Surgery obstruction of twisted products

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1. Introduction. Let \((G, \chi)\) be a pair of a finite group \(G\) and a homomorphism \(\chi: G \to \{\pm 1\}\). Then we call an oriented closed PL (or smooth) \(G\)-manifold \(L^m\) a \(G\)-\(\chi\)-manifold when the action of \(g \in G\) preserves the orientation of \(L^m\) if and only if \(\chi(g) = +1\). We can define the cobordism group \(\Omega^s_\chi(G)\) of \(G\)-\(\chi\)-manifolds as in \([1]\). Moreover, let \((\pi, w)\) be a pair of a finitely presentable group \(\pi\) and a homomorphism \(w: \pi \to \{\pm 1\}\). Then the Wall group \(L^s_\pi(\pi, w)\) is defined in \([7]\). Its element \(\sigma\) can be represented as the surgery obstruction \(\sigma(f)\) of a normal map of degree one \(f: M^n \to N^n\) between compact PL (or smooth) manifolds to deform to a simple homotopy equivalence, where \(\pi_1(N^n) = \pi\) and \(w: \pi_1(N^n) \to \{\pm 1\}\) is the characteristic map of the orientation bundle of \(N^n\).

Now, assume that there is an epimorphism \(\phi: \pi \to G\). Then we can define a homomorphism

\[ \Omega^s_\chi(G) \otimes L^s_\pi(\pi, w) \to L^s_{\pi + \pi}(\pi, w_\chi) \]

\((w_\chi: \pi \to \{\pm 1\})\) is the homomorphism defined by \(w_\chi(h) = w(h)(\chi(\phi(h)))\) as follows: For \(\sigma(f) \in L^s_\pi(\pi, w)\) of \(f: M^n \to N^n\), consider the covering map \(\tilde{f}: \tilde{M}^n \to \tilde{N}^n\), where \(\tilde{N}^n\) is the universal covering of \(N^n\) and \(\tilde{M}^n\) is the covering of \(M^n\) induced from \(\tilde{N}^n\) by \(f\). Further, let \(L^m\) be a \(G\)-\(\chi\)-manifold. Then \(\pi\) acts on \(L^m\) through \(\phi\) and the product manifolds \(\tilde{M}^n \times L^m\) and \(\tilde{N}^n \times L^m\) have the diagonal \(\pi\)-actions. Thus we have a map of degree one, \(\tilde{f} \times \pi_1: \tilde{M}^n \times \pi_1 L^m \to \tilde{N}^n \times \pi_1 L^m(1 = 1_{L^m}: L^m \to L^m)\), between the orbit spaces of the diagonal \(\pi\)-actions. This map has a natural structure of a normal map of degree one, and the characteristic map of the orientation bundle of \(\tilde{N}^n \times \pi_1 L^m\) is given by \((w_\chi)p_\pi\), where \(p_\pi: \pi_1(\tilde{N}^n \times \pi_1 L^m) \to \pi_1(N^n) = \pi\) is the map induced by the projection \(p\). Thus \(\sigma(\tilde{f} \times \pi_1) \in L^s_{\pi + \pi}(\pi_1(\tilde{N}^n \times \pi_1 L^m), (w_\chi)p_\pi)\), and we denote by the same letter \(\sigma(\tilde{f} \times \pi_1) \in L^s_{\pi + \pi}(\pi, w_\chi)\) its image under the homomorphism induced by \(p\). We define a desired homomorphism by sending \((L_m, \sigma(f))\) to \(\sigma(\tilde{f} \times \pi_1)\).

On the other hand, we can define the \(G\)-\(\chi\)-equivariant Witt group \(W^s_\chi(G, Z)\) (cf. \(\S 3\)) and a homomorphism

\[ \rho: \Omega^s_\chi(G) \to W^s_\chi(G, Z) \]

by setting

\[ \rho([L]^{2k}) = \langle H^k(L^{2k}, Z) / \text{Tor}, \text{the intersection form} \rangle \]
\[ \rho([L^{2k+1}]) = \langle \text{Tor } H^{*+1}(L^{2k+1}, Z) \rangle, \]
where Tor denotes the \( Z \)-torsion subgroup. An algebraic action of \( W^*_h(G, Z) \) on the Wall group of \( \pi \) is defined by the tensor product, \( W^*_h(G, Z) \otimes L^*_h(\pi, w) \rightarrow L^*_h(m+n)(\pi, w\chi) \) (cf. §8).

Our main result is Theorem 2 of §9, which claims that the following diagram is commutative:
\[ \begin{array}{c}
\Omega^*_h(G) \otimes L^*_h(\pi, w) \\
\downarrow \rho \otimes 1 \\
W^*_h(G, Z) \otimes L^*_h(\pi, w) \\
\end{array} \rightarrow \begin{array}{c}
L^*_h(m+n)(\pi, w\chi) \\
\end{array} \]

This can be considered to be a generalization of the product formula of J. Morgan [4] to the equivariant case. The proof is not analogous to Morgan. We use the algebraic surgery theory due to A. Ranicki [5, 6]. The construction of the equivariant analogue \( L^*_{\chi, x}(Z) \) of Ranicki’s symmetric Poncaré cobordism group \( L^*(Z) \) and the isomorphism \( L^*_{\chi, x}(Z) \cong W^*_h(G, Z) \) (Theorem 1, §7) are the main steps to the proof of Theorem 2.

The paper is organized as follows. In §2, we discuss the normal structure of the map \( f \times \pi 1 \). In §§3 and 4, we define the \( G: \chi \)-equivariant Witt group \( W^*_h(G, Z) \) and \( G: \chi \)-equivariant symmetric Poncaré cobordism group \( L^*_{\chi, x}(Z) \). In §§5 and 6, we define homomorphisms
\[ \Phi : W^*_h(G, Z) \rightarrow L^*_{\chi, x}(Z) \quad \text{and} \quad \Psi : L^*_{\chi, x}(Z) \rightarrow W^*_h(G, Z), \]
which will be shown to be the mutual inverses in §7. In §8, we define the algebraic pairing \( W^*_h(G, Z) \otimes L^*_h(\pi, w) \rightarrow L^*_h(\pi, w\chi) \) which is mentioned above concerning the main theorem. In §§9 and 10, the main theorem is presented and proved.

2. Twisted product of a normal map with a \( G: \chi \)-manifold. Let \( f : M^n \rightarrow N^n \) be a map of degree one between \( n \)-dimensional compact PL (or smooth) manifolds with \( \pi_1(N^n) = \pi \). Let \( F : \nu_M \rightarrow \xi \) be a bundle map covering \( f \), where \( \nu_M \) is the stable normal bundle of \( M^n \) and \( \xi \) is a bundle over \( N^n \). The map \( f : M^n \rightarrow N^n \) of degree one equipped with the bundle map data \( F : \nu_M \rightarrow \xi \) is called a normal map of degree one. In case the boundaries \( \partial M^n \) and \( \partial N^n \) are not empty, we assume that the restriction of \( f \) to the boundaries, \( f|\partial M^n : \partial M^n \rightarrow \partial N^n \), is a simple homotopy equivalence. The surgery obstruction \( \delta(f) \in L^*_h(\pi, w) \) of \( f \) to deforming \( f \) to a simple homotopy equivalence relative boundary is defined in [7], where \( w : \pi \rightarrow \{ \pm 1 \} \) is the characteristic map of the orientation bundle of \( N^n \).
Let \( L^m \) be an \( m \)-dimensional closed \( G \times \mathcal{X} \)-manifold and \( \tilde{f} \times \pi_1 : \tilde{M}^n \times \pi_1 L^m \to \tilde{N}^n \times \pi_1 L^m \) the map of degree one defined in §1. We make \( \tilde{f} \times \pi_1 \) a normal map of degree one as follows. For a manifold \( W, rW \) denotes its tangent bundle. Let \( \rho_M : \tilde{M}^n \times \pi_1 L^m \to M^n \) and \( \rho_N : \tilde{N}^n \times \pi_1 L^m \to N^n \) be the projections to the first factors. Then \( \tau(\tilde{M}^n \times \pi_1 L^m) \) is isomorphic to the Whitney sum \( \rho_M^*(\tau M) \oplus \tilde{M} \times \pi_1 L = \rho_M^*(\tau M) \oplus (\tilde{f} \times \pi_1)^*(\tilde{N} \times \pi_1 L) \). Let \( \eta \) be a bundle over \( \tilde{N}^n \times \pi_1 L^m \) such that the Whitney sum \( (\tilde{N} \times \pi_1 L) \oplus \eta \) is trivial. Then the bundle \( (\tilde{M} \times \pi_1 L \oplus (\tilde{f} \times \pi_1)^* \eta) = (\tilde{f} \times \pi_1)^*(\tilde{N} \times \pi_1 L \oplus \eta) \) is trivial. Hence the bundle \( \tau(\tilde{M} \times \pi_1 L) \oplus (\rho_M^*(\nu_M) \oplus (\tilde{f} \times \pi_1)^* \eta) = \rho_M^*(\tau M) \oplus \nu_M \times \pi_1 L \oplus (\tilde{f} \times \pi_1)^* \eta \) is trivial. Therefore we may take the bundle \( \rho_M^*(\nu_M) \oplus (\tilde{f} \times \pi_1)^* \eta \) as the stable normal bundle of \( \tilde{M}^n \times \pi_1 L^m \). The bundle map \( F: \nu_M \to \xi \) can be lifted to the bundle map \( \tilde{F} : \rho_M^*(\nu_M) \to \rho_M^*(\xi) \) canonically, and we obtain the following bundle map.

\[
\begin{array}{ccc}
\rho_M^*(\nu_M) \oplus (\tilde{f} \times \pi_1)^* \eta & \xrightarrow{\tilde{F} + (\tilde{f} \times \pi_1)} & \rho_M^*(\xi) \oplus \eta \\
\tilde{M}^n \times \pi_1 L^m & \xrightarrow{\tilde{f} \times \pi_1} & \tilde{N}^n \times \pi_1 L^m 
\end{array}
\]

where the vertical maps are the bundle projections. Since \( \rho_M^*(\nu_M) \oplus (\tilde{f} \times \pi_1)^* \eta \) is the stable normal bundle of \( \tilde{M}^n \times \pi_1 L^m \), the above diagram gives a structure of a normal map of degree one to the map \( \tilde{f} \times \pi_1 \).

Now the bundle \( \rho_M^*(\nu_N) \oplus \eta \) may be regarded as the stable normal bundle of \( \tilde{N}^n \times \pi_1 L^m \). The difference bundle \( ((\rho_M^*(\nu_N) \oplus \eta) - (\rho_M^*(\xi) \oplus \eta)) \) is the induced virtual bundle \( \rho_N^*(\nu_N - \xi) \). This means that the normal invariant of \( \tilde{f} \times \pi_1 \) endowed with the above bundle data is the image of the normal invariant of \( f \) endowed with \( F \) under the map induced by \( \rho_N, \rho_N^*: [N^n, G/PL] \to [\tilde{N}^n \times \pi_1 L^m, G/PL] \) (or \( \rho_N^*: [N^n, G/O] \to [\tilde{N}^n \times \pi_1 L^m, G/O] \) in the smooth case).

3. **G-\( \mathcal{X} \)-equivariant Witt group.** We denote the ring of integers by \( Z \), the field of rational numbers by \( Q \) and the quotient map from \( Q \) to \( Q/Z \) by \( \omega \). The dual module \( V^* \) of a finitely generated (abbreviated f.g.) free \( Z \)-module \( V \) is defined by \( V^* = \text{Hom}_Z(V, Z) \), and the dual module \( W^* \) of a f.g. torsion \( Z \)-module \( W \) by \( W^* = \text{Hom}_Z(W, Q/Z) \). For a morphism \( \beta : V_1 \to V_2, \beta^* : V_2^* \to V_1^* \) denotes the dual morphism of \( \beta \), where \( V_1 \) and \( V_2 \) are both together either f.g. free \( Z \)-modules or f.g. torsion \( Z \)-modules.

