Some polynomial identities and commutativity of s-unital rings

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SOME POLYNOMIAL IDENTITIES AND COMMUTATIVITY OF s-UNITAL RINGS

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Throughout this paper, $R$ will represent an (associative) ring (with or without identity 1). $C = C(R)$ the center of $R$, $D = D(R)$ the commutator ideal of $R$, and $N = N(R)$ the set of all nilpotent elements in $R$.

A ring $R$ is called $s$-unital if $x \in Rx \cap xR$ for any $x \in R$. As stated in [13], if $R$ is $s$-unital, then for any finite subset $F$ of $R$ there exists an element $e$ in $R$ such that $ex = xe = x$ for all $x \in F$. Such an element $e$ will be called a pseudo-identity of $F$ (in $R$).

Let $n$ be a positive integer. We consider the following ring-properties:

$P_1(n)$: $(xy)^n = x^n y^n$ and $(xy)^{n+1} = x^{n+1} y^n$ for all $x, y \in R$.

$P_2(n)$: $(xy)^n = x^n y^n = y^n x^n$ for all $x, y \in R$.

$P_3(n)$: $(xy)^n = (yx)^n$ for all $x, y \in R$.

$P_4(n)$: $[x, (xy)^n] = 0$ for all $x, y \in R$.

$P_5(n)$: $[x, (yx)^n] = 0$ for all $x, y \in R$.

$P_6(n)$: $[x, y^n] = 0$ for all $x, y \in R$.

$P_7(n)$: $[x, x^n] = [x^n, y]$ for all $x, y \in R$.

$P_8(n)$: There is a polynomial $\psi(\lambda)$ with integer coefficients such that $[x, y^n] = [\psi(x), y]$ for all $x, y \in R$.

$P_9(n)$: $[x, (x+y)^n - x^n] = 0$ for all $x, y \in R$.

$P_{10}(n)$: $[x^n, y^n] = 0$ for all $x, y \in R$.

$Q(n)$: For any $x, y \in R$. $n[x, y] = 0$ implies $[x, y] = 0$.

The properties $P_1(n), P_3(n), P_6(n), P_7(n)$ and $P_{10}(n)$ have been considered by many authors. The main objective of this paper is to prove the following

**Theorem 1.** Let $m_1, \cdots, m_t$ and $n_1, \cdots, n_t$ be (fixed) positive integers such that $1 \leq m_i \leq 9$ and $2 \leq n_i$ for $i = 1, \cdots, t$. Let $d = (n_1, \cdots, n_t)$. If an $s$-unital ring $R$ has the (conjunctive) property $P_{m_1}(n_1) \wedge \cdots \wedge P_{m_t}(n_t) \wedge Q(d)$, then $R$ is commutative.

In preparation for the proof of our theorem, we introduce here some definitions. Let $P$ be a ring-property. If $P$ is inherited by every subring and every homomorphic image, $P$ is called an $h$-property. More weakly, if $P$ is inherited by every finitely generated subring and every natural
homomorphic image modulo the annihilator of a central element, $P$ is called an $H$-property. And, a ring-property $P$ such that a ring has the property $P$ if and only if all its finitely generated subrings have $P$, is called an $F$-property. Finally, $P$ is called a $C(n)$-property if every ring with 1 having the property $P \land Q(n)$ is commutative.

Obviously, $P(n) - P_0(n)$ are $h$-properties and $Q(n)$ is an $H$-property. These properties are also $F$-properties and the property "being commutative" is an $F$-property.

To our end, we shall prove three propositions. The first one enables us to reduce some problems of $s$-unital rings into those of rings with 1.

**Proposition 1.** Let $P$ be an $H$-property and $P'$ an $F$-property. If every ring with 1 having the property $P$ has the property $P'$, then every $s$-unital ring having $P$ has $P'$.

