On powers of artinian rings without identity

Ichiro Murase\textsuperscript{*} \quad Hisao Tominaga\textsuperscript{†}

\textsuperscript{*}Japan Women’s University
\textsuperscript{†}Okayama University

ON POWERS OF ARTINIAN RINGS WITHOUT IDENTITY

ICHIRO MURASE and HISAO TOMINAGA

0. Introduction. Throughout this paper an Artinian (Noetherian) ring means a left Artinian (Noetherian) ring, i.e., an associative ring with minimum (maximum) condition on left ideals. The existence of an identity is not assumed.

Recently L. S. Levy [3] proved that there is a surprising abundance of indecomposable Artinian, non-Noetherian rings; moreover they can be nonnilpotent. Here an indecomposable ring means a ring which is not the ring-direct sum of two nonzero rings, and this restriction aims to rule out uninteresting trivial cases. According to Hopkins' famous theorem ([2], p. 728)), every Artinian, non-Noetherian ring can not have a left or right identity, because an Artinian ring $A$ is necessarily Noetherian if $A$ contains such an identity. Therefore we have a large class of nonnilpotent rings which can not contain a left or right identity by the nature of themselves. The present paper is motivated by this interesting result.

Let $A$ be any Artinian ring, and consider the descending chain of left ideals: $A \supseteq A^2 \supseteq A^3 \supseteq \cdots$. Then after a finite number of terms we have equalities only. We are interested in the subrings $A^k$. We shall prove that all $A^k$ for $k \geq 2$ are Artinian and Noetherian even if $A$ is non-Noetherian. It will be further proved that if $A \neq A^2$ then every $A^k$ can not contain a (two-sided) identity.

As is well known, in an Artinian, non-Noetherian ring $A$ the additive group of $A$ contains a divisible torsion subgroup ([1], p. 285). There exists a unique maximal divisible, torsion subgroup $D$ of $A$. The subgroup $D$ is contained in the total annihilator $W$ of $A$, i.e., $DA=0=AD$ ([1], p. 281). We shall consider such a ring $A$ and prove the following theorems. But, $N$ denotes the radical of $A$.

First, if $A$ is indecomposable, then $A \neq A^2$, $N^2 \neq 0$, $A^2 \cap D \neq 0$ and $A^2$ contains no left or right identity. Next, according to Levy [3], if $A$ is indecomposable then the ring $S=A/D$ can not have a left or right identity. However, it can be proved more generally that $A/W$ can not have a left or right identity, whether $A$ may be non-Noetherian or not, provided that $A$ is indecomposable. Further, if $S=S^2$ and if $A$ is indecomposable, then every $A^k$ can not contain a left or right identity.
1. Every $A^k$ is Artinian.

Theorem 1. If $A$ is an Artinian (Noetherian) ring, then every subring $A^k$ is Artinian (Noetherian).

Proof. We state the proof only for Artinian case. The slight modification needful in Noetherian case is obvious.

Assume that $A^k$ is Artinian for some integer $k$. Let $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ be any descending chain of left ideals of $A^{k+1}$. We claim that after a finite number of terms only the equality holds.

Consider first the descending chain of left ideals of $A^k$, $A^k L_1 \supseteq A^k L_2 \supseteq \cdots$. Then by assumption there exists an integer $m$ such that $A^k L_m = A^k L_{m+1} = \cdots = M$.

Consider next the following two chains:

$$L_n + M \supseteq L_{n+1} + M \supseteq \cdots,$$

$$L_n \cap M \supseteq L_{n+1} \cap M \supseteq \cdots.$$

All $L_j + M$ and $L_j \cap M$ ($j = m, m+1, \cdots$) are left ideals of $A^k$, because $A^k (L_j + M) = A^k (L_j + A^k L_j) \subseteq M + A^{k+1} L_j \subseteq M + L_j$, and similarly $A^k (L_j \cap M) \subseteq M \cap L_j$. Therefore there exists an integer $n \geq m$ such that

$$L_n + M = L_{n+1} + M = \cdots,$$

$$L_n \cap M = L_{n+1} \cap M = \cdots.$$

Then, using the modular law, we can have

$$L_n = L_n \cap (L_n + M) = L_n \cap (L_{n+1} + M) = L_{n+1} + (L_{n+1} \cap M) = L_{n+1}.$$

The proof can be now completed by induction on $k$.

