On the $p'$-section sum in a finite group ring

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Let $G$ be a finite group, and $p$ a prime number. Let $C_1 = \{1\}, C_2, \ldots, C_r$ be all the $p$-regular classes of $G$. We denote by $S_i$ the $p$'-section containing $C_i$, namely $S_i = \{ \sigma \in G | \sigma' \in C_i \}$, where $\sigma'$ denotes the $p$'-component of the element $\sigma$ of $G (i = 1, 2, \ldots, r)$. In particular, $S_1$ is the set consisting of all the $p$-elements of $G$. Let $k$ be a field of characteristic $p$, and $J$ the Jacobson radical of the group ring $k[G]$.

Recently, the author is informed by T. Okuyama that in 1955 R. Brauer stated the following without proof.

Theorem (Brauer [2]). Let $\hat{S}_i = \sum \sigma$, where the summation is taken over all $\sigma \in S_i$. Then there holds that $\cap_{i=1}^r (0 : \hat{S}_i) = J$.

We recall here the previous paper [6]. There, we showed that $(0 : \hat{S}_1) \supset J$, while the inclusion $\cap_{i=1}^r (0 : \hat{S}_i) \subset J$ is an easy deduction of Proposition 1 in [6]. So that, we should like to provide a new proof of the above Theorem along with the arguments used in the proofs of these results.

As a consequence of the Theorem, we have that if $e$ is a primitive idempotent of $k[G]$, then $k[G] \hat{S}_e$ is the socle of $k[G] e$, provided $\hat{S}_e \neq 0$. The condition will be described by the value on $C_1$ of the Brauer character afforded by $k[G] e$. On the other hand, Okuyama's proof of the Theorem is different from ours. There, the condition $\hat{S}_e \neq 0$ is discussed in connection with the coefficients $a_i$'s which appear in the expression $e = \sum_{i=1}^r a_i \sigma$, $a_i \in k$. We refer to it at the end of this paper.

In the proof of the Theorem, from the beginning, we may assume that $k$ is a splitting field for $G$. In addition to the notations introduced above, we shall use the following.

Let $p$ be a prime divisor of $p$ in an algebraic number field containing the $|G|$-th roots of unity, and $\nu$ the exponential valuation associated with $p$ multiplied by a factor to make $\nu(p) = 1$. We assume henceforth $k$ is the residue class field of $\nu$. If $\alpha$ is a $p$-integer, then $\overline{\alpha}$ denotes the residue class of $\alpha$ in $k$. Let $\{\eta_1, \eta_2, \ldots, \eta_r\}$ and $\{\phi_1, \phi_2, \ldots, \phi_r\}$ be the set of the principal indecomposable Brauer characters of $G$ and the set of
the irreducible Brauer characters of $G$ respectively, in which we arrange the indices so that $(\eta_j, \phi_j) = \delta_{ij}$ for all $i, j$ ($i, j = 1, 2, \cdots, r$). The $k$-algebra $k[G]/J$ is isomorphic to a direct sum of full matrix algebras over $k$; $k[G]/J \cong \sum_{i=1}^{r} M(n_i, k)$. We assume that under the isomorphism the simple component corresponding to the irreducible $k$-character $\varphi_i$ is mapped onto $M(n_i, k)$ and so $n_i = \phi_i(1)$. If $I$ is a subset of $k[G]$, then $(0 : I)$ denotes the set of the right annihilators of $I$ in $k[G]$. Finally, we put $\lambda(\sum_{a \in k} a_a) = a_1$, where $a_a \in k$ and $1$ denotes the identity of $G$.

Now we enter into the proof of the Theorem. Let $S_i$ be a fixed $p'$-section and $\sigma \in C_i$. There holds that $\nu(\eta_i(\sigma)) \geq \nu(|C_0(\sigma)|)$ for all $\eta_i$ ([3], (84.14)). After a suitable change of indices if necessary, we may assume that the first $\eta_1, \eta_2, \cdots, \eta_i$ are all that enjoy the equality sign in the above. We put $\eta_i(\sigma) = \eta_i(\sigma)/\mu \sigma$, where $|C_0(\sigma)| = \mu \sigma h$, $(\mu, h) = 1$.