**Proof.** Let $R$ be an $s$-unital ring having the property $P$. We show that if $F$ is a finite subset of $R$, then the subring $\langle F \rangle$ generated by $F$ has the property $P'$. To see this, choose a pseudo-identity $e$ of $F$ and a pseudo-identity $e'$ of $F \cup \{e\}$. Obviously, $e$ is a central element of $S = \langle F \cup \{e, e'\} \rangle$. Let $A$ be the annihilator of $e$ in $S$. Then the factor ring $S/A$ has the identity $e' + A$. Since $\langle F \rangle \cap A = 0$, $\langle F \rangle$ may be regarded as a subring of $S/A$. Thus, by hypothesis, $\langle F \rangle$ has the property $P'$.

Some known results on rings with 1 can be extended to $s$-unital rings by Proposition 1. For example, by [11, Theorem 3] and [4, Theorem 1] we obtain

**Corollary 1.** Let $R$ be an $s$-unital ring.

1. Let $k$ be a positive integer. Suppose that for each pair of elements $x, y$ in $R$ there exist positive integers $m, n$ such that $\langle x^m, y^n \rangle = 0$. Then $D$ is a nil ideal.

2. Suppose that for each pair of elements $x, y$ in $R$ there exists an integer $n \geq 2$ such that $(xy)^n = x^n y^m$ and $(xy)^{n+1} = x^{n+1} y^{n+1}$. Then $D$ is a nil ideal.

Next, we reprove a theorem of Kezlan [10].

**Proposition 2.** Let $f$ be a polynomial in non-commuting indeterminates $x_1, \ldots, x_k$ with integer coefficients. Then the following statements are equivalent:

1. $f(0, 0, \ldots, 0) = 0$.
2. For each $i$, there exists a polynomial $g_i(x)$ such that $f(x_1, \ldots, x_i, \ldots) = g_i(x_i)$.
3. The polynomial $f$ can be written in the form $f(x_1, \ldots, x_k) = g(x_1) + \sum_{i=2}^{k} h_i(x_1, x_i)$, where $g(x_1)$ is a polynomial in $x_1$ and $h_i(x_1, x_i)$ is a polynomial in $x_1$ and $x_i$ with integer coefficients.

The proofs of these propositions are similar to those in [11].
1) For any ring $R$ satisfying $f = 0$, $D$ is a nil ideal.
2) Every semiprime ring satisfying $f = 0$ is commutative.
3) For every prime $p$, $(\text{GF}(p))^2$ fails to satisfy $f = 0$.

Proof. Since 2) $\Rightarrow$ 1) $\Rightarrow$ 3) are immediate, it remains only to prove that 3) implies 2). Obviously, the coefficients of $f$ are relatively prime. Since every semiprime ring is a subdirect sum of prime rings, it suffices to show that every prime ring $R$ satisfying $f = 0$ is commutative. Now, by a theorem of Amitsur [1, Theorem 7 (6)], the (classical) quotient ring $R^*$ of $R$ is an Artinian simple ring satisfying $f = 0$. Hence, by 3) (and Posner's theorem), $R^*$ is a central division algebra of finite rank $m^2$ over $C^* = C(R^*)$. Suppose that $R^*$ is not commutative, namely $m \geq 2$, and choose a maximal subfield $K$ of $R^*$. Then again by the theorem of Amitsur, $R^* \otimes_{C^*} K \cong (K)_m$ satisfies $f = 0$. But this contradicts 3). Thus, $R^*$, and therefore $R$ is commutative.

Corollary 2 (cf. [5, Theorems 1, 2, 3] and [9, Theorem]). Let $R$ be a semiprime ring, and $\nu$ a (fixed) positive integer.

1) If for each pair of elements $x, y$ in $R$ there exists an integer $n$ such that $2 \leq n \leq \nu$ and $[x, (xy)^n - x^n y^n] = 0$ (resp. $[x, (xy)^n - (yx)^n] = 0$), then $R$ is commutative.

2) Suppose that for each pair of elements $x, y$ in $R$ there exists an integer $n$ such that $2 \leq n \leq \nu$ and $[x, [x^n y] - [x, y^n]] = 0$. Then $R$ is commutative.