2. Principal Peirce decompositions of $A$ and $A^k$. Let $A$ be any nonnilpotent Artinian ring, and $N$ the radical of $A$. Then the identity of the ring $A/N$ can be lifted to an idempotent $e$ of $A$, which will be called a principal idempotent of $A$. We have the Peirce decompositions:

$$A = Ae + L, \quad A = eA + R,$$

where $L$ is the left annihilator of $e$ in $A$, and $R$ the right annihilator of $e$ in $A$. Naturally both $L$ and $R$ are contained in $N$, and so we
have

\[ N = Ne + L, \quad N = eN + R. \]

Since \( R = Re + R \cap L \) and \( L = eL + L \cap R \),

(1)

\[ A = eAe + Re + eL + L \cap R, \]

\[ N = eNe + Re + eL + L \cap R. \]

We call (1) the principal Peirce decomposition of \( A \) with respect to \( e \).

Let \( T = L \cap R \). Then we have

\[ RL = (Re + T) (eL + T) = ReL + T^2, \]

\[ RT^k = (Re + T) T^k = T^{k+1} = T^kL \]

for all positive integers \( k \).

**Theorem 2.** Let \( A \) be any nonnilpotent Artinian ring, and let \( T_k = ReL + T^k \) \((k \geq 1)\). Then there hold the following:

(i) \( A^* = eAe \oplus Re \oplus eL \oplus T_k \).

(ii) Let \( N_k = eNe \oplus Re \oplus eL \oplus T_k \). Then \( N_k \) is the radical of \( A^* \).

(iii) \( N_k = A^{k-1}N + NA^{k-1} \). But, here \( k \geq 2 \).

(iv) \( A^*/N_k \cong A/N \) (a ring-isomorphism).

**Remark.** Here the notation \( \oplus \) means a module-direct sum, while \( + \) means merely a linear sum. However, when it is self-evident and there is no fear of confusion, we write \( + \) also for \( \oplus \).

**Proof.** (i) Recall (1). Then it is obvious that (i) holds for \( k = 1 \). Therefore it can be proved by induction on \( k \). Assume that it holds for some integer \( k \). Then we have

\[ A^{k+1} = (eA + R) (eAe + Re + eL + ReL + T^k) \]

\[ = eA (eAe + Re) + R (eAe + Re) \]

\[ + eA (eL + ReL + T^k) + R (eL + ReL + T^k) \]

\[ = eA + Re + eL + ReL + T^{k+1}. \]

(ii) – (iv) We have

\[ A^{k-1}N + NA^{k-1} = (eAe + Re + eL + ReL + T^{k-1}) (Ne + L) \]

\[ + (eN + R) (eAe + Re + eL + ReL + T^{k-1}) \]

\[ = (eAe + eL)Ne + (Re + ReL + T^{k-1})Ne \]

\[ + (eAe + eL)L + (Re / ReL + T^{k-1}) L \]

\[ + eN(eAe + Re) + eN(eL + ReL + T^{k-1}) \]

\[ + R(eAe / Re) + R(eL + ReL + T^{k-1}). \]

Deleting redundant terms, we get
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\[ A^{k-1}N + NA^{k-1} = eNe + eL + Re + ReL + T^k = N_k. \]

Clearly \( N_k \) is a nilpotent two-sided ideal of \( A^k \). Moreover it is easy
to see the following ring-isomorphisms:

\[ A^k/N_k \cong eAe/eNe \cong A/N. \]

Hence \( N_k \) is the radical of the Artinian ring \( A^k \).

Remember that (i) is the principal Peirce decomposition of \( A^k \) with
respect to \( e \).

Consider the descending chain of left ideals:

\[ A \supseteq A^2 \supseteq A^3 \supseteq \ldots. \]

Let \( \rho \) be the nilpotency exponent of \( N \). Then \( T^\rho = 0 \), because \( T = L \cap R \subseteq N \), and so

\[ A^\rho = eAe + Re + eL + ReL. \]

Besides, \( A^k = A^\rho \) for all \( k \geq \rho \).

**Theorem 3.** For a non-nilpotent Artinian ring \( A \), we have \( A^k = A^{k+1} \)
if and only if \( T^k \subseteq ReL \).

**Proof.** Clearly \( A^k = A^{k+1} \) is equivalent to \( A^k = A^\rho \), which holds if
and only if \( ReL \rightarrow T^\rho = ReL \). This is equivalent to \( T^k \subseteq ReL \).

**Theorem 4.** Let \( A \) be a non-nilpotent Artinian ring. Then there
hold the following:

(i) \( A = A^2 \) if and only if \( R \cap L = RL \).

(ii) \( A^2 = A^3 \) if and only if \( RL = ReL \).

**Proof.** (i) By Theorem 3, \( A = A^2 \) if and only if \( R \cap L \subseteq ReL \).

Note the following relation:

\[ R \cap L \supseteq RL \supseteq ReL. \]

Then (i) is obvious.