Let \( G \) be a finite group with a homomorphism \( \chi : G \to \{ \pm 1 \} \). Then the integral group ring \( Z[G] \) has the involution \( - \) defined by \( \sum n_g g = \sum n_g \chi(g) g^{-1} \) for \( n_g \in Z \) and \( g \in G \). A f.g. \( G \)-module \( V \) means a left
$\mathbb{Z}[G]$-module which is a f.g. $\mathbb{Z}$-module. For a f.g. $\mathbb{Z}$-free $G$-module $V$, the dual module $V^*$ has a structure of a f.g. $\mathbb{Z}$-free $G$-module defined by $(xu)(v) = u(\bar{x}v)$ for $u \in V^*, v \in V$ and $x \in \mathbb{Z}[G]$. Similarly for a f.g. $\mathbb{Z}$-torsion $G$-module $U$, the dual module $U^*$ is also a f.g. $\mathbb{Z}$-torsion $G$-module. For a f.g. $\mathbb{Z}$-free or $\mathbb{Z}$-torsion $G$-module $V$, the dual module of $V^*$ is canonically identified with $V$ as a $G$-module. A $G$-map between f.g. $G$-modules means a $\mathbb{Z}[G]$-map between them.

Now we define the $G$-$\chi$-equivariant Witt group.

(1) Even dimensional case. The following definition is due to A. Dress [2] when $\chi = 1$. For $\varepsilon = \pm 1$, let us define an $\varepsilon$-symmetric $G$-$\chi$-equivariant form $(V, \alpha)$ to be a f.g. $\mathbb{Z}$-free $G$-module $V$ together with a $G$-isomorphism $\alpha : V \rightarrow V^*$ such that $\alpha = \varepsilon \alpha^*$. In other words, there is a non-singular bilinear pairing $\tilde{\alpha} : V \times V \rightarrow \mathbb{Z}$ such that $\alpha(gv, gv') = \chi(g)\alpha(v, v')$, $\alpha(v, v') = \varepsilon \tilde{\alpha}(v', v)$ and $ad \tilde{\alpha} = \alpha$, where $ad \tilde{\alpha}$ is the adjoint of $\tilde{\alpha}$, $ad \tilde{\alpha}(v) (v') = \tilde{\alpha}(v', v)$, for $v, v' \in V$ and $g \in G$. For any two such forms $(V_1, \alpha_1)$ and $(V_2, \alpha_2)$, one has an orthogonal sum $(V_1, \alpha_1) \oplus (V_2, \alpha_2)$ which is an $\varepsilon$-symmetric $G$-$\chi$-equivariant form as well. A $G$-isomorphism $\beta : (V_1, \alpha_1) \rightarrow (V_2, \alpha_2)$ satisfying $\beta^* \alpha_2 \beta = \alpha_1$ is an isomorphism in our setting. One may form the half-group of isomorphism classes of $\varepsilon$-symmetric $G$-$\chi$-equivariant form with respect to orthogonal sum and its associated universal group $\mathfrak{y}_\varepsilon^G(G, \mathbb{Z})$. One now defines a $G$-lagrangean $P$ of $(V, \alpha)$ to be a $\mathbb{Z}[G]$-submodule of $V$ which coincides with its orthogonal complement $V' = \ker(i^* \alpha : V \rightarrow P^*)$, where $i : P \rightarrow V$ is the inclusion. If $(V, \alpha)$ has a $G$-lagrangean, it is called a split form.

For each integer $k \geq 0$, we define the $G$-$\chi$-equivariant Witt group $W_{2k}(G, \mathbb{Z})$ to be the residue class group of $\mathfrak{y}_{2k}^G(G, \mathbb{Z})$ with respect to the subgroup generated by all split $(-1)^k$-symmetric $G$-$\chi$-equivariant form in $\mathfrak{y}_{2k}^G(G, \mathbb{Z})$.

(2) Odd dimensional case. For $\varepsilon = \pm 1$, an $\varepsilon$-symmetric $G$-$\chi$-equivariant linking form $(S, \lambda)$ is a f.g. $\mathbb{Z}$-torsion $G$-module $S$ together with a $G$-isomorphism $\lambda : S \rightarrow S^*$ such that $\lambda = \varepsilon \lambda^*$. This means that there is a non-singular bilinear pairing $\tilde{\lambda} : S \times S \rightarrow \mathbb{Q}/\mathbb{Z}$ such that $\tilde{\lambda}(gs, gs') = \chi(g)\tilde{\lambda}(s, s')$, $\tilde{\lambda}(s, s') = \varepsilon \tilde{\lambda}'(s', s)$ and $ad\tilde{\lambda} = \lambda$ for $s, s' \in S$ and $g \in G$.

**Definition.** A $G$-resolution of length 1 of an $\varepsilon$-symmetric $G$-$\chi$-equivariant linking form $(S, \lambda)$ is a short exact sequence of $G$-modules

$$0 \longrightarrow U \xrightarrow{\beta} V \xrightarrow{\gamma} S \longrightarrow 0$$

together with a bilinear pairing $\Lambda : V \times V \longrightarrow \mathbb{Q}$ such that

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(i) \(U\) and \(V\) are both f.g. \(\mathbb{Z}\)-free \(G\)-modules.

(ii) \(\Lambda(gv, gv') = \chi(g)\Lambda(v, v')\), \(\Lambda(\beta(u), v) \in \mathbb{Z}\) and \(\Lambda(v, \beta(u)) \in \mathbb{Z}\).

(iii) \(\tilde{\lambda}(\gamma(v), \gamma(v')) = \omega(\Lambda(v, v')) \in \mathbb{Q}/\mathbb{Z}\),

for \(v, v' \in V,\ u \in U\) and \(g \in G\).

**Lemma 3.1.** Let \((S, \lambda)\) be an \(\varepsilon\)-symmetric \(G\times\)equivariant linking form. Then, there is a \(G\)-resolution of length 1 of \((S, \lambda)\).

**Proof.** There are a f.g. \(\mathbb{Z}[G]\)-free module \(V\) and a \(G\)-epimorphism \(\gamma: V \to S\). Let \(U\) be the kernel of \(\gamma\) and \(\beta: U \to V\) the inclusion. Since \(V \otimes_{\mathbb{Z}} V\) is a f.g. \(\mathbb{Z}[G]\)-free module by the diagonal \(G\)-action, there is a bilinear form \(\Lambda: V \times V \to \mathbb{Q}\) such that \(\omega(\Lambda(v, v')) = \tilde{\lambda}(\gamma(v), \gamma(v'))\) and \(\Lambda(gv, gv') = \chi(g)\Lambda(v, v')\) for \(v, v' \in V\). Since \(\beta(U) = \ker \gamma,\ \Lambda(\beta(u), v) \in \mathbb{Z}\) and \(\Lambda(v, \beta(u)) \in \mathbb{Z}\) for \(u \in U\) and \(v \in V\). q.e.d.

For any two \(\varepsilon\)-symmetric \(G\times\)equivariant linking forms \((S_1, \lambda_1)\) and \((S_2, \lambda_2)\), one has an orthogonal sum \((S_1, \lambda_1) + (S_2, \lambda_2)\) which is an \(\varepsilon\)-symmetric \(G\times\)equivariant linking form as well. Two \(\varepsilon\)-symmetric \(G\times\)equivariant linking forms \((S_1, \lambda_1)\) and \((S_2, \lambda_2)\) are called isomorphic if there is a \(G\)-isomorphism \(\delta: S_1 \to S_2\) such that \(\delta^* \lambda_2 \delta = \lambda_1\). One may form the half group of isomorphism classes of \(\varepsilon\)-symmetric \(G\times\)equivariant linking forms with respect to orthogonal sums and its associated universal group \(w_\varepsilon^G(G, \mathbb{Z})\).

Consider the following two conditions on an \(\varepsilon\)-symmetric \(G\times\)equivariant linking form \((S, \lambda)\):

(a) There is a \(G\)-resolution of length 1, \(0 \to U \xrightarrow{\beta} V \xrightarrow{\gamma} S \to 0\), such that the map \(\Lambda V: V \to U^*,\) defined by \(\Lambda V(v)(u) = \Lambda(v, \beta(u))\) for \(v \in V\) and \(u \in U\), is an isomorphism.

(b) There is a \(\mathbb{Z}[G]\)-submodule \(Q\) of \(S\) which coincides with its orthogonal complement \(S^* = \ker(i^* \lambda: S \to Q^*)\), where \(i: Q \to S\) is the inclusion.

For each integer \(k \geq 0\), the \(G\times\)equivariant Witt group \(W^G_{2k-1}(G, \mathbb{Z})\) is defined to be the residue class group of \(w^G_{2k-1}(G, \mathbb{Z})\) with respect to the subgroup generated by those \((-1)^{k+1}\)-symmetric \(G\times\)equivariant linking forms which satisfy either (a) or (b).

4. Equivariant symmetric algebraic Poincaré cobordism group. Let \(G\) be a finite group with a homomorphism \(\chi: G \to \{\pm 1\}\). An \(n\)-dimensional \(G\)-chain complex \((C, d_c)\) is a chain complex.
such that each $C_r$ is a f.g. $\mathbb{Z}$-free $G$-module and each $d_r$ is a $G$-homomorphism. A $G$-chain map $f: C \to D$ between two $G$-chain complexes is a chain map such that each $f_r: C_r \to D_r$ is a $G$-homomorphism. For a $G$-chain complex $(C, d_C)$, the cochain complex $(C^*, d_C^*)$ is a $G$-chain complex, where $C^r = (C_r)^*$ is a f.g. $\mathbb{Z}$-free $G$-module as in §3 for each $r$. The homology and cohomology groups of a $G$-chain complex are f.g. $G$-modules. Given two $G$-chain complexes $(C, d_C)$ and $(D, d_D)$, $\text{Hom}_Z(C^*, D_*)$ is the $G$-chain complex such that $(\text{Hom}_Z(C^*, D_*))_r = \sum_{\rho + \varphi = r} \text{Hom}_Z(C^\rho, D_\varphi)$ with $G$-action defined by $(g\psi)(c) = g(\psi(\chi(g^{-1})c))$ for $g \in G$, $\psi \in \text{Hom}_Z(C^\rho, D_\varphi)$ and $c \in C^\rho$, and the differential is given by $d(\psi) = d_D^*\psi + (-1)^{\rho}d_C^*\psi$ for $\psi \in \text{Hom}_Z(C^\rho, D_\varphi)$. Let $\text{Hom}_Z^G(C^*, D_*)$ be the subcomplex of $\text{Hom}_Z(C^*, D_*)$ consisting of all the $G$-module maps, that is, $\text{Hom}_Z^G(C^*, D_*) = \{ \psi \in \text{Hom}_Z(C^*, D_*) \mid g\psi = \psi \text{ for any } g \in G \}$.

For a $G$-chain complex $(C, d_C)$, the generator $T \in \mathbb{Z}_2$ acts on $\text{Hom}_Z^G(C^*, C_*)$ by the transposition involution

$$T(\{ \psi : C^p \to C_0 \}) = \{ (-1)^p \psi^* : C^0 \to C_p \}. $$

Let $W$ be the free $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex given by

$$W_r = \begin{cases} \mathbb{Z}[\mathbb{Z}_2] & (r \geq 0) \\ 0 & (r < 0) \end{cases} \quad d_r = \begin{cases} 1 + (-1)^rT & (r > 0) \\ 0 & (r \leq 0) \end{cases}.$$

For a $G$-chain complex $(C, d_C)$, we define the equivariant $\mathbb{Z}_2$-hypercohomology group by $Q^G_\mathbb{Z}(C) = H_\mathbb{Z}(\text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \text{Hom}_Z^G(C^*, C_*)))$. An element $\psi \in Q^G_\mathbb{Z}(C)$ is represented by a collection of $G$-chain maps $\{ \psi_s \in \text{Hom}_Z^G(C^{n-r+s}, C_r) \mid r, s \geq 0 \}$ such that

$$d_C\psi_s + (-1)^{r}\psi_s d_C^* + (-1)^{n+s-1}(\psi_{s-1} + (-1)^s T\psi_{s-1}) = 0 \quad (s \geq 0, \psi_{s-1} = 0).$$

An $n$-dimensional symmetric Poincaré $G$-complex $(C, \psi)$ is an $n$-dimensional $G$-chain complex $(C, d_C)$ together with an element $\psi \in Q^G_\mathbb{Z}(C)$ such that the chain map $\psi_0 : C^{n-r} \to C_r$ is a chain equivalence (forgetting the $G$-actions) with $(C^{n-r})_r = C_r$, and $d_C^{n-r} = (-1)^r d_C^{n-r} : C^{n-r} \to C^{n-r+1}$.