Proof. (1) In fact, $R$ satisfies the identity

$f(x, y, z) = [x, (xy)^2 - x^2 y^2]z[x, (xy)^3 - x^3 y^3]z \cdots [x, (xy)^\nu - x^\nu y^\nu] = 0$

(resp. $f(x, y, z) = [x, (xy)^2 - (yx)^2]z[x, (xy)^3 - (yx)^3]z \cdots [x, (xy)^\nu - (yx)^\nu] = 0$),

but $f(E_{12}, E_{21}, E_{21}) \neq 0$ in $(\text{GF}(p))^2$.

(2) $R$ satisfies the identity

$f(x, y, z) = [x, [x^2, y] - [x, y^2]]z[x, [x^3, y] - [x, y^3]]z \cdots [x, [x^\nu, y] - [x, y^\nu]] = 0$,

but $f(E_{11}, E_{12}, E_{21}) \neq 0$ in $(\text{GF}(p))^2$.

According to Proposition 2, as Corollary 2 shows, various kinds of semiprime PI-rings (especially, semiprime rings having any one of the properties $P_1(n) - P_{10}(n)$ ($n \geq 2$)) are commutative. However, if we remove the hypothesis "semiprime", even under some extra hypothesis, say that $R$ has 1 or that $R$ is $n$-torsion free, we have not yet obtained definite results concerning the precise commutativity of $R$.

In the subsequent study, we shall use freely the following well-known
results: Let $a, b \in R$, and $n$ a positive integer.

(I) If $[a, [a, b]] = 0$ then $[a^n, b] = na^{n-1}[a, b]$.

(II) If $R$ contains $1$ and $a^n b = (a+1)^n b = 0$, then $b = 0$.

Now, in advance of exposing the relationship among the properties $P_1(n) - P_6(n)$, we state the following lemma.

Lemma 1. Let $n \geq 2$. If $[x, y] \in C$ for all $x, y \in R$, then $P_7(n)$ implies $P_6(n^4)$.

Proof. We claim that $[x, y^{n^2}]x^{(n-1)^2} = [x, y^{n^2}]$ for all $x, y \in R$. Indeed, by (I) we have $[x, y^{n^2}]x^{(n-1)^2}x^{n-1} = nx^{n^1}x^{(n-1)^2} = nx^{n^1}x^{(n-1)^2} = [x, y^{n^2}]$. Now, by making use of the argument employed in the proof of [7, Lemma 5], we can prove that the subring $\langle x^{n^2} | x \in R \rangle$ is commutative. This implies that $[x^{n^2}, y^{n^2}] = 0$ for all $x, y \in R$.

Proposition 3. (i) If $R$ is s-unital, then $P_1(n) \Leftrightarrow P_2(n) \Rightarrow P_4(n) \Leftrightarrow P_5(n) \Rightarrow P_6(n) \Rightarrow P_7(n)$ and $P_6(n) \Rightarrow P_7(n)$.

(ii) If $n \geq 2$, then $P_7(n) \Leftrightarrow P_6(n) \Leftrightarrow P_5(n) \Rightarrow P_6(n^a)$ for some positive integer $a$.

Proof. (i) In view of Proposition 1, we may assume that $R$ has 1. Obviously, $P_3(n) \Rightarrow P_4(n) \cap P_5(n), P_2(n) \Rightarrow P_1(n) \cap P_3(n)$, and $P_6(n) \Rightarrow P_4(n) \cap P_5(n) \cap P_7(n) \cap P_6(n)$. Furthermore, $P_1(n)$ together with $P_5(n)$ implies $P_2(n)$, and so we prove that $P_1(n) \Rightarrow P_4(n)$ (resp. $P_5(n) \Rightarrow P_6(n)$).