From the orthogonality relation

$$\sum_{i=1}^{l} \eta_i(\sigma) \phi_i(\tau^{-1}) = \begin{cases} |C_0(\sigma)| & \text{if } \tau \text{ is conjugate to } \sigma \\ 0 & \text{otherwise} \end{cases}$$

and that $\varphi_i(\tau) = \varphi_i(\tau^\sigma)$ for any element $\tau$ of $G$, we get (reducing mod $\mu$)

(*) \hspace{1cm} \sum_{i=1}^{l} \eta_i(\sigma) \varphi_i(\tau^{-1}) = \begin{cases} h & \text{if } \tau \in S_i \\ 0 & \text{otherwise.} \end{cases}$

Let $U_j = \sum_{i} k[G] e + J$, where $e$ runs over the primitive idempotents such that $k[G] e$ affords a Brauer character $\eta_i$ with $s > t$. Then $U_j$ is a two sided ideal of $k[G]$. In fact, it is the inverse image of $\sum_{i=1}^{l} M(n_i, k)$ by the composite map $k[G] \rightarrow k[G]/J = \sum_{i=1}^{l} M(n_i, k)$. We identify $k[G]/U_j$ with $\sum_{i=1}^{l} M(n_i, k)$ and denote by $\rho_i$ the projection of $k[G]/U_j$ onto $M(n_i, k)$. If we put $\mu = \sum_{i=1}^{l} \eta_i(\sigma) \rho_i$, then $\mu$ is a (symmetric) non-singular linear function on $k[G]/U_j$. Hence by Theorem 9 of Nakayama [3] (or see [1], (55.11)), there exists an element $c$ of $k[G]$ such that $(0 : U_j) = k[G] c$ and $\eta_i(\sigma) = \lambda(\eta_i(\sigma))$ for all $x \in k[G]$, where $\phi$ is the natural map $k[G] \rightarrow k[G]/U_j$. From this, by making use of (*), we get easily that $c = h \hat{S}_j$ and hence $(0 : \hat{S}_j) = U_j$, as $k[G]$ is Frobeniusian. It is clear that $\cap_{j=1}^{l} U_j = J$ (namely, for any $i$, there exists a $p$-regular element $\sigma$ such that $\nu(\eta_i(\sigma)) = \nu(|C_0(\sigma)|)$. This follows easily, for instance, from the relation $(\eta_i, \phi_j) = 1)$. Thus we conclude that $\cap_{j=1}^{l} (0 : \hat{S}_j) = J$ and the proof is complete.

From the above argument, we get also

**Corollary A.** Let $e$ be a primitive idempotent of $k[G]$, and let $\eta$
be the Brauer character afforded by $k[G]e$. If $S_j$ is a $p'$-section of $G$, then the following are equivalent:

1. $\hat{S}_j e \neq 0$.
2. $\nu(\eta(\sigma)) = \nu(|C_\sigma(\sigma)|)$, where $\sigma$ is a $p$-regular element in $S_j$.

We continue our argument to give an alternative proof to the following result of Okuyama.

**Corollary B** (Okuyama [5]). Under the same notation as in Corollary A, let $e = \sum_{\sigma \in S_j} a_\sigma e$. Then the following are equivalent:

1. $\hat{S}_j e \neq 0$.
2. $\sum a_\sigma \neq 0$, where the summation is taken over all $\tau \in S_j^{-1} = \{\sigma^{-1} | \sigma \in S_j\}$.

**Proof.** Recall that if $\xi$ is a symmetric linear function on $M(n, k)$, then there exists some $a \in k$ such that $\xi(x) = a \cdot \text{tr}(x)$ for all $x \in M(n, k)$ (since the set $\{xy - yx | x, y \in M(n, k)\}$ spans the subspace consisting of the elements of trace zero). In particular, we have $\xi(e) \neq 0$ for any primitive idempotent $e$. Keeping the notation used in the proof of the Theorem, we know that $\mu$ is a symmetric, non-singular linear function on $k[G]/U_j = \sum_{-1}^{j-1} M(n, k)$ and hence $\mu(e) \neq 0$ for any primitive idempotent $e$ of $k[G]$ not contained in $U_j$. And if $e = \sum a_\sigma e$, then $\mu(e) = \sum a_\sigma \eta(\sigma) \overline{\phi_\eta(\tau)} = (\sum a_\sigma) \overline{h}$, where the second summation is taken over all $\tau \in S_j^{-1}$. From these observations, we get the above assertion.

**Remark.** According to a result of Okuyama [5], there holds that the summation $\sum a_\sigma$ in the above (2) is equal to the restricted summation $\sum a_\sigma$, where $\tau$ runs over all $\tau \in S_j^{-1}$.

**REFERENCES**


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