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ON AN INDIVIDUAL ERGODIC THEOREM

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1. Introduction and the theorem. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and let $L_p(\mu)=L_p(X, \mathcal{F}, \mu)$, $1 \leq p \leq \infty$, denote the (real or complex) Banach spaces defined as usual with respect to $(X, \mathcal{F}, \mu)$. If T is a bounded linear operator on $L_1(\mu)$, we denote by τ its linear modulus [2]. In [5] (see also [6]) we have proved the following

Theorem A. *If $(X, \mathcal{F}, \mu)$ is a finite measure space, and if the linear modulus τ of a bounded linear operator T on $L_1(\mu)$ satisfies the norm conditions*:

$$(1) \qquad \sup_n \left\| \frac{1}{n} \sum_{i=0}^{n-1} \tau^i \right\|_1 < \infty,$$

$$(2) \qquad \sup_n \left\| \frac{1}{n} \sum_{i=0}^{n-1} \tau^i \right\|_\infty < \infty.$$

then for any $f \in L_\infty(\mu)$ the individual ergodic limit

$$\lim_n \frac{1}{n} \sum_{i=0}^{n-1} T^i f(x)$$

exists for almost all x in X.

On the other hand, Derriennic and Lin [3] have shown by an example that given an $\varepsilon > 0$ there exists a positive linear operator T on L_1 of a finite measure space, with $T1=1$ and $\|T^n\|_1=1+\varepsilon$ for all $n \geq 1$, such that for some f in L_1 the above individual ergodic limit does not exist for almost all x in a certain measurable subset of positive measure. This shows that Theorem A fails to hold if the function f in $L_\infty(\mu)$ is replaced by $f \in L_1(\mu)$. (So far it is not known whether the function $f \in L_\infty(\mu)$ can be replaced by $f \in L_p(\mu)$, with $1 < p < \infty$.)

In this note, however, we shall prove the following

Theorem. *Let T be a bounded linear operator on $L_1(\mu)$, where $(X, \mathcal{F}, \mu)$ is a σ-finite measure space. Assume that the linear modulus τ of T satisfies the above condition* (1) *and the next condition*: *For some constant M,*

$$(2') \qquad \sup_n \left\| \frac{1}{n} \sum_{i=0}^{n-1} \tau^i f \right\|_\infty \leq M \|f\|_\infty \quad \text{for all } f \in L_1(\mu) \cap L_\infty(\mu).$$

Assume, in addition, that for every $A \in \mathcal{F}$ with $0 < \mu(A) < \infty$

$$\limsup_n \frac{1}{n} \sum_{i=0}^{n-1} \tau^i 1_A(x) \not\equiv 0. \tag{3}$$

Then for any $f \in L_1(\mu)$ *the limit*

$$\lim \frac{1}{n} \sum_{i=0}^{n-1} T^i f(x) \tag{4}$$

exists for almost all x *in* X.

As an immediate corollary to the Theorem we have the

Corollary. *If* T *is an invertible positive linear operator on* $L_1(\mu)$, *with* μ *finite, such that*

$$\sup_{-\infty<n<\infty} \|T^n\|_1 < \infty \quad and \quad \sup_{n\geq 1} \|T^n\|_\infty < \infty,$$

then for any $f \in L_1(\mu)$ *the ergodic limit* (4) *exists for almost all* x *in* X.

2. Proof of the Theorem. Let T^* and τ^* denote the adjoint operators of T and τ, respectively. Recalling that $\tau f = \sup\{|Tg| : g \in L_1(\mu)$ with $|g| \leq f\}$ for every $0 \leq f \in L_1(\mu)$, we get $|T^* f| \leq \tau^*|f|$ for every $f \in L_\infty(\mu)$. Furthermore, choosing a sequence $g_1, g_2, \cdots$ in $L_1(\mu)$, with $0 \leq g_n \leq g_{n+1}$ and $\lim_n g_n = 1$, we get, by (2′),

$$\begin{aligned}\int \tau^*|f|\, d\mu &= \lim_n \int g_n \tau^*|f|\, d\mu = \lim_n \int (\tau g_n)|f|\, d\mu \\ &\leq M \int |f|\, d\mu \quad (f \in L_1(\mu) \cap L_\infty(\mu)).\end{aligned}$$

Thus T^* and τ^* can be extended to bounded linear operators R and ρ on $L_1(\mu)$, respectively. It is then easily seen that

$$\sup_n \left\|\frac{1}{n} \sum_{i=0}^{n-1} \rho^i\right\|_1 \leq M, \tag{5}$$

$$R^* = T \text{ and } \rho^* = \tau \text{ on } L_1(\mu) \cap L_\infty(\mu), \tag{6}$$

and hence that the linear modulus of R is ρ.