(ii) By Theorem 3, \( A^2 = A^3 \) if and only if \( T^2 \subseteq ReL \), which is
equivalent to \( RL = ReL \), because \( RL = ReL + T^\rho \).

3. **Every** \( A^k (k \geq 2) \) **is Noetherian.** In the previous paper [4], we
have proved the following theorem: An Artinian ring \( A \) is Noetherian if
and only if \( R \) is a finite set. But, let \( R = A \) if \( A \) is nilpotent.

Now, recall the theorem of Hopkins that \( Re \) is a finite set, which
was also reproved in [4]. Then, since \( R = Re + R \cap L = Re + T \),
we can restate the above theorem as follows.

**Theorem 5.** An Artinian ring \( A \) is Noetherian if and only if \( T \) is
a finite set. But, let \( T = A \) if \( A \) is nilpotent.

For further study we cite also the following result of the previous paper \([4]\): In an Artinian ring \( A \), the group \((R, +)\) satisfies the minimum condition on subgroups. But, let \( R = A \) if \( A \) is nilpotent.

**Theorem 6** (cf. the proof of \([3]\), Proposition 2.6). In a nonnilpotent Artinian ring \( A \), \( ReL \) is a finite set.

**Proof.** The elements of \( ReL \) are finite sums \( \sum a_i b_i, a_i \in Re, b_i \in eL \). Therefore \( ReL \) forms a subgroup of the group \((R, +)\), and it satisfies the minimum condition on subgroups. Besides, the group \( ReL \) is of bounded order, because \( Re \) is a finite subgroup of \((R, +)\). Now, as is well known, an additive Abelian group \( G \) of bounded order is a direct sum of cyclic groups. If \( G \) moreover satisfies the minimum condition, then the number of the summands must be finite, and hence \( G \) is finite. By this reason, \( ReL \) is finite.

**Theorem 7** (Levy \([3]\), p. 281). Let \( A \) be an Artinian ring. If \( A = A^2 \) then \( A \) is Noetherian.

**Proof.** Clearly we can assume that \( A \) is nonnilpotent. If \( A = A^2 \), then we have \( T = RL = ReL \) by Theorem 4. Hence \( T \) is finite, and so \( A \) is Noetherian by Theorem 5.

According to Fuchs \([1]\), an Artinian ring \( A \) is Noetherian if and only if the group \((A, +)\) contains no quasicyclic \( p \)-group. A quasicyclic \( p \)-group is a group of type \( Z(p^\infty) \), and so it is a divisible torsion group. Furthermore such a group belongs to the total annihilator \( W \) of \( A \).

Let \( A \) be an Artinian, non-Noetherian ring. Then by the above theorem, \( A \) contains a divisible torsion subgroup, and the subgroup is contained in \( R \), because naturally \( W \subseteq R \).

Recall now the following theorem of Kuroš ([1], p. 65): The subgroups of an additive Abelian group \( G \) satisfy the minimum condition if and only if \( G \) is a direct sum of a finite number of quasicyclic and/or cyclic \( p \)-groups.

Since the group \((R, \div)\) satisfies the minimum condition on subgroups, the theorem of Kuroš can be applied to the subgroup \( T \) of \( R \). Thus we have

\[
(3) \quad T = T_0 + D, \quad T_0 \cap D = 0,
\]

where \( T_0 \) is a finite subgroup and \( D \) is the direct sum of a finite number
of quasicyclic $p$-groups. Therefore we can write
\[ A = A_0 + D, \quad A_0 \cap D = 0, \]
where $A_0 = eAe + Re + eL + T_0$. Then it is obvious that $D$ is a unique maximal divisible, torsion subgroup of the additive group of $A$.

Remark. The following theorem of Szász-Levy ([3], p. 281) is worthy of note: If an Artinian, non-Noetherian ring $A$ is indecomposable, then the additive group of $A$ is primary. It implies that the subgroups of type $Z(p^n)$ of $A$ are of the same prime $p$.

**Theorem 8.** Let $A$ be any Artinian ring. Then every subring $A^k$ is Noetherian for $k \geq 2$.

**Proof.** If $A$ itself is Noetherian, then every $A^k$ is Noetherian. It is already proved in Theorem 1. Therefore there remains the case where $A$ is non-Noetherian. In this case the group $(A, +)$ contains the maximal divisible, torsion subgroup $D$.