$P_1(n) \Rightarrow P_7(n)$. Since $x^{n^2}y^{n^2} = (xy)^{n^2} = x^{n^2}y^{n^2}$, we get $x[x^{n^2}, y^{n^2}] = 0$. Hence $x[x^{n^2}, y^{n^2}] = 0$ by (II), and in particular $x[x^{n^2}, y^{n^2}] = 0$. So we have

$$[x, (xy)^n] = x[(xy)^n - (yx)^n] = x[x^{n^2}, y^{n^2}] = 0.$$

$P_4(n) \Rightarrow P_6(n)$. By [12, Theorem], there exists a positive integer $k$ such that $kD = 0$. If $u$ is in $N$, then for any $x \in R$ we have

$$[x, u] = [(1 + u)(1 + u)^{-1}x]^{n^1} + u = 0.$$

Hence, noting that $D \subseteq N$ by Proposition 2, we see that $[x^n, [x, y]] = 0$.

Next, if $R$ has $P_7(n)$, then

$$[x, (x + y)^n - y^n] = [\psi(x), (x + y) - y] = 0.$$
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\[ [x,y^n] - [x^n,y] = [x(x+y)^n] - [(x+y)^n,y] = [x+y, (x+y)^n] = 0. \]

We have thus seen the equivalence of \( P_7(n) - P_6(n) \).

Now, suppose \( R \) has the property \( P_5(n) \). By [7, Lemma 1] there exists a positive integer \( h \) such that \([x,y]^h = 0 \) for all \( x, y \in R \). Choose a positive integer \( k \) such that \( n^k \geq h \), and let \( T = \langle x^n | x \in R \rangle \). Since \([[(x,y), z^n] = [[x,y]^n, z] = 0 \) for all \( x, y, z \in R \), Lemma 1 shows that \([s^n, t] = 0 \) for all \( s, t \in T \). It therefore follows that \([x^{n^{k+1}}, y] = [x^n, y^n] = 0 \) for all \( x, y \in R \).

**Remark 1.** Let \( i, j \) be non-negative integers. Let us consider the following ring-property:

\[ P(i,j;n): [x(x'yx'^i)^n] = 0 \] for all \( x, y \in R \).

Obviously, \( P(1,0;n) = P_4(n) \), \( P(0,1;n) = P_5(n) \) and \( P(0,0;n) = P_6(n) \). From the proof of Proposition 3 (i), we can easily see that \( P(i,j;n) \) is equivalent to \( P_6(n) \) for any \( i, j \geq 0 \).

Obviously, if the power map \( \pi_n: x \mapsto x^n \) is a ring-homomorphism of \( R \) then \( R \) has the property \( P_5(n) \). In [6, Theorem 3], it is shown that if \( \pi_n \) is a surjective ring-homomorphism of \( R \) for some \( n \geq 2 \) then \( R \) is commutative. On the other hand, in [3, Theorem 3], it is shown that if a ring \( R \) with 1 has the property \( P_1(n) \) and is generated by \( \{x^n | x \in R \} \) or \( \{x^{n(n+1)} | x \in R \} \) then \( R \) is commutative. The next improves these results as well as [2, Corollary 2] (see also [7, Corollary 2]).

**Corollary 3.** Let \( n \geq 2 \). Let \( R \) be an \( s \)-unital ring having one of the properties \( P_1(n) - P_6(n) \) or a ring having one of the properties \( P_1(n) - P_5(n) \), and let \( T = \langle x^n | x \in R \rangle \). If the centralizer of \( T \) in \( R \) coincides with \( C \), then \( R \) is commutative.

**Proof.** If an \( s \)-unital ring \( R \) has one of the properties \( P_1(n) - P_6(n) \), then it has the property \( P_6(n) \). So the assertion is clear. If a ring \( R \) has one of the properties \( P_1(n) - P_5(n) \), then it has the property \( P_6(n^a) \) for some positive \( a \). Hence \([x^{n^{a-1}}, y^n] = [x^n, y] = 0 \) for all \( x, y \in R \). Then, \([x^{n^{a-1}}, y] = 0 \) by hypothesis. We can thus continue the same procedure to obtain the conclusion \([x,y] = 0 \).