A $G$-chain map $f : C \to D$ induces the chain map $\text{Hom}_Z^G(f) : \text{Hom}_Z^G(C^*, C_*) \to \text{Hom}_Z^G(D^*, D_*)$ defined by $\text{Hom}_Z^G(f)(\psi) = \psi f$ for $\psi \in \text{Hom}_Z^G(C^*, C_*)$. This is a $\mathbb{Z}[\mathbb{Z}_2]$-chain map since $T \in \mathbb{Z}_2$ acts as the transposition. Hence, it induces a homomorphism $f^* : Q^G_\mathbb{Z}(C) \to Q^G_\mathbb{Z}(D)$. Let $(C, \psi_C)$ and $(D, \psi_D)$ be two $n$-dimensional symmetric Poincaré $G$-complexes. A $G$-isomorphism $f$ from $(C, \psi_C)$ to $(D, \psi_D)$ is a $G$-chain isomorphism $f$ such that $f^* \psi_C = \psi_D$. 

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A $G$-chain map from $(C, \psi_C)$ to $(D, \psi_D)$ is called a $G$-quasi equivalence if $f^*\psi_C = \psi_D$ and $f$ induces an isomorphism of the homology groups in each dimension. Two $n$-dimensional symmetric Poincaré $G$-complexes $(C, \psi_C)$ and $(D, \psi_D)$ are called $G$-quasi equivalent if there is a sequence of $n$-dimensional symmetric Poincaré $G$-complexes $(C_1, \psi_1), \ldots, (C_m, \psi_m)$ such that $(C_1, \psi_1) = (C, \psi_C), (C_m, \psi_m) = (D, \psi_D)$ and there is a $G$-quasi equivalence either $f_i : C_i \to C_{i+1}$ or $f_i : C_{i+1} \to C_i$ for each $i$ ($i = 1, \ldots, m-1$).

Let $f : C \to D$ be a $G$-chain map. Let $C(\text{Hom}^G(f))$ be the algebraic mapping cone of the chain map $\text{Hom}^G(f)$, that is, the chain complex defined by $(C(\text{Hom}^G(f)))_r = \text{Hom}^G(D^*, D_r) \oplus \text{Hom}^G(C^*, C_r)$, and $d(\theta, \psi) = (d\theta + (-1)^{r-1}\text{Hom}^G(f)(\psi), d\psi)$, where $\theta \in \text{Hom}^G(D^*, D_r)$, and $\psi \in \text{Hom}^G(C^*, C_r)$. This becomes a $\mathbb{Z}[\mathbb{Z}_2]$-chain complex in an obvious way. We define the $(n+1)$-dimensional relative $Q_G$-group by $Q_G^{n+1}(f) = H_{n+1}(\text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C(\text{Hom}^G(f))))$. An element $(\delta\psi, \psi) \in Q_G^{n+1}(f)$ is represented by a collection of $G$-chain map pairs

$$(\delta\psi, \psi)_s = (\delta\psi_s \in \text{Hom}^G(D^{n+1-r+s}, D_r) \oplus \text{Hom}^G(C^{n-r+s}, C_r)), r, s \geq 0,$$

satisfying the following two conditions with $s \geq 0$, $\delta\psi_{-1} = 0$ and $\psi_{-1} = 0$:

(*') $d\delta\psi_s + (-1)^r\delta\psi_s d^* + (-1)^{n-s}(\delta\psi_{s-1} + (-1)^s T\delta\psi_{s-1}) + (-1)^n \text{Hom}^G(f)(\psi_s) = 0$, and

(**') $d\psi_s + (-1)^r\psi_s d^* + (-1)^{n+s-1}(\psi_{s-1} + (-1)^s T\psi_{s-1}) = 0$.

An $(n+1)$-dimensional connected symmetric $G$-pair $(f : C \to D, (\delta\psi, \psi))$ is a $G$-chain map $f$ from an $n$-dimensional Poincaré $G$-complex $C$ to an $(n+1)$-dimensional $G$-chain complex $D$ together with a class $(\delta\psi, \psi) \in Q_G^{n+1}(f)$ which satisfies the condition:

(***) $H_0(C(\Delta)) = 0$, $\Delta : D^{n+1-*} \longrightarrow C(f)_*$,

where $C(f)$ is the algebraic mapping cone of the chain map $f$ and $C(\Delta)$ is the algebraic mapping cone of the chain map $\Delta : D^{n+1-*} \to C(f)_*$ defined by $\Delta(c) = (\delta\psi_0(c), \psi_0 f^*(c)) \in D_* \oplus C_{*-1} = C(f)_*$ for $c \in D^{n+1-*}$. Remark that the condition $H_0(C(f)) = 0$ implies (**'), since we have the exact sequence $\longrightarrow H_0(D^{n+1-*}) \to H_0(C(f)) \to H_0(C(\Delta)) \to 0$.

For an $(n+1)$-dimensional connected symmetric $G$-pair $(f : C \to D, (\delta\psi, \psi))$, define the $n$-dimensional symmetric Poincaré $G$-complex $(C', \psi')$ as follows:

$$d_{C'} = \begin{bmatrix}
    d_C & 0 & (-1)^{n+1} \psi_0 f^* \\
    (-1)^r f & d_D & (-1)^r \delta\psi_0 \\
    0 & 0 & (-1)^r d\psi
\end{bmatrix}$$

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\[ C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1} \rightarrow C'_r = C_{r-1} \oplus D_r \oplus D^{n-r+2} \]

\[ \psi'_0 = \begin{bmatrix} -1 \cdot f & (\psi_{s+1})' \end{bmatrix} \]

\[ C'^{n-r} = C'^{n-r+1} \oplus D_{r+1} \rightarrow C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1} \]

\[ \psi''_s = \begin{bmatrix} 0 & (\psi_{s+1})' \end{bmatrix} \]

\[ C'^{n-r+s} = C'^{n-r+s+1} \oplus D_{r+s+1} \rightarrow C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1} (s \geq 1) \]

We call \((C', \psi')\) the \(n\)-dimensional symmetric Poincaré G-complex obtained from \((C, \psi)\) by symmetric G-surgery on a connected \((n+1)\)-dimensional symmetric G-pair \((f : C \rightarrow D, (\delta \psi, \psi))\). It may be verified that performing symmetric G-surgery using a different cycle representative of \((\delta \psi, \psi) \in Q^{n+1}(f)\) leads to an isomorphic symmetric Poincaré G-complex \((C', \psi')\).

In the above situation, the following two conditions are equivalent,

1. \((C')\) is acyclic,
2. the relative homology class \((\delta \psi_0, \psi_0) \in H_{n+1} G(f)\) induces the isomorphisms \(H_n D, C) = H_{n+1} G(f)\) induces the isomorphisms \(H_n D, C) = H_{n+1} (D, \delta \psi, \psi)\).

In such a case, \((f : C \rightarrow D, (\delta \psi, \psi))\) is called an \((n+1)\)-dimensional Poincaré G-pair with boundary \((C, \psi)\), and \((C, \psi)\) is called G-null-cobordant.

The direct sum of \(n\)-dimensional symmetric Poincaré G-complexes \((C, \psi)\) and \((C', \psi')\) is an \(n\)-dimensional symmetric Poincaré G-complex \((C \oplus C', \psi \oplus \psi')\), where \((\psi \oplus \psi')_s = \psi_s \oplus \psi'_s : C'^{n-r+s} \oplus C'^{n-r+s} \rightarrow C_r \oplus C'_r (s, r \geq 0)\).

**Lemma 4.1.** Let \((C', \psi')\) be an \(n\)-dimensional symmetric Poincaré G-complex obtained from an \((n+1)\)-dimensional connected symmetric G-pair \((f : C \rightarrow D, (\delta \psi, \psi))\). Then the direct sum \((C, \psi) \oplus (C', \psi')\) is G-null-cobordant.

**Proof.** Define an \((n+1)\)-dimensional symmetric G-pair \((h : C \oplus C' \rightarrow D', (0, (0, \psi \oplus (- \psi')))\) by \(D'_r = C_r \oplus D^{n-r+1}\),

\[ d_D = \begin{bmatrix} d_C & (-1)^{n+1} \psi_0 f^* \\ 0 & (-1)^r d^*_0 \end{bmatrix} : D_r \rightarrow D_{r-1}, \]

and

\[ h(c) = (c, 0) \text{ for } c \in C \]
\[ h(c') = (c_1, c_3) \text{ for } c' = (c_1, c_2, c_3) \in C'_r = C_r \oplus D_{r+1} \oplus D^{n-r+1}. \]
Let \((C', \psi')\) be the symmetric Poincaré \(G\)-complex obtained from \((C, \psi) \oplus (C', -\psi')\) by symmetric \(G\)-surgery on the above \(G\)-pair. Then one can verify that \(C'\) is acyclic. Hence the above \(G\)-pair is an \((n+1)\)-dimensional Poincaré \(G\)-pair and \((C, \psi) \oplus (C', -\psi')\) is \(G\)-null-cobordant. q.e.d.

We form the half group of the isomorphism classes of \(n\)-dimensional symmetric Poincaré \(G\)-complexes with respect to orthogonal sums and its associated universal group \(X^n_{\mathbb{Z}}(\mathbb{Z})\). Let \(U^n_{\mathbb{Z}}(\mathbb{Z})\) be the subgroup of \(X^n_{\mathbb{Z}}(\mathbb{Z})\) generated by the isomorphism classes of \(n\)-dimensional symmetric Poincaré \(G\)-complexes which are \(G\)-quasi equivalent to \(n\)-dimensional \(G\)-null-cobordant symmetric Poincaré \(G\)-complexes. Let us define the \(n\)-dimensional \(G\)-equivariant symmetric algebraic cobordism group \(L^n_{\mathbb{Z}}(\mathbb{Z})\) by \(L^n_{\mathbb{Z}}(\mathbb{Z}) = X^n_{\mathbb{Z}}(\mathbb{Z})/U^n_{\mathbb{Z}}(\mathbb{Z})\). Not that by Lemma 4.1., if \((C', \psi')\) is obtained from \((C, \psi)\) by symmetric \(G\)-surgery, they represent the same element in \(L^n_{\mathbb{Z}}(\mathbb{Z})\).

5. The map \(\Phi\). We define a homomorphism \(\Phi: \mathfrak{W}_G(G, \mathbb{Z}) \to L^n_{\mathbb{Z}}(\mathbb{Z})\).

(1) Even dimensional case. Let \((V, a)\) be a \((-1)^k\)-symmetric \(G\)-equivariant form as in §3 \((k \geq 0)\). Define a \(2k\)-dimensional symmetric Poincaré \(G\)-complex \((C_V, \psi_a)\) by

\[
(C_V)_r = \begin{cases} V^* & (r = k) \\ 0 & (r \neq k) \end{cases}, \quad d_{C_V} = 0
\]

and

\[
(\psi_a)_0 = a: (C_V)^k = V \to (C_V)_k = V^*, \\
(\psi_a)_s = 0 & (s \geq 1).
\]

**Lemma 5.1.** If \((V, a)\) is split, then \((C_V, \psi_a)\) is \(G\)-null-cobordant.

**Proof.** Let \(P\) be a \(G\)-lagrangian of \(V\), and \(i: P \to V\) the inclusion. Let \((f: C \to D, (0, \psi_a))\) be the \((2k+1)\)-dimensional connected symmetric \(G\)-pair defined by

\[
f = \begin{cases} i^* : (C_V)_k = V^* \to D_h = P^* \\ 0 : (C_V)_r = 0 \to D_r = 0 \quad (r \neq k). \end{cases}
\]

The conditions (*) and (**) in §4 are verified, because the composition \(i^* a i\) is trivial and hence \(\text{Hom}^G(f) = 0\). And we have easily \(H_0(C(f)) = 0\) and the condition (***) in §4. Let \((C', \psi')\) be the \(2k\)-dimensional symmetric Poincaré \(G\)-complex obtained from \((C_V, \psi_a)\) by symmetric \(G\)-surgery on the above \(G\)-pair. Then \(C'\) has the form...
... \rightarrow 0 \rightarrow P \xrightarrow{-ai} V^* \xrightarrow{(-1)^k\iota^*} P^* \rightarrow 0 \rightarrow ...

and it is acyclic. q.e.d.