First, assume that $A$ is non-nilpotent, and consider the Peirce decompositions (1) of $A$ and (i) of $A^k$ in Theorem 2. The term $T$ can be written as (3). Note that $T_0$ is a finite subgroup of $T$. Then obviously $T^k = T_0^k$, and it is finite. Therefore, by Theorem 6, $T_k = ReL + T^k$ is finite, too. Hence $A^k$ is Noetherian by Theorem 5.

Next, assume that $A$ is nilpotent. Then, since the group $(A, +)$ satisfies the minimum condition, by the theorem of Kuroš we can write $A = A_0 + D, \quad A_0 \cap D = 0$, where $A_0$ is a finite subgroup. Then clearly $A^k = A_0^k$, and it is finite. Hence $A^k$ is Noetherian.


**Theorem 9.** Let $A$ be an Artinian ring such that $A \neq A^2$. Then every $A^k$ can not contain an identity.

**Proof.** In case $A$ is nilpotent, it is trivial. Further, if $A$ is decomposable and $A \neq A^2$, then for some indecomposable direct summand $A_i$ we have $A_i \neq A_i^2$. Therefore it is clear that $A$ may be assumed to be indecomposable and non-nilpotent.

Under this assumption, suppose that $A^k$ contains an identity $e'$. We first claim that $e'$ is a principal idempotent of $A$.

Consider (1) and write an element $a$ of $A$ as
\[ a = a_{11} + a_{01} + a_{10} + a_{00}, \]
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\[ a_{11} \subseteq eAe, \ a_{01} \subseteq Re, \ a_{10} \subseteq eL, \ a_{00} \subseteq T. \]

Then \( a + N = a_{11} + N \). Since \( a_{11} \subseteq A^k \), we have

\[
(e' + N)(a + N) = (e' + N)(a_{11} + N) = e'a_{11} + N = a_{11} + N = a + N.
\]

Therefore \( e' + N \) is the identity of the semisimple ring \( A/N \), and hence \( e' \) is a principal idempotent of \( A \).

Take now \( e' \) as the principal idempotent \( e \) for (1), and consider (i) of Theorem 2. Then we must have \( A^k = eAe \), and so \( Re = 0 \) and \( eL = 0 \). Consequently,

\[ A = eAe + T, \quad A^2 = eAe + T^2. \]

Here \( T \neq 0 \), because \( A \neq A^2 \). Moreover, \( A = eAe + T \) is clearly a ring-direct sum. It is a contradiction.

**Theorem 10.** Let \( A \) be a nonnilpotent Artinian, non-Noetherian ring. If \( A \) is indecomposable, then there hold the following:

(i) \( A \neq A^2 \).

(ii) \( N^2 \neq 0 \).

(iii) \( A^2 \cap D \neq 0 \).

(iv) \( A^2 \) contains no left or right identity.

**Proof.** (i) It is clear by Theorem 7.

(ii) Suppose \( N^2 = 0 \). Then by (i) of Theorem 2, we have \( A^2 = eAe + Re + eL \), because both \( ReL \) and \( T^2 \) are contained in \( N^2 \). Therefore we have \( A = A^2 + T \) and \( A^2 T = 0 = TA^2 \). It follows that \( A \) is the ring-direct sum of \( A^2 \) and \( T \), contradictory to assumption.

(iii) Suppose \( A^2 \cap D = 0 \). Note that \( D \) is a direct summand of \( A \) and that the complementary summand can be so chosen as to contain \( A^2 \) ([1], p. 63). Therefore we can write

\[ A = B + D, \quad B \cap D = 0, \quad A^2 \subseteq B. \]

Then \( B^2 = (B + D)^2 = A^2 \subseteq B. \) Hence \( A = B \oplus D \), a ring-direct sum. It is a contradiction.

(iv) By reason of (iii) \( A^2 \) contains a nonzero element of \( D \). It annihilates every element of \( A^2 \). Therefore \( A^2 \) can not contain a left or right identity.

**Theorem 11.** Let \( A \) be a nonnilpotent Artinian ring with radical \( N \) of nilpotency exponent \( \rho \geq 2 \). If \( A^{\rho - 1} \neq A^\rho \) and \( A^{\rho - 1} \) is indecomposable, then \( A^\rho \) contains no left or right identity.

**Proof.** Suppose that \( A^\rho \) contains a left identity \( e' \). By the same
argument as that in the proof of Theorem 9, we can see that $e'$ is a principal idempotent of $A$. Take $e'$ as $e$ for (1), and consider (2). Then we must have $A^p = e Ae + e L$, and

$$A^{p-1} = (e Ae + e L) + T^{p-1} = A^p + T^{p-1}.$$  

Here $(e Ae) T^{p-1} = 0 = T^{p-1} (e Ae)$, $(e L) T^{p-1} \subseteq N^p = 0$ and $T^{p-1} (e L) = 0$. Therefore $A^{p-1}$ is the ring-direct sum of $A^p \neq 0$ and $T^{p-1} \neq 0$. It is a contradiction.

