)$.

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