By the above lemma, we obtain a well defined homomorphism $\Phi : W_{2k}(G, Z) \rightarrow L^2_{2k}(Z)$ by putting $\Phi((V, \alpha)) = [(C_v, \psi_v)]$.

(2) Odd dimensional case. Let $(S, \lambda)$ be a $(-1)^{k+1}$-symmetric equivariant linking form. Take a resolution $0 \rightarrow U \xrightarrow{\beta} V \xrightarrow{\gamma} S \rightarrow 0$ and a bilinear pairing $\Lambda : V \times V \rightarrow Q$ satisfying (i), (ii) and (iii) of (2) in §3. For $v, v' \in V$, put $\mu(v, v') = \Lambda(v, v') - (-1)^{k+1} \Lambda(v', v)$. Then $\mu(v, v') \in Z$ and $\mu(v, v') = (-1)^k \mu(v', v)$. Let $\Lambda_U : U \rightarrow V^*$ and $\Lambda_V : V \rightarrow U^*$ be the maps defined by $(\Lambda_U(u))(v) = \Lambda(\beta(u), v)$ and $(\Lambda_V(v))(u) = \Lambda(v, \beta(u))$ for $u \in U$ and $v \in V$ respectively.

Let $(C, \psi)$ be the $(2k+1)$-dimensional symmetric Poincaré $G$-complex defined by

$$C_r = \begin{cases} V^* & (r = k+1) \\ U^* & (r = k) \\ 0 & (r \neq k, k+1) \end{cases}$$

$$d_r = \begin{cases} \beta^* & (r = k+1) \\ 0 & (r \neq k+1) \end{cases}$$

$$\psi_0 = \begin{cases} \Lambda_U : \hat{U} \rightarrow V^* & (r = k+1) \\ \Lambda_V : V \rightarrow U^* & (r = k) \\ 0 & (r \neq k, k+1) \end{cases}$$

$$\psi_1 = \begin{cases} \text{ad} \mu : V \rightarrow V^* & (r = k+1) \\ 0 & (r \neq k+1) \end{cases}$$

$$\psi_s = 0 \quad (s \geq 2).$$

**Lemma 5.2.** If $(S, \lambda)$ satisfies the condition (b) in (2), §3, then $(C, \psi)$ is $G$-null-cobordant.

**Proof.** Let $Q$ be a $Z[G]$-submodule of $S$ as in (2), §3. Put $V_i = \gamma^{-1}(Q)$ and $U_i = \gamma^{-1}(V_i)$. Then there is a short exact sequence $0 \rightarrow U_1 \xrightarrow{\beta_1} V_1 \xrightarrow{\gamma_1} Q \rightarrow 0$, where $\beta_1 = \beta|U_1$ and $\gamma_1 = \gamma|V_1$ are the restrictions. Let $i : Q \rightarrow S, i_\nu : V_1 \rightarrow V$ and $i_\nu : U_1 \rightarrow U$ be the inclusions. Since $i^* \lambda i : Q \rightarrow Q^*$ is trivial, the pairing takes integral values on $V_1 \times V_1$. Denote this pairing by $\Lambda_1 : V_1 \times V_1 \rightarrow Z$. Define the adjoint map $\Lambda_1 : V_1 \rightarrow V_1^*$ by $(\text{ad} \Lambda_1(v))(v') = \Lambda_1(v, v')(v', v' \in V_1)$. Let $(f : C \rightarrow D, (\delta \psi, \psi))$ be the $(2k+2)$-dimensional connected symmetric $G$-pair defined by

$$f = \begin{cases} i_\nu^* : C_{k+1} = V^* \rightarrow D_{k+1} = V_1^* & (r = k+1) \\ i_\nu^* : C_k \rightarrow U^* \rightarrow D_k = U_1^* & (r = k) \\ 0 \rightarrow D_r & (r \neq k, k+1) \end{cases}$$

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$\delta_D = \beta_i^k (r = k+1)$ and $0 (r \neq k+1)$.

$$(\delta \psi)_0 = \begin{cases} \text{ad } \Lambda_1 : D^{k+1} = V_1 \to D_{k+1} = V_1^* & (r = k+1) \\ 0 & (r \neq k+1) \end{cases}$$

$(\delta \psi)_s = 0 (s \geq 1)$.

Let $(C', \psi')$ be the $(2k+1)$-dimensional symmetric Poincaré $G$-complex obtained from $(C, \psi)$ by $G$-surgery on the above $G$-pair. Then $C'$ has the form

$$\cdots \to 0 \to U_1 \xrightarrow{(1)} V^* \xrightarrow{(2)} V_1 \oplus U_1^* \xrightarrow{(3)} U_1^* \oplus V_1 \xrightarrow{(4)} V^* \xrightarrow{(5)} U_1^* \xrightarrow{(6)} U_1^* \to 0 \to \cdots$$

where the maps are given as follows: $(1) = \Lambda_1 v_i v$, $(2) = (-1)^{k+2} \beta$, $(3) = (-1)^{k+1} i^2$, $(4) = (-1)^{k+1} \Lambda_1 v_i v$, $(5) = (-1)^{k+1} \text{ad } \Lambda_1$ and $(6) = (-1)^k i^2$. Since $(S, \lambda)$ is a non-singular linking form, this chain complex is acyclic. q.e.d.

**Lemma 5.3.** The class $[(C, \psi)]$ in $L^{2k+1}_G(S)$ does not depend on the particular choice of a resolution of $(S, \lambda)$.

**Proof.** Let $0 \to U \xrightarrow{\beta} V \xrightarrow{\gamma} S \to 0$ and $0 \to U' \xrightarrow{\beta'} V' \xrightarrow{\gamma'} S \to 0$ be two resolutions of $S$ with the associated bilinear pairings $\Lambda : V \times V \to Q$ and $\Lambda' : V' \times V' \to Q$, respectively. Let $(C, \psi)$ and $(C', \psi')$ be the $(2k+1)$-dimensional symmetric Poincaré $G$-complexes corresponding to the two resolutions respectively constructed as before (Lemma 5.2.). The exact sequence $0 \to U \oplus U' \xrightarrow{\beta \oplus \beta'} V \oplus V' \xrightarrow{\gamma \oplus \gamma'} S \oplus S \to 0$ and the bilinear pairing $\Lambda \oplus (-\Lambda') : V \times V' \oplus V \times V' \to Q$ gives a resolution of $(S, \lambda) \oplus (S, -\lambda)$. The $(2k+1)$-dimensional symmetric Poincaré $G$-complex corresponding to this resolution is $(C, \psi) \oplus (C', -\psi')$ which is $G$-null-cobordant by Lemma 5.2., since $(S, \lambda) \oplus (S, -\lambda)$ has a $Z[G]$-submodule $Q = \{(s, s) \in S \oplus S \mid s \in S\}$ satisfying (b) in (2), §3. This implies that $[(C, \psi)] = [(C', \psi')]$ in $L^{2k+1}_G(S)$. q.e.d.

For a $(-1)^{k-1}$-symmetric $G$-equivariant linking form $(S, \lambda)$, the above $(2k+1)$-dimensional symmetric Poincaré $G$-complex $(C, \psi)$ is denoted by $(C_s, \psi)$.

**Lemma 5.4.** If $(S, \lambda)$ satisfies the condition (a) in (2), §3, then
(Cs, ψₙ) is G-null-cobordant.

Proof. Let $0 \to U \xrightarrow{β} V \xrightarrow{γ} S \to 0$ be a resolution of $S$ satisfying the condition (a). Let $(f : C_s \to D, (0, ψₙ))$ be the $(2k+2)$-dimensional connected symmetric $G$-pair defined by

$$f = \begin{cases} 1 : (C_s)_{k+1} = V^* \to D_{k+1} = V^* & (r = k+1) \\ 0 : (C_s)^e_0 = U^* \to D_k = 0 & (r = k) \\ 0 : (C_s)^e_0 = D_r = 0 & (r \neq k, k+1). \end{cases}$$

Let $(C', ψ')$ be the $(2k+1)$-dimensional symmetric Poincaré $G$-complex obtained from $(Cs, ψₙ)$ by $G$-surgery on the above $G$-pair. Then $C'$ Has the form

$$\cdots \to 0 \to V \xrightarrow{γ} U^* \to 0 \to \cdots$$

$\oplus \xrightarrow{β^*} \oplus$

$$V^* \xrightarrow{(-1)^k+1} V^*$$

and it is acyclic. Hence $(Cs, ψₙ)$ is G-null-cobordant. q.e.d.

From Lemma 5.2., 5.3. and 5.4., we obtain a well-defined homomorphism $Φ : W^G_{2k+1}(G, Z) \to L^G_{2k+1}(Z)$ by putting $Φ((S, λ)) = [(Cs, ψₙ)].$

6. The map $Ψ.$ We define a homomorphism $Ψ : L^G_{2k+1}(Z) \to W^G_{2k+1}(G, Z).$

(1) Even dimensional case. Let $(C, ψ)$ be a 2$k$-dimensional Poincaré $G$-complex. Put $\hat{H}^k(C) = H^k(C)/Tor$, where $H^k(C)$ is the $k$-th cohomology group of $C$ and Tor is its torsion subgroup. Let $α : \hat{H}^k(C) \to (\hat{H}^k(C))^*$ be the map defined by $α(x)(y) = c'(ψ₀(c))$, where $x, y \in \hat{H}^k(C)$ and $x = [c], y = [c']$ for $c, c' \in C^k$. Then, the pair $(\hat{H}^k(C), α)$ defines a $(-1)^k$-symmetric $G$-$χ$-equivariant form.

Lemma 6.1. The correspondence $(C, ψ) \to (\hat{H}^k(C), α)$ induces a well-defined homomorphism $Ψ : L^G_{2k+1}(Z) \to W^G_{2k+1}(G, Z).$

Proof. Let us assume that $(C, ψ)$ is $G$-null-cobordant. There is a $(2k+1)$-dimensional symmetric Poincaré $G$-pair with boundary $(C, ψ), (f : C \to D, (δψ, ψ)).$ By $(*)$ in §4,

$$dδψ₀ + (-1)^rδψ₀d^* = (-1)^{Hom^G(ψ₀)} : D^{n-r} \to D_r \quad (0 \leq r \leq n+1),$$

the following diagram is commutative up to sign:
\[ \cdots \to H^k(D) \xrightarrow{f^*} H^k(C) \xrightarrow{f_*} H^{k+1}(D, C) \to \cdots \]
\[ \cdots \to H_{k+1}(D, C) \xrightarrow{f_*} H_k(C) \xrightarrow{f_*} H_k(D) \to \cdots \]

where the two horizontal sequences are the exact sequences of the homology and cohomology groups of the pair \((f : C \to D)\), and the vertical maps are the isomorphisms induced by \(\delta \psi_0\) and \(\psi_0\). The standard argument of the Poincaré duality shows that \(f^*(H^k(D))/\text{Tor}\) is a \(G\)-lagrangean of \((\widetilde{H}^k(C), a)\).

Finally it is clear that, if \((C, \psi)\) and \((C', \psi')\) are \(G\)-quasi equivalent, then \((\widetilde{H}^k(C), a)\) and \((\widetilde{H}^k(C'), a)\) are mutually isomorphic. q.e.d.

(2) Odd dimensional case. Let \((C, \psi)\) be a \((2k+1)\)-dimensional symmetric Poincaré \(G\)-complex. Put \(Z' = \ker(d^* : C^r \to C^{r+1})\). \(V = \ker(q : Z^{k+1} \to H^{k+1}(C)/\text{Tor})\) \((q\) is the projection) and \(U = C^k/Z^k\). Then \(d^*\) induces a \(G\)-homomorphism \(\beta : U \to V\), and \(V/\beta(U)\) is isomorphic to \(\text{Tor} H^{k+1}(C)\). Define a bilinear pairing \(\Lambda : V \times V \to Q\) by \(\Lambda(v, v') = (1/m)c(\psi_0(v))\), where \(v, v' \in V, mv = d^*c(m : \text{integer } \neq 0, c \in C^k = (C_k)^*)\) and \(\psi_0 : C^{k+1} \to C_k\). If \(v\) or \(v'\) is in the image \(d^*(C^k)\), then \(\Lambda(v, v') \in \mathbb{Z}\).