5. The ring $S = A/D$.

**Theorem 12.** Let $A$ be a nonnilpotent Artinian ring with the total annihilator $W \neq 0$. If $A$ is indecomposable, then the ring $A/W$ contains no left or right identity.

**Proof.** Suppose that $A/W$ contains a left identity $f + W$, $f \in A$. Then $f^2 \equiv f \pmod W$, and $f$ acts on the elements of $A$ as a left identity modulo $W$. Since $W$ is contained in the radical $N$ of $A$, the element $f$ acts on $A$ as a left identity modulo $N$. Hence $f + N$ is the identity of $A/N$.

Let $w = f^2 - f$ and $e = f + w$. Then $e^2 = f^2 = f + w = e$, and $e \equiv f \pmod N$. Therefore $e$ is a principal idempotent of $A$. Consider the Peirce decomposition of $A$ with respect to $e$, and let it be (1). Then we have naturally $W \subseteq R \cap L$.

For every element $x$ of $R$,

$$fx = (e - w)x = ex - wx = 0.$$  

On the other hand we have $fx \equiv x \pmod W$. Therefore $x \in W$, and we see $R \subseteq W$. It follows that $Re = 0$ and $R \cap L \subseteq W$. Hence $W = R \cap L$. Therefore we have

$$A = (e Ae + e L) \oplus W.$$  

This is a ring-direct sum. It is a contradiction.

**Theorem 13** (Levy [3], p. 290). If a nonnilpotent Artinian, non-Noetherian ring $A$ is indecomposable, then the ring $S = A/D$ contains no left or right identity.

**Proof.** Suppose that $S$ has a left identity $f + D$. Then $f$ acts on the elements of $A$ as a left identity modulo $D$, and also modulo $W$, because $D \subseteq W$. Then $f + W$ is a left identity of $A/W$, contradictory to Theorem 12.
Theorem 14. Let $A$ be a nonnilpotent Artinian, non-Noetherian ring, and let $S = A/D$. Then $S = S^2$ if and only if $T = ReL + D$.

Proof. Consider (1) and (2):

$$A = e Ae + Re + eL + T,$$

$$A^p = e Ae + Re + eL + ReL.$$  

The condition $S = S^2$ is equivalent to $S = S^p$, i.e.,

$$A/D = (A/D)^p = (A^p + D)/D.$$  

Therefore, if $S = S^2$, then we have $A = A^p + D$, which implies $T = ReL + D$.

Theorem 15. Let $S = A/D$ under the same assumption as that in Theorem 14. If $S = S^2$, and if $A$ is indecomposable, then every $A^p$ can not contain a left or right identity.

Proof. By Theorem 14, $T = ReL + D$. Here we have $ReL \cap D \neq 0$, because otherwise we would have

$$A = A^p \oplus D$$  

(a ring-direct sum), contradictory to assumption. Therefore every $A^p$ contains a nonzero element of $D$, and hence it can not contain a left or right identity.

As for the case of $S \neq S^2$, it is not yet certain to the authors whether $A^p (= A^{p+1} = \cdots)$ can contain a one-sided identity or not. Remember that $A^p$ is an idempotent Artinian, Noetherian ring. In general, some of such rings can contain a one-sided identity, others can not.

Example. Let $K = \mathbb{Z}/p\mathbb{Z}$ ($p$ a prime), and consider the ring $A$ expressed as follows:

$$A = \begin{pmatrix} K & 0 & 0 \\ K & 0 & 0 \\ K & K & K \end{pmatrix}.$$  

It is an Artinian, Noetherian ring with $A = A^2$. The radical of $A$ is

$$N = \begin{pmatrix} 0 & 0 & 0 \\ K & 0 & 0 \\ K & K & 0 \end{pmatrix}.$$  

We have $AN \neq N$ and $NA \neq N$. Therefore $A$ contains neither left nor right identity.
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DEPARTMENT OF MATHEMATICS
JAPAN WOMEN'S UNIVERSITY
2-8-1, MEJIRODAI, BUNKYO-KU
TOKYO, 112 JAPAN

AND

DEPARTMENT OF MATHEMATICS
OKAYAMA UNIVERSITY
3-1-1, TSUSHIMA-NAKA
OKAYAMA, 700 JAPAN

(Received April 3, 1978)