Next, take a function w in $L_1(\mu)$ with $\int w\, d\mu = 1$ and $w > 0$ a.e. on X, and put

$$Pf = w^{-1}\rho(fw) \qquad (f \in L_1(w\, d\mu)).$$

Since $L_1(w\, d\mu)$ is isomorphic to $L_1(\mu)$ by the mapping : $f \longrightarrow fw$, P on $L_1(w\, d\mu)$ is a representation of ρ on $L_1(\mu)$. Since

$$\int (Pf) gw\, d\mu = \int \rho(fw) g\, d\mu = \int f(\rho^* g) w\, d\mu$$

for $f \in L_1(w\,d\mu)$ and $g \in L_\infty(w\,d\mu) = L_\infty(\mu)$, it follows that

$$P^* = \rho^* \qquad \text{on} \quad L_\infty(w\,d\mu) = L_\infty(\mu).$$

Therefore we may apply Theorem 3.2 in [3], together with (5), (6) and (3), to infer that there exists a strictly positive P-invariant function in $L_1(w\ d\mu)$, which, in turn, implies that there exists a function v in $L_1(\mu)$ with $v > 0$ a.e. on X and $\rho v = v$.

Define

$$g_0(x) = \min\{v(x),\ 1\} \qquad (x \in X).$$

By a mean ergodic theorem (see e.g. [4], Theorem VIII.5.1) it follows that

$$\lim_n \left\| \frac{1}{n} \sum_{i=0}^{n-1} \rho^i g_0 - h \right\|_1 = 0$$

for some $0 \leq h \in L_1(\mu)$, with $\rho h = h$. Since $\rho = \tau^*$ on $L_\infty(\mu)$, we deduce from (1) that $h \in L_\infty(\mu)$. Further $h > 0$ a.e. on X because $\rho v = v$ and $v > 0$ a.e. on X.

To complete the proof, let us fix an $f \in L_1(\mu)$. Then we have

$$\begin{aligned}\int |Tf| h\ d\mu &\leq \int (\tau|f|) h\ d\mu = \int |f| \tau^* h\ d\mu \\ &= \int |f| \rho h\ d\mu = \int |f| h\ d\mu < \infty;\end{aligned}$$

therefore T can be regarded as a contraction operator on $L_1(h\,d\mu)$. Since $L_1(\mu) \subset L_1(h\,d\mu)$ and since $\frac{1}{n} \sum_{i=0}^{n-1} \rho^i u$ converges in the norm topology of $L_1(\mu)$ for every $u \in L_1(\mu)$, we finally apply Chacon's general ratio theorem [1] and Theorem 1 in [6] to infer that the limit

$$\begin{aligned}\lim_n \frac{1}{n} \sum_{i=0}^{n-1} T^i f(x) &= \left(\lim_n \frac{\sum_{i=0}^{n-1} T^i f(x)}{\sum_{i=0}^{n-1} \tau^i h(x)} \right) \left(\lim_n \frac{1}{n} \sum_{i=0}^{n-1} \tau^i h(x) \right) \\ &= \left(\lim_n \frac{\sum_{i=0}^{n-1} T^i f(x)}{\sum_{i=0}^{n-1} \tau^i h(x)} \right) \left(\lim_n \frac{1}{n} \sum_{i=0}^{n-1} \rho^{*i} h(x) \right)\end{aligned}$$

exists for almost all x in X with respect to the measure $h\,d\mu$, which is equivalent to μ because $h > 0$ a.e. on X. This establishes the Theorem.

3. Proof of the Corollary. By the Theorem it suffices to show that T satisfies condition (3). To do this, fix an $A \in \mathcal{F}$ with $\mu(A) > 0$. Then we get $\inf_{-\infty<n<\infty} \|T^n 1_A\|_1 > 0$, because $\sup_{-\infty<n<\infty} \|T^n\|_1 < \infty$. Therefore

$$\lim_n \inf \int \frac{1}{n} \sum_{i=0}^{n-1} T^i 1_A \, d\mu > 0,$$

and since μ is finite, we then apply Fatou's lemma and get

$$\lim_n \sup \frac{1}{n} \sum_{i=0}^{n-1} T^i 1_A(x) \not\equiv 0.$$

This completes the proof.

4. Remark. Lef T be as in the Theorem. By (6), T can be extended to a bounded linear operator on $L_\infty(\mu)$. Then by the Riesz convexity theorem T is again extended to a bounded linear operator on each $L_p(\mu)$, with $1 < p < \infty$. Let h be the function in the proof of the Theorem. Since $h \in L_1(\mu) \cap L_\infty(\mu)$, it follows that $L_p(\mu) \subset L_1(h\, d\mu)$ for every $1 \leq p \leq \infty$. Thus the proof of the Theorem shows that for every $1 \leq p \leq \infty$ and every $f \in L_p(\mu)$ the ergodic limit

$$\lim \frac{1}{n} \sum_{i=0}^{n-1} T^i f(x)$$

exists for almost all x in X.

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