Hence \(\Lambda_\beta(u, v) \in \mathbb{Z}\) and \(\Lambda(v, \beta(u)) \in \mathbb{Z}\) for \(u \in U\) and \(v \in V\), and \(\Lambda\) induces a well-defined pairing \(\tilde{\lambda} : \text{Tor} H^{k+1}(C) \times \text{Tor} H^{k+1}(C) \to Q/\mathbb{Z}\) by \(\tilde{\lambda}(x, y) = \omega(\Lambda(v, v'))\) for \(x = \gamma(v)\) and \(y = \gamma(v')\), where \(\gamma : V \to V/\beta(U) = \text{Tor} H^{k+1}(C)\) is the projection and \(\omega : Q \to Q/\mathbb{Z}\) is the quotient map. From the equation

\[ \psi_0 + (-1)^k \psi_0^* = -(d\psi_1 + (-1)^k \psi_1 d^*) : C^{k+1} \to C_k. \]

it follows that \(\Lambda(v, v') + (-1)^k \Lambda(v, v) \in \mathbb{Z}\) for \(v, v' \in V\). Hence \(\tilde{\lambda}(x, y) = (-1)^{k+1} \tilde{\lambda}(y, x)\) for \(x, y \in \text{Tor} H^{k+1}(C)\). Since \(\psi_0\) is a \(G\)-map, \(\Lambda(gv, gv') = \chi(g) \Lambda(v, v')\) and \(\tilde{\lambda}(gx, gy) = \chi(g) \tilde{\lambda}(x, y)\) for \(g \in G, v, v' \in V\) and \(x, y \in \text{Tor} H^{k+1}(C)\). Let \(\lambda : \text{Tor} H^{k+1}(C) \to (\text{Tor} H^{k+1}(C))^*\) be the adjoint map of \(\tilde{\lambda}\). Then \(\lambda\) is an isomorphism, since \(\psi_0\) induces an isomorphism \(H^{k+1}(C) \to H_k(C)\) and \(\text{Tor} H_k(C) \to (\text{Tor} H^{k+1}(C))^*\) by the universal coefficient theorem. Consequently the pair \((\text{Tor} H^{k+1}(C), \lambda)\) is a \((-1)^{k+1}\)-symmetric \(G\)-equivariant linking form in the sense of §3.

Lemma 6.2. The correspondence \((C, \psi) \mapsto (\text{Tor} H^{k+1}(C), \lambda)\) induces a well-defined homomorphism \(\Psi : L_{2k+1}(Z) \to W_{2k+1}(G, Z)\).

Proof. Let us assume that \((C, \psi)\) is \(G\)-null-cobordant. There is a \((2k+2)\)-dimensional symmetric Poincaré \(G\)-pair with boundary \((C, \psi), (f : C \to D, (\delta \psi, \psi))\). Consider the following commutative diagram,
where the horizontal middle sequence is the cohomology exact sequence of the pair \((f: C \to D)\), and the upper vertical maps are the inclusions and the lower ones are the quotient maps. Let \(Q\) be the image \(f^*(\text{Tor} \, H^{k+1}(D))\) and \(j: Q \to \text{Tor} \, H^{k+1}(C)\) the inclusion. The orthogonal complement of \(Q\) with respect to \(\lambda\). \(Q^\perp = \ker(f^* \lambda: \text{Tor} \, H^{k+1}(C) \to Q^\perp)\), coincides with \(f^*(H^{k+1}(D)) \cap \text{Tor} \, H^{k+1}(C)\). Let \(S\) be the quotient module \(Q^\perp/Q\). Let \(\lambda_S: S \to S^* = \text{Hom}_2(S, Q/Z)\) be the map defined by \((\lambda_S(s))(s') = \lambda(s)(s')\) for \(s, s' \in S\). This is well-defined and a \(G\)-isomorphism by the duality. Put \(V = \ker(f^*: H^{k+1}(D)/\text{Tor} \to H^{k+1}(C)/\text{Tor})\). There is an epimorphism \(\gamma: V \to S\). Put \(U = f^*(H^{k+1}(D)/\text{Tor}) \cap V\). Let \(\beta: U \to V\) be the inclusion. There is a short exact sequence of \(G\)-modules \(0 \to U \overset{\beta}{\to} V \overset{\gamma}{\to} S \to 0\).

The duality maps \((\psi_0)_*: H^{k+1}(D) \to H_{k+1}(D, C)\) and \((\psi_0)_*: H^{k+1}(D, C) \to H_{k+1}(D)\) induce \(G\)-isomorphisms \(\Lambda_U: U \to V^*\) and \(\Lambda_V: V \to U^*\) such that \(\Lambda_V = (-1)^{k+1}\Lambda_U^*\). Since \(\beta \otimes Q: U \otimes Q \to V \otimes Q\) is an isomorphism, these define a bilinear pairing \(\Lambda: V \times V \to Q\) such that \(\Lambda = (-1)^{k+1}\Lambda^*\) and \(\Lambda(gv, gv') = \chi(g)\Lambda(v, v')\) for \(v, v' \in V\) and \(g \in G\). By the construction, \(\omega(\Lambda(v, v')) = \lambda_S(\gamma(v), \gamma(v')) \in Q/Z\), for \(v, v' \in V\). Hence these give a \(G\)-resolution of length 1 of \((S, \lambda_S)\) satisfying the condition (a) in (2), §3. Now the direct sum \((S, -\lambda_S) \oplus (\text{Tor} \, H^{k+1}(C), \lambda)\) has a \(Z[G]\)-submodule \(Q' = \langle (\bar{x}, x) \in S \oplus \text{Tor} \, H^{k+1}(C) \mid x \in Q \rangle\) and \(\bar{x}\) is the class of \(x\) in \(Q'/Q\) and it satisfies the condition (b) in (2), §3. Therefore \((\text{Tor} \, H^{k+1}(C), \lambda)\) represents 0 in \(W_{k+1}(G, Z)\).

Finally it is clear that if \((C, \psi)\) and \((C', \psi')\) are \(G\)-quasi equivalent, then \((\text{Tor} \, H^{k+1}(C), \lambda)\) and \((\text{Tor} \, H^{k+1}(C'), \lambda')\) are isomorphic. q.e.d.

7. \(\Phi\) and \(\Psi\) are mutual inverses. In the preceding two sections, we have constructed the two homomorphisms \(\Phi: W^G_k(G, Z) \to L^*_G(Z)\) and \(\Psi: L^*_G(Z) \to W^G_k(G, Z)\). By the definitions, \(\Psi \Phi = \text{the identity}\). In this section, we shall prove \(\Phi \Psi = \text{the identity}\). (1) Even dimensional case. Let \((C, \psi)\) be a \(2k\)-dimensional symmetric
Poincaré $G$-complex. Put $Z_k = \ker(d : C_k \to C_{k-1})$, $B = C_k/Z_k$. These are f.g. $\mathbb{Z}$-free $G$-modules. Let $p : C_k \to B$ be the projection. There is an injective $G$-homomorphism $\tilde{d} : B \to C_{k-1}$ such that $d = dp : C_k \to C_{k-1}$.

**Lemma 7.1.** Let $p_\# : \text{Hom}(C^*, B) \to \text{Hom}(B^*, B)$ be the map induced by $p$. Then $p_\# \psi_0 = 0$, where $\psi_0 : C^* \to C_k$ is the $k$-th component of $\psi_0$.

**Proof.** Since coker $(d^* : (C_{k-1})^* \to B^*)$ is a torsion group, for each $c \in B^*$ there is an integer $m \neq 0$ and $c' \in C^{k-1}$ such that $mc = d^*c'$ and so $mp^*c = d^*c'$. Then $mp_0(p^*c) = \psi_0(d^*c') = (-1)^{k+1}d(\psi_0(c'))$ is in $Z_k$. Hence $mp_0(p^*c) = 0$, and so $p_\# \psi_0(p^*c) = 0$. q.e.d.

Consider the $(2k+1)$-dimensional connected symmetric $G$-pair $(g : C \to D, (0, \psi))$ defined by

$$f = \begin{cases} 0 : C_r \to D_r = 0 & (r \geq k+1) \\ p : C_k \to D_k = B & (r = k) \\ 1 : C_r \to D_r = C_r & (r \leq k-1) \end{cases}$$

where the differential of $D, d_\psi$ is given by $(d_\psi)_r = 0$ for $r = k+1, (d_\psi)_k = \tilde{d}$ and $(d_\psi)_r = (d_\phi)_r$ for $r \leq k-1$. By the above lemma, this $G$-pair is well defined. Let $(C', \psi')$ be the $2k$-dimensional symmetric Poincaré $G$-complex obtained from $(C, \psi)$ by $G$-surgery on the above $G$-pair.

**Lemma 7.2.** $C_k' = C_k$ and $\psi_0' = \psi_0 : C'^k \to C_k = C_k$. The homology groups of $C'$ are given by $H_k(C') = H_k(C)/\text{Tor}$ and $H_r(C') = 0$ for $r \neq k$.

**Proof.** The first assertion is clear from the definition. Now, $C'$ has the form

$$\cdots \to C_{k+2} \to C_{k+1} \to C_k \to C_{k-1} \to C_{k-2} \to C_{k-3} \to \cdots$$

where the differential of $D, d_\psi$ is given by $(d_\psi)_r = 0$ for $r = k+1, (d_\psi)_k = \tilde{d}$ and $(d_\psi)_r = (d_\phi)_r$ for $r \leq k-1$. By the above lemma, this $G$-pair is well defined. Let $(C', \psi')$ be the $2k$-dimensional symmetric Poincaré $G$-complex obtained from $(C, \psi)$ by $G$-surgery on the above $G$-pair.

and it follows that $H_r(C') = 0$ for $r \leq k-1$, hence by duality $H_r(C') = 0$ for $r \geq k+1$, and $H_k(C')$ is $\mathbb{Z}$-free. The $k$-th cycle group $Z_k = \ker(d_{C'} : C'_k \to C'_{k-1})$ coincides with $Z_k$. Therefore $H_k(C')$ is isomorphic to some quotient module of $H_k(C)$. But it may be seen that $\psi_0(B^*) \subset \ker$ (the quotient map $Z_k \to H_k(C)/\text{Tor}$), hence $H_k(C')$ must be isomorphic to $H_k(C)/\text{Tor}$. q.e.d.
Put $Z'^k = \ker (d\xi : C'_{\infty} \to C'^{k+1})$. Let $i : Z'^k \to C'^k$ be the inclusion, and $i^* : C'_k = (C'^k)^* \to (Z'^k)^*$ its dual map. There is a $G$-homomorphism $d' : (Z'^k)^* \to C'_{k-1}$ such that $d_{C'} = d' i^* : C'_k \to C'_{k-1}$. Let $q : Z'^k \to H^k(C') = H^k(C)/\Tor$ be the projection. Then the sequence $0 \to (H^k(C)/\Tor)^* \xrightarrow{q^*} (Z'^k)^* \xrightarrow{d'} C'_{k-1}$ is exact. Define the $G$-chain map $h : C' \to C''$ by

$$h = \begin{cases} 0 : C'_r \to C''_r = 0 & (r \geq k+1) \\ i^* : C'_k = C_k \to C''_k = (Z'^k)^* & (r = k) \\ 1 : C'_r \to C''_r = C'_r & (r \leq k-1) \end{cases}$$

where the differential of $C''$ is given by $(d_{C''})_r = 0 (r \geq k+1)$, $(d_{C''})_k = d'$ and $(d_{C''})_r = (d_{C'})_r (r \leq k-1)$. Then $h$ induces the isomorphisms of the homology groups. Put $\psi'' = h^* \psi' \in Q^k(C'')$. Then $\psi'' = i^* \psi_0 i : C'^k = Z'^k \to C''_k = (Z'^k)^*$ and $\psi''_0 = 0 (s \geq 1)$. $(C'', \psi'')$ is a $2k$-dimensional Poincaré $G$-complex and $h : (C', \psi') \to (C'', \psi'')$ is a $G$-quasi equivalence. Now, set $\Phi \Psi(C, \psi) = (C, \psi')$. By definition, $C'_k = (H^k(C)/\Tor)^*$ and $\overline{\psi}_0 = (\psi_0)_* : H^k(C) \to H_k(C)/\Tor = (H^k(C))^*$ and $\overline{\psi}_s = 0$ for $s \geq 1$. Define the $G$-chain map $e : C' \to C''$ by

$$e = \begin{cases} 0 : C'_r \to C''_r = 0 & (r \geq k+1) \\ q^* : C'_k = (H^k(C))^* \to C''_k = (Z'^k)^* & (r = k) \\ 0 : C'_r \to C'_r & (r \leq k-1) \end{cases}$$

Then $e$ induces the isomorphisms of the homology groups.