**Corollary 4.** If \( n \geq 2 \), then the properties \( P_1(n) - P_6(n) \) are \( C(n) \)-properties.
Proof. Let $R$ be a ring with 1 having the property $P_i(n) \land Q(n)$. If $1 \leq i \leq 6$ then, according to Proposition 3 (i), we may assume that $i = 6$. Given $u \in N$, by an easy induction on the nilpotency index of $u$, we can show that $u \in C$, and therefore $D \subseteq C$ by Proposition 2. Now, by (1), for any $x, y \in R$ we have $ux^{n-1}[x, y] = [x^n, y] = 0$, whence $[x, y] = 0$ follows by (II). On the other hand, if $7 \leq i \leq 9$, then $R$ has $P_6(n^a)$ and $Q(n^a) = Q(n)$ for some positive integer $a$ (Proposition 3 (ii)). Hence, $R$ is commutative by what was just proved above.

Proof of Theorem 1. In virtue of Proposition 1, we may assume that $R$ has 1. Since $P_1(n) - P_6(n)$ are $C(n)$-properties, the proof of our theorem is now immediate by [8, Proposition 1].

Corollary 5. Let $m_1, \ldots, m_t$ and $n_1, \ldots, n_t$ be (fixed) positive integers such that $1 \leq m_i \leq 9$, $2 \leq n_i$ $(i = 1, \ldots, t)$ and $(m_1, \ldots, n_t) = 1$. If an $s$-unital ring $R$ has the property $P_{m_1}(n_1) \land \cdots \land P_{m_t}(n_t)$, then $R$ is commutative.

Remark 2. Let $n \geq 2$, and $m$ a strictly proper divisor of $n$. Then, the properties $P_3(n) - P_{10}(n)$ are not $C(m)$-properties. In this sense, the results on $P_3(n) - P_5(n)$ in Corollary 4 are best possible. To see this, we take a prime divisor $p$ of $n$ such that $p \nmid m$. Let $S$ be a non-commutative algebra over $GF(p)$ such that $S^3 = 0$. Let $R$ be the ring whose additive group is the direct sum of $GF(p)$ and $S$ with multiplication given by $(k, s)(k', s') = (kk', ks' + k's + ss')$. It is easy to see that $R$ has the properties $P_3(n) - P_3(n)$ and $Q(n)$, but $R$ is not commutative. In the same way, the properties $P_1(n)$ and $P_2(n)$ are not $C(m)$-properties when $n$ is odd. However, as is easily seen, $P_2(2)$ and $P_3(2)$ are $C(1)$-properties. In general, when $n$ is even, we can not deny the possibility that $P_1(n)$ and $P_2(n)$ can be $C(n/2)$-properties (see [12, Examples 3 and 4]).

Remark 3. So far we did say little about $P_{10}(n)$. It is easy to see that $P_{10}(2)$ is a $C(2)$-property. However, $P_{10}(n)$ is not a $C(n)$-property if $n$ has a divisor of the form $1 + p^r + p^{2r} + \cdots + p^{sr}$, where $r$ and $s$ are positive integers and $p$ is a prime not dividing $n$. In fact, let $n$ have such a divisor and let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a^p \end{pmatrix} \middle| a, b \in GF(p^{(s+1)r}) \right\}.$$  

Then, $R$ is an $n$-torsion free ring with 1 and has the property $P_{10}(n)$, but
$R$ is not commutative. Thus, in particular, $P_{10}(n)$ is not a $C(n)$-property if $3 \leq n \leq 10$. What about $P_{10}(11)$?

**Remark 4.** In view of Remark 3, it seems unavoidable to exclude the property $P_{10}(n)$ from the statement in Corollary 5. However, we have the following: If an $s$-unital ring $R$ has the property $P_i(m) \wedge P_j(n)$ for some positive integers $i$, $j$, $m$ and $n$ such that $1 \leq i, j \leq 10$, $2 \leq m, n$ and $(m, n) = 1$, then $R$ is commutative. In fact, if $R$ has this property, then $R$ has the property $P_{10}(mA) \wedge P_{10}(n)$ for some positive integer $a$ (Proposition 3 (ii)). Then $R$ is commutative by Proposition 1 and [14. Theorem] (cf. the proof of [8. Theorem 1]).

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