**Lemma 7.3.** $e^* \overline{\psi} = \psi''$.

**Proof.** It suffices to show that the right hand square of the following diagram is commutative:

\[
\begin{array}{ccc}
C^* & \xrightarrow{i} & Z'^k \\
\downarrow \psi_0 & & \downarrow \psi'' \\
C'_k & \xrightarrow{j^*} & (Z'^k)^* \\
\end{array}
\quad \begin{array}{ccc}
q & \quad & q \\
\quad & \equiv & \\
H^k(C) & \xrightarrow{\psi_0} & (H^k(C))^* \\
\end{array}
\]

Clearly the left hand square is commutative. For each $b \in \ker q \subset Z'^k$, there exists an integer $m \neq 0$ and $c \in C'^{k-1}$ such that $d^* c = mb$. Then, for each $a \in Z'^k$, $ma(\psi'_0(b)) = a(\psi'_0(d^*c)) = (-1)^{k+1} a d\psi'_0(c) = (-1)^{k+1} (d^*a)(\psi'_0(c)) = 0$. Hence $a(\psi'_0(b)) = 0$, and so $\psi'_0(\ker q) = 0$. Similarly, $mb(\psi'_0(a)) = (d^*c)(\psi'_0(a)) = c (d\psi'_0(a)) = (-1)^k c (\psi'_0 (d^*a)) = 0$. This implies $b(\psi'_0(a)) = 0$, and so $\psi'_0(\ker q) = 0$. Consequently, $\psi'' = q^* \overline{\psi} q$ for some
\[ \tilde{\psi} : H^k(C) \to (H^k(C))^\ast \]. Since \( \psi_0^\ast \) induces the same map as \( \psi_0 \) on \( H^k(C) \), we get \( \tilde{\psi} = \tilde{\psi_0} \). q.e.d.

By the above lemma, \( e \) gives a \( G \)-quasi equivalence from \( \Phi \Psi(C, \psi) \) to \( (C''', \psi''') \), and this proves that \( \Phi \Psi = \) the identity.

(2) Odd dimensional case. Let \( (C, \psi) \) be a \((2k+1)\)-dimensional symmetric Poincaré \( G \)-complex. Put \( Z_k = \ker(d : C_k \to C_{k-1}) \). \( R = \ker \) (the quotient map: \( Z_k \to H_k(C)/\text{Tor} \)), and \( K = C_k/R \). \( K \) is a f.g. \( \mathbb{Z} \)-free \( G \)-module. Let \( \varphi : C_k \to K \) be the quotient map. There is a \( G \)-homomorphism \( \varphi : K \to C_{k-1} \) such that \( d = \varphi \cdot \psi : C_k \to C_{k-1} \). Let \( (f : C \to D, (0, \psi)) \) be the \((2k+2)\)-dimensional connected symmetric \( G \)-pair defined by

\[
\begin{align*}
f & = \begin{cases} 
0 : C_r \to D_r = 0 & (r \geq k+1) \\
\varphi : C_k \to D_k \to K & (r = k) \\
1 : C_r \to D_r \to C_r & (r \leq k-1) 
\end{cases}
\end{align*}
\]

where the differential of \( D, d_D \), is given by \((d_D)_{r} = 0 (r \leq k+1), (d_D)_{r} = \varphi \) and \((d_D)_{r} = (dc_{r})_{r} (r \leq k-1) \). Let \( (C', \psi') \) be the \((2k+1)\)-dimensional symmetric Poincaré \( G \)-complex obtained from \( (C, \psi) \) by \( G \)-surgery on the above \( G \)-pair.

**Lemma 7.4.** \( C_k = C_{k+1} \). \( C_{k+1} = C_{k+1} \). and \( \psi_0 = \psi_0 : C^{k+1} \to C_k \) and \( \psi_0 = \psi_0 : C^k \to C_{k+1} \). The homology groups of \( C' \) are given by \( H_k(C') = \text{Tor} \ H_k(C) \) and \( H_k(C') = 0 (r \neq k) \).

**Proof.** The first assertion is clear from the definition. Now, \( C' \) has the form

\[
\begin{array}{ccccccc}
\cdots & \rightarrow & C_{k+3} & \rightarrow & C_{k+2} & \rightarrow & C_{k+1} & \rightarrow & C_k & \rightarrow & C_{k-1} & \rightarrow & C_{k-2} & \rightarrow & \cdots \\
& & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \\
& \psi_0 & & \psi_0 & & (\ast \varphi) & & (\ast \varphi) & & (\ast \varphi) & & (\ast \varphi) & & \\
\cdots & \rightarrow & C_{k-1} & \rightarrow & K^* & & K & \rightarrow & C_{k-1} & \rightarrow & \cdots
\end{array}
\]

and it follows that \( H_r(C') = 0 \) for \( r \leq k-1 \). By duality, \( H_r(C') = 0 \) for \( r \geq k+2 \). Now, \( \ker (d_C : C_k \to C_{k-1}) = \ker \) (the quotient map: \( Z_k \to H_k(C)/\text{Tor} \)), so that \( H_k(C') \) is isomorphic to \( \text{Tor} \ H_k(C) \) and \( H_{k+1}(C') = 0 \). q.e.d.

Put \( Z' = \ker (d_C^* : C_r \to C_r^*), \) and \( V = Z'^{k+1} \). Let \( i : V \to C'^{k+1} \) be the inclusion, and \( i^* : C_{k+1} = (C'^{k+1})^* \to V^* \) its dual map. There is an injective \( G \)-homomorphism \( \tilde{d}^* : V^* \to C_k \) such that \( d_c = \tilde{d}^* i^* : C_{k+1} \to C_k \).

Let \( h : C' \to C'' \) be the \( G \)-chain map defined by
where the differential of $C''$, $d_{C''}$, is given by $(d_{C''})_r = 0$ for $r \geq k+2$. Then $h$ induces the isomorphism of the homology groups. Put $\psi'' = h^*\psi' \in Q_{k+1}^{k+1}(C'')$. Then $\psi''_0 = i^*\psi_0 : C'' = C^k \rightarrow C''_{k-1} = V^*$ and $\psi'' = \psi_0 i : C''_{k+1} = V \rightarrow C'_k = C_k$, and $\psi''_s = 0$ for $s \geq 1$. $(C'', \psi'')$ is a $(2k+1)$-dimensional symmetric Poincaré $G$-complex and $h : (C', \psi') \rightarrow (C'', \psi'')$ is a $G$-quasi equivalence.

Note here that $C'_r = C'_k = C_k$. Put $U = C^k / Z^k$. $U$ is a f.g. $Z$-free $G$-module. Let $q : C'' \rightarrow U$ be the quotient map, and $q^* : U^* \rightarrow (C'')^* = C_k$ its dual map. There is an injective $G$-homomorphism $\beta : V^* \rightarrow U^*$ such that $d'' = q^* \beta : V^* \rightarrow C_k = C_k$ and $U^*/\beta(V^*) = \text{Tor } H_k(C)$.

**Lemma 7.5.** There is a $G$-homomorphism $\psi_0 : U \rightarrow V^*$ such that $\psi''_0 = \psi_0 q : C'' = C^k \rightarrow C''_{k+1} = V^*$.

**Proof.** Since $H^{k+1}(C'') = H^{k+1}(C')$ is a torsion group, for each $v \in V$ there is an integer $m \neq 0$ and $c \in C^k = C_k$ such that $mv = d^*c$. For each $z \in Z^k = \ker q$, $mv(\psi''_0(z)) = (d^*c)(\psi''_0(z)) = c(d\psi''_0(z)) = (-1)^{k+1}c(\psi''(d^*z)) = 0$. Hence $v(\psi''_0(z)) = 0$. This proves the lemma. q.e.d.

Since $\psi''_s = 0$ for $s \geq 1$. $T \psi'' = \psi''$, where $T$ is the transposition involution. This implies that $\psi''_0 = i^*\psi_0 : C''_{k+1} = V \rightarrow C'_k = C_k$ is equal to $i^*\psi_0$. Hence $\psi''_0 = \psi_0 i = q^*\psi''_0$, where $\psi''_0 : V \rightarrow U^*$ is the dual map of $\psi''$. Define the $G$-chain map $e : C \rightarrow C''$ by

$$e = \begin{cases} 0 : C_r = 0 \rightarrow C'_r = 0 & (r \geq k+2) \\ 1 : C_{k+1} = V^* \rightarrow C''_{k-1} = V^* & (r = k+1) \\ q^* : C_k = U^* \rightarrow C'_k = C_k & (r = k) \\ 0 : C_r = 0 \rightarrow C''_r = C_r & (r \leq k-1) \end{cases}$$

where the differential of $C''$ is given by $(d_{C''})_{k+1} = \beta$ and $(d_{C''})_r = 0$ for $r = k+1$. Then $e$ induces the isomorphisms of the homology groups. Define $\tilde{\psi} \in Q_{2k+1}(\tilde{C})$ by $(\tilde{\psi})_0 = (\psi_0 : U \rightarrow V^*)$. Then $(\tilde{C}, \tilde{\psi})$ is a $(2k+1)$-dimensional symmetric Poincaré $G$-complex. By Lemma 7.5 and the above remark, it may be seen that $e^*\tilde{\psi} = \psi''$. Hence $e$ induces a $G$-quasi equivalence from $(\tilde{C}, \tilde{\psi})$ to $(C'', \psi'')$. Now $(\tilde{C}, \tilde{\psi})$ represents the class $\Phi_{\tilde{\Psi}}(C, \psi)$ in $L_{2k+1}(\mathbb{Z})$. This proves that $\Phi_{\tilde{\Psi}} = \text{the identity}$.

Consequently, we obtain the following
Theorem 1. The maps $\Phi : W^*(G, \mathbb{Z}) \rightarrow L^*_r(G, \mathbb{Z})$ and $\Psi : L^*_r(G, \mathbb{Z}) \rightarrow W^*(G, \mathbb{Z})$ are isomorphisms.

8. The action of $W^*_r(G, \mathbb{Z})$ on Wall groups. First we describe the Wall group $L^*_r(\pi, w)$ by Ranicki's quadratic Poincaré complexes [5]. Let $\pi$ be a multiplicative group with a homomorphism $w : \pi \rightarrow \{\pm 1\}$. Then the integral group ring $\mathbb{Z}[\pi]$ has an involution defined by $\sum n_h w(h) h^{-1}$, where $n_h \in \mathbb{Z}$ and $h \in \pi$. An $n$-dimensional based f.g. free $\mathbb{Z}[\pi]$ chain complex is a chain complex

$$C_* : C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} \cdots \xrightarrow{d} C_1 \xrightarrow{d} C_0$$

such that each $C_r$ is a based f.g. free $\mathbb{Z}[\pi]$-module and each $d$ is a $\mathbb{Z}[\pi]$-homomorphism. The cochain group $C^*$ of $C_*$ is a based f.g. free $\mathbb{Z}[\pi]$ chain complex

$$C^* : C^0 \xrightarrow{d^*} C^1 \xrightarrow{d^*} \cdots \xrightarrow{d^*} C^{n-1} \xrightarrow{d^*} C^n$$

such that $C^r = \text{Hom}_{\mathbb{Z}[\pi]}(C_r, \mathbb{Z}[\pi])$ ($1 \leq r \leq n$) and $d^*$ is the dual homomorphism of $d$, where $C_r$ is a $\mathbb{Z}[\pi]$-module by the action $(sf)(c) = f(sc)$ ($s \in \mathbb{Z}[\pi]$, $c \in C_r$, $f \in C^r$) and it is based by the dual base of $C_r$. The generator $T \in \mathbb{Z}_2$ acts on $\text{Hom}_{\mathbb{Z}[\pi]}(C^*, C_*)$ by

$$T : \text{Hom}_{\mathbb{Z}[\pi]}(C^p, C_a) \rightarrow \text{Hom}_{\mathbb{Z}[\pi]}(C^q, C_p)$$

$$f \xrightarrow{T} (-1)^{pa} f^*$$

For a based f.g. free $\mathbb{Z}[\pi]$-module chain complex $C$, define the $\mathbb{Z}_2$-hyperhomology group $Q_\pi(C) = H_\pi(W \otimes_{\mathbb{Z}[\pi]} \text{Hom}_{\mathbb{Z}[\pi]}(C^*, C_*))$, where $W$ is the free $\mathbb{Z}[\mathbb{Z}_2]$-resolution of $\mathbb{Z}$. An element of $Q_\pi(C)$ is an equivalence class of collection

$$\{\theta_s \in \text{Hom}_{\mathbb{Z}[\pi]}(C^{n-r-s}, C_r) \mid r \geq 0, s \geq 0\}$$

such that

$$d\theta_s + (-1)^r \theta_s d^* + (-1)^{n-s-1}(\theta_{s+1} + (-1)^{s-1} T \theta_{s+1}) = 0 \quad (s \geq 0).$$

An $n$-dimensional quadratic Poincaré complex over $\mathbb{Z}[\pi]$ ($C, \theta$) is an $n$-dimensional based f.g. free $\mathbb{Z}[\pi]$-module chain complex together with an element $\theta \in Q_\pi(C)$ such that the cycle $(1 + T)\theta_0 \in \{\text{Hom}_{\mathbb{Z}[\pi]}(C^{n-r}, C_r), r \geq 0\}$ gives a simple chain equivalence $C^{n-*} \rightarrow C_*$. The quadratic $L$-groups $L_n(\pi)$ ($n \geq 0$) are defined to be the algebraic Poincaré cobordism groups of $n$-dimensional quadratic Poincaré complexes over $\mathbb{Z}[\pi]$. The quadratic $L$-groups are 4-periodic, $L_n(\pi) = L_{n+4}(\pi)$, being equal to the Wall surgery
obstruction groups $L^2_\pi(\pi, \omega)$.

Let $(G, \chi)$ be a pair of a finite group $G$ and a homomorphism $\chi : G \to \{\pm 1\}$. Let $\phi : \pi \to G$ be an epimorphism. We denote the composite map $\chi \phi$ by $\chi$. If $M$ is an f.g. $\mathbb{Z}$-free $G$-module, then $M$ is also a f.g. $\mathbb{Z}$-free $\mathbb{Z}[\pi]$-module by $hu = \phi(h)u$ ($h \in \pi, u \in M$).

**Lemma 8.1.** Let $M$ be a f.g. $\mathbb{Z}$-free $G$-module, and $P$ a f.g. free $\mathbb{Z}[\pi]$-module. Then $H \otimes \mathbb{Z}P$ with the diagonal $\pi$-module structure, $h(\Sigma u \otimes b) = \Sigma hu \otimes hb$ ($h \in \pi, u \in M, \ b \in P$) is a f.g. free $\mathbb{Z}[\pi]$-module.

**Proof.** Let $\{u_1, \cdots, u_s\}$ be a base over $\mathbb{Z}$ of $M$, and $\{b_1, \cdots, b_t\}$ a base over $\mathbb{Z}[\pi]$ of $P$. Then $\{u_i \otimes b_j, 1 \leq i \leq s, 1 \leq j \leq t\}$ form a base over $\mathbb{Z}[\pi]$ of $M \otimes \mathbb{Z}P$. q.e.d.

Let $(C, \psi)$ be an $m$-dimensional symmetric Poincaré $G$-complex. Let $(D, \theta)$ be an $n$-dimensional quadratic Poincaré complex over $\mathbb{Z}[\pi]$. Consider the chain complex $C \otimes \mathbb{Z}D$.

$$(C \otimes \mathbb{Z}D)_r = \Sigma C_k \otimes D_{r-k}, d(x \otimes y) = x \otimes dy + (-1)^k dx \otimes y (x \otimes y \in C_k \otimes D_r)$$

We consider $C \otimes \mathbb{Z}D$ as a $\mathbb{Z}[\pi]$-module chain complex by diagonal $\pi$-action. Then $C \otimes \mathbb{Z}D$ is a f.g. free $\mathbb{Z}[\pi]$ chain complex by Lemma 8.1. Now $D_{r-k}$ has a preferred base over $\mathbb{Z}[\pi]$, $\{b_1, \cdots, b_t\}$. Let $\{u_1, \cdots, u_s\}$ be a base over $\mathbb{Z}$ of $C_k$. Then $\{u_i \otimes b_j, 1 \leq i \leq s, 1 \leq j \leq t\}$ form a base over $\mathbb{Z}[\pi]$ of $C_k \otimes D_{r-k}$. We take this base as a preferred base of $C_k \otimes D_{r-k}$. The simple equivalence class of $C \otimes \mathbb{Z}D$ endowed with these bases does not depend on the particular choice of the base over $\mathbb{Z}$ of $C_k$, for $Wh(\mathbb{Z}) = 0$.

Now, let

$$\psi \in \text{Hom}_\mathbb{Z}(C^{m+\tau+r}, C_r) \mid r \geq 0, s \geq 0$$

and

$$\theta \in \text{Hom}_{\mathbb{Z}[\pi]}(D^{n-r-s}, D_r) \mid r \geq 0, s \geq 0$$

be collections of chains resenting $\psi$ and $\theta$, respectively. Put

$$(\psi \otimes \theta)_s = \Sigma \psi_{r} \otimes T^r \theta_{s+r} \ (s \geq 0)$$

where $\psi_r \otimes T^r \theta_{s+r} \in \text{Hom}_\mathbb{Z}(C^*, C^*) \otimes \text{Hom}_{\mathbb{Z}[\pi]}(D^*, D_r)_{n-s-r}$. There is a natural inclusion

$$\kappa : \text{Hom}_\mathbb{Z}(C^*, C^*)_{m+r} \otimes \text{Hom}_{\mathbb{Z}[\pi]}((D^*, D_r)_{n-s-r}$$

$$\to \text{Hom}_{\mathbb{Z}[\pi]}(C^* \otimes D^*, C^* \otimes D_r)_{m-n-s}$$

defined by $(\kappa(u \otimes v))(c \otimes d) = u(c) \otimes v(d) \ (u \in \text{Hom}_\mathbb{Z}(C^{m+r}, C_k))$. 

http://escholarship.lib.okayama-u.ac.jp/mjou/vol24/iss1/9
\[ v \in \text{Hom}_{\mathbb{Z}[\pi]}(D^{n-s-r-j}, D^j), \quad c \in C^{m-r-k}, \quad d \in D^{n-s-r-j}, \] and \( \pi \) acts on \( C^* \otimes \mathbb{Z} \) by \( (h(u \otimes v))(c \otimes d) = (u \otimes v)(\chi(h)h^{-1}c \otimes w(h)h^{-1}d) = (u \otimes v)(\chi(h)w(h)h^{-1}(c \otimes d)) = (u \otimes v)(w(h)h^{-1}(c \otimes d)) \) \( (h \in \pi) \). The collection of chains \( \{ \kappa((\psi \otimes \theta)_s) \}, s \geq 0 \) represents an element of \( Q_{m+n} \) \( (C \otimes \mathbb{Z} D) \), where the involution of \( \mathbb{Z}[\pi] \) is given by \( \sum n_h h \rightarrow \sum w\chi(h)n_h h^{-1} \) \( (n_h \in \mathbb{Z}, h \in \pi) \). Hence we obtain an \((m+n)\)-dimensional quadratic Poincaré complex over \( \mathbb{Z}[\pi], (C \otimes \mathbb{Z} D, \kappa(\psi \otimes \theta)) \). We denote this quadratic Poincaré complex by \( (C, \psi) \otimes (D, \theta) \).

**Lemma 8.2.** Let \( f : (C, \psi) \rightarrow (C', \psi') \) be a \( G \)-quasi equivalence between \( m \)-dimensional symmetric Poincaré \( G \)-complexes. Let \( (D, \theta) \) be an \( n \)-dimensional quadratic Poincaré complex over \( \mathbb{Z}[\pi] \). Then \( f \) induces a simple chain equivalence over \( \mathbb{Z}[\pi] \). \( f \otimes 1 \) from \( (C, \psi) \otimes (D, \theta) \) to \( (C', \psi') \times (D, \theta) \).

**Proof.** Since \( f : C \rightarrow D \) induces the isomorphisms of the homology groups, \( f \otimes 1 : C \otimes \mathbb{Z} D \rightarrow C' \otimes \mathbb{Z} D \) also induces the isomorphisms of the homology groups by Künmeth formula. Since the chain complexes \( C \otimes \mathbb{Z} D \) and \( C' \otimes \mathbb{Z} D \) are both f.g. free \( \mathbb{Z}[\pi] \)-module chain complexes, \( f \otimes 1 \) is a chain equivalence over \( \mathbb{Z}[\pi] \) which is clearly simple, as \( \text{WH}(\mathbb{Z}) = 0 \). q.e.d.

Now it can be seen that if an \( m \)-dimensional symmetric Poincaré \( G \)-complex \( (C, \psi) \) is a boundary of an \((m+1)\)-dimensional symmetric Poincaré \( G \)-pair, then for any \( n \)-dimensional quadratic Poincaré complex over \( \mathbb{Z}[\pi], (D, \theta), (C, \psi) \otimes (D, \theta) \) is a Poincaré boundary of an \((m+n+1)\)-dimensional quadratic Poincaré pair over \( \mathbb{Z}[\pi] \).

Consequently the above construction gives a pairing \( L_{m,n}^{\delta}(\mathbb{Z}) \otimes L_{n}^{\delta}(\pi, w) \rightarrow L_{m+n}^{\delta}(\pi, w\chi) \). By the isomorphism \( \Phi : W(G, Z) \rightarrow L_{n}^{\delta}(\pi, w\chi) \), we obtain a pairing

\[ W_{m}^{\delta}(G, Z) \otimes L_{n}^{\delta}(\pi, w) \rightarrow L_{m+n}^{\delta}(\pi, w\chi) \).

**9. Main theorem.** We return to the surgery problem in §1. Let \( L^m \) be an \( m \)-dimensional closed PL (or smooth) \( G \)-\( \chi \)-manifold. By equivariant triangulation theorem [3], we can assume that there is a triangulation \( t(L^m) = \{ \tau \mid \tau : \text{open simplex} \} \) of \( L^m \) such that \( (1) \) for each \( \tau \in t(L^m) \) and \( g \in G, \; g\tau \in t(L^m) \) and \( (2) \) if \( g\tau = \tau \), then \( g \) fixes each point of \( \tau \). We choose and fix such a triangulation \( t(L^m) \). Let \( \{ C_{\ast}(L), \partial_{\ast} \} \) be the \( G \)-chain complex defined by this triangulation. The manifold \( L^m \times L^m \) has the \( G \)-CW structure \( \{ \tau \times \nu \mid \tau, \nu \in t(L^m) \} \). Let the generator \( T \in \mathbb{Z}_2 \) acts on
Let $L^m \times L^m$ by $T(x, y) = (y, x)((x, y) \in L^m \times L^m)$. Then $L^m \times L^m$ is a $\mathbb{Z}_2 \times G$-manifold and the above CW structure is a $\mathbb{Z}_2 \times G$-CW structure. Let $S^m$ be the infinite dimensional sphere on which $\mathbb{Z}_2$ acts by the antipodal involution. Let $\{e_s, Te_s \mid s = 1, 2, \cdots \}$ be the standard $\mathbb{Z}_2$-CW structure of $S^m$, where $\dim e_s = s$ and $Te_s$ is $e_s$ transformed by $T$. The manifold $S^m \times L^m$ has the $\mathbb{Z}_2 \times G$-action defined by $(T, g)(a, x) = (Ta, gx)(T \in \mathbb{Z}_2, g \in G, (a, x) \in S^m \times L^m)$, and $\{e_s \times r, Te_s \times r \mid s = 1, 2, \cdots, r \in t(L^m) \}$ gives a $\mathbb{Z}_2 \times G$-CW structure of $S^m \times L^m$. Let $F': S^m \times L^m \to L^m \times L^m$ be the map defined by $F'(a, x) = (x, x)((a, x) \in S^m \times L^m)$. $F'$ is a $\mathbb{Z}_2 \times G$-equivariant map. Let $F: S^m \times L^m \to L^m \times L^m$ be a $\mathbb{Z}_2 \times G$-equivariant cellular approximation of $F'$ with respect to the above $\mathbb{Z}_2 \times G$-CW structures of $S^m \times L^m$ and $L^m \times L^m$. Then $F$ induces the $\mathbb{Z}_2 \times G$-equivariant chain map $F_*: W_* \otimes C_*(L^m) \to C_*(L^m) \otimes C_*(L^m)$, where $W_*$ is the chain complex $C_*(S^m)$ defined by the above standard CW structure. $W_*$ is a free $\mathbb{Z}[\mathbb{Z}_2]$-resolution of $S^m$. Now, the chain complex $C_*(L^m) \otimes C_*(L^m)$ is identified with the chain complex $\text{Hom}_G(C_*(L^m), C_*(L^m))$ by the map $c \otimes d \to (u \to u(c)d)$ $(c, d \in C_*(L^m), u \in C^*(L^m))$. Hence $F_*$ induces the map denoted by the same letter.

$$F_*: W_* \otimes C_*(L^m) \to \text{Hom}_G(C_*(L^m), C_*(L^m)).$$

Let $[L^m] \in C_m(L^m)$ be the fundamental cycle of $L^m$. Then $g[L^m] = \chi(g)[L^m]$ for $g \in G$. Hence, for each $s \geq 0$, $F_*(e_s \otimes [L^m]) = \psi_s$ is an element of $\text{Hom}_G(C_*(L^m), C_*(L^m))_{m+s}$. where $G$ acts on $C_*(L^m)$ by $(gu)(c) = u(\chi(g)c^{-1})$ $(g \in G, u \in C^*(L^m), c \in C_*(L^m))$. The pair $(C_*(L^m), \psi = \{\psi_s\})$ is an $m$-dimensional symmetric poincaré $G$-complex.

Let $\mathcal{Q}_G(G)$ be the equivariant PL (resp. smooth) bordism group of closed PL (resp. smooth) $G$-manifolds. Then the above construction induces a well-defined homomorphism $\rho : \mathcal{Q}_G(G) \to L^G_*, \chi(Z)$. By the isomorphism $\Psi : L^G_*, \chi(Z) \to W^G_*(G, Z)$, we obtain the homomorphism $\rho = \Psi \rho : \mathcal{Q}_G(G) \to W^G_*(G, Z)$ which is given by

$$[L^{2k}] \to \langle H^k(L^{2k})/\text{Tor. the intersection form} \rangle$$

and

$$[L^{2k+1}] \to \langle \text{Tor } H^{k+1}(L^{2k+1}) \rangle,$$

Theorem 2. Let $\pi$ be a finitely presented group with a homomorphism $w : \pi \to \{\pm 1\}$. Let $(G, \chi)$ be a pair of a finite group $G$ and a homomorphism $\chi : G \to \{\pm 1\}$. Let $\phi : \pi \to G$ be an epimorphism, and denote the composite $\chi \phi$ by $\chi$. Then the following diagram is commutative:
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\[ \Omega^\infty_\pi(G) \otimes L_\pi^\infty(\pi, w) \longrightarrow L^\infty_{\pi-n}(\pi, w_\pi) \]
\[ W_\pi^\infty(G, \mathbb{Z}) \otimes L_\pi^\infty(\pi, w) \longrightarrow L^\infty_{\pi+n}(\pi, w_\pi) \quad (m, n \geq 0) \]

where the upper map is the map defined by the construction in §1 and the lower map is the map defined in §8.

10. Proof of Theorem 2. The proof of Theorem 2 is essentially same as that of the product formula of Ranicki [6, §8]. For each element \( x \in L_\pi^\infty(\pi, w) \quad (n \geq 5) \), there is an \( n \)-dimensional normal map of degree one, \( f : M^n \rightarrow N^n \), such that \( \pi_1(N^n) = \pi \) and \( \sigma(f) = x \). We consider the map \( \tilde{f} \times 1 : M^n \times_\pi L^m \rightarrow M^n \times_\pi L^m \) defined in §1, where \( L^m \) is an \( m \)-dimensional \( G, \chi \)-manifold.

First we assume that \( M^n \) and \( N^n \) are closed manifolds. Let \( F : \nu_M \rightarrow \xi \) be the bundle map associated to \( f \), where \( \nu_M \) is the stable normal bundle of \( M^n \) and \( \xi \) is some bundle over \( N^n \). \( F \) induces the bundle map \( \tilde{F} : \tilde{\nu}_M \rightarrow \tilde{\xi} \), where \( \tilde{\nu}_M \) (resp. \( \tilde{\xi} \)) is the bundle over \( \tilde{M}^n \) (resp. \( \tilde{N}^n \)) induced by the projection from \( \nu_M \) (resp. \( \xi \)). Let \( T(\tilde{F}) : T(\tilde{\nu}_M) \rightarrow T(\tilde{\xi}) \) be the map induced by \( \tilde{F} \) between the Thom spaces of \( \tilde{\nu}_M \) and \( \tilde{\xi} \). Following [6], there is a stable \( \pi \)-map \( H : \sum^n \tilde{N}^n \rightarrow \sum^n \tilde{M}^n \) which is an equivariant S-dual of \( T(\tilde{F}) \), where \( \tilde{N}^n \) (resp. \( \tilde{M}^n \)) denotes the disjoint union of \( \tilde{N}^n \) (resp. \( \tilde{M}^n \)) and one \( \pi \)-fixed point. The map \( H \) is called a geometric Umker map for the normal map \( f \). \( H \) defines the composite \( \pi \)-map

\[ \theta_H : N_\pi \rightarrow \text{adjoin}(H) \rightarrow \Omega^\infty_\pi \sum^n \tilde{M}^n \rightarrow \text{stable homotopy projection} \rightarrow S^\infty_\pi \wedge \tilde{M}^n \wedge \tilde{M}^n, \]

where the generator \( T \in \mathbb{Z}_2 \) acts on \( S^w \) by the antipodal map and on \( \tilde{M}^n \wedge \tilde{M}^n \) by the transposition \((a, b) \rightarrow (b, a)\). Let \( C(M) \) be the f.g. free \( Z[\pi] \)-chain complex of \( \tilde{M}^n \). Then \( \theta_H \) induces the homomorphism

\[ \theta_H : H_n(N^n, wZ) \rightarrow H_n(W \otimes_{\pi \mathbb{Z}} C(M) \otimes_{\pi \mathbb{Z}} C(M)) \]

where \( wZ \) denotes the twisted coefficient associated to \( w : \pi \rightarrow \{ \pm 1 \} \), \( W \) is the \( Z[\mathbb{Z}] \)-chain complex \( C(S^w) \) defined in §9, and \( T \in \mathbb{Z}_2 \) acts on \( C(M) \otimes_{\pi \mathbb{Z}} C(M) \) by the signed transposition \( a \otimes b \rightarrow (-1)^{a-b} b \otimes a \) (\( a \in C_p(M) \) and \( b \in C_0(M) \)). \( \theta_H \) depends only on the stable \( \pi \)-equivariant homotopy class of \( H \).

Define the Umker \( Z[\pi] \)-module chain map \( f^1 : C(\tilde{N}) \rightarrow C(M) \) to be the composite \( Z[\pi] \)-chain map

\[ f^1 : C(\tilde{N}) \xrightarrow{((N) \cap -)^{-1}} C(\tilde{N})^{\pi-*} \xrightarrow{\tilde{f}^*} C(\tilde{M})^{\pi-*} \xrightarrow{[M] \cap -} C(M) \]

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where $C(\tilde{N})$ is the free $\mathbb{Z}[\pi]$-chain complex of $\tilde{N}$, $[M] \in H_n(M^n, wZ)$ and $[N] \in H_n(N^n, wZ)$ are the fundamental classes of $M^n$ and $N^n$ respectively, and $\cap$ denotes the cap products. Let $C(f^i)$ be the algebraic mapping cone of $f^i$.

Let $e : C(\tilde{M}) \to C(f^i)$ be the projection. Then $e$ induces the map

$$e_\pi : H_n(W \otimes_{Z[\pi]} C(M) \otimes_{Z[\pi]} C(M)) \to H_n(W \otimes_{Z[\pi]} (C(f^i) \otimes_{Z[\pi]} C(f^i))).$$

Put $\theta = e_\pi \theta([N])$. Now the chain complex $C(f^i) \otimes_{Z[\pi]} C(f^i)$ is isomorphic to the chain complex $\text{Hom}_{Z[\pi]}(C(f^i), C(f^i)_\pi)$, and $\theta$ is considered as an element of $Q_n(C(f^i))$. The pair $(C(f^i), \theta)$ is an $n$-dimensional quadratic Poincaré complex over $Z[\pi]$ and represents the surgery obstruction of $f$.

Next we consider the surgery obstruction of the normal map $\tilde{f} \times \pi : \tilde{M} \times \pi L^m \to \tilde{N} \times \pi L^m$. Note that $\pi$ acts on the covering spaces $\tilde{M} \times L^m$ and $\tilde{N} \times L^m$ by the diagonal actions. If $H : \Sigma^\infty \tilde{N} \to \Sigma^\infty \tilde{M}$ is a geometric Umker map for $f$, then

$$H \cap 1 : \Sigma^\infty(\tilde{N} \times L^m)_+ = \Sigma^\infty(\tilde{N} \times L^m) \to \Sigma^\infty(\tilde{M} \times L^m)_+ = \Sigma^\infty(\tilde{M} \times L^m)_+$$

is a geometric Umker map for $\tilde{f} \times \pi$. The composite $\pi$-map

$$\theta_{H\cap 1} : (\tilde{N} \times L^m)_+ \xrightarrow{\text{adjoint} (H \cap 1)} Q^\infty \Sigma^\infty(\tilde{M} \times L^m)_+ \xrightarrow{\text{stable projection}} S^\infty \times_{Z\pi} (\tilde{M} \times L^m)_+ \cap (\tilde{M} \times L^m) = S^\infty \times_{Z\pi} M^m \cap M^m \times L^m$$

is given by $\theta_{H \cap d}$, where $d$ is the diagonal map $L^m \to L^m \times L^m$.

Let $C(L)$ be the $G$-chain complex defined by an equivariant triangulation $t(L^m)$ as in §9. Then $\theta_{H\cap 1}$ induces the homomorphism

$$\theta_{H\cap 1} : H_{n+m}((\tilde{N} \times \pi L^m) \otimes Z) \to H_{n+m}(W \otimes_{Z[\pi]} (C(\tilde{M}) \otimes C(L)) \otimes_{Z[\pi]} (C(\tilde{M}) \otimes C(L))).$$

The Umker $Z[\pi]$-chain map for $f \times \pi$ is given by

$$f^i \otimes 1 : C(\tilde{N}) \otimes C(L) \to C(\tilde{M}) \otimes C(L)$$

and the algebraic mapping cone $C((\tilde{f} \times \pi))$ is $Z[\pi]$-chain equivariant to $C(f^i) \otimes C(L)$ on which $\pi$ acts diagonally. The chain map $e \otimes 1 : C(\tilde{M}) \otimes C(L) \to C(f^i) \otimes C(L)$ induces the homomorphism

$$(e \otimes 1)_\pi : H_{n+m}(W \otimes_{Z[\pi]} (C(\tilde{M}) \otimes C(L)) \otimes_{Z[\pi]} C(\tilde{M}) \otimes C(L))) \to H_{n+m}(W \otimes_{Z[\pi]} (C(f^i) \otimes C(L))) \otimes_{Z[\pi]} (C(f^i) \otimes C(L))).$$

Put $\theta' = (e \otimes 1)_\pi([\tilde{N} \times \pi L])$, where $[\tilde{N} \times \pi L]$ is the fundamental class of $\tilde{N} \times \pi L^m$. Then the pair $(C(f^i) \otimes C(L), \theta')$ is an $(n+m)$-dimensional
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quadratic Poincaré complex over \( \mathbb{Z}[\pi] \), and it represents the surgery obstruction \( \delta(f \times \pi) \). By the same argument as in [6, §8] we see that \((C(f^1) \otimes C(L), \theta^1)\) is equivalent to \((C(f^1), \theta) \oplus (C(L), \psi)\).

When \( M^n \) and \( N^n \) have non-empty boundaries, a similar construction can be made by the use of the homotopy Umker map pair

\[
H : (\Sigma^m \tilde{N}, \Sigma^m \partial \tilde{N}) \to (\Sigma^m \tilde{M}, \Sigma^m \partial \tilde{M}).
\]

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(Received June 5, 1981)