Covers of abelian groups

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COVERS OF ABELIAN GROUPS

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Introduction. Recently papers have appeared in the literature \([4, 5, 9]\) dealing with rings and finite unions of subrings or ideals. This paper deals with abelian groups and finite unions of subgroups.

If \(G\) is an abelian group and \(G = H_1 \cup \cdots \cup H_n\) where each \(H_i\) is a proper subgroup of \(G\) and \(n\) is finite, the set \(\{H_i\}\) is called a cover of \(G\). In this paper we assume all groups are abelian and that a cover ceases to be a cover if any member of it is omitted. We call a cover maximal if no member of it has a cover. If groups \(G\) and \(G'\) have covers \(C\) and \(C'\), respectively, then \(C\) and \(C'\) are said to be isomorphic if the members of \(C\) are isomorphic (as groups) to the members of \(C'\) in a one-to-one manner. An element or cyclic subgroup of a group \(G\) is called special if it is in exactly one cyclic subgroup of \(G\). We ask the questions: (1) What groups have maximal covers? (2) If two groups have isomorphic maximal covers, are the groups isomorphic? (3) If a group has no cover, what is its structure? The answers are found in the theorems.

1. General facts. Let a group \(G\) have the cover \(\{H_i\}\). we have the following facts.

(a) The integer \(n\) cannot be 1 or 2.
(b) If \(n=3\), \(G/\cap H_i\) is the Klein 4-group.
(c) A necessary and sufficient condition for a group \(G\) to have a cover is that \(G/pG\) is not cyclic for some prime \(p\).
(d) \(G/H_i\) is finite.
(e) If the cover is maximal and \(H\) is a subgroup of finite index in some \(H_i\), then \(H_i/H\) is cyclic.
(f) If \(G\) is finite and the cover is maximal, then \(\cap H_i\) is the direct sum of the primary components of \(G\) which are cyclic.
(g) If the cover is maximal, it is unique.

The proof of (a) is trivial. The proofs of (b), (c), and (d) may be found in \([3], [1]\), and \([7]\) (or in \([8]\), page 227), respectively. To show (e), suppose \(H_i/H\) is not cyclic. Since it is finite, it has a non-cyclic primary component and, by (c), a cover, say \(\{K_i/H\}\), \(K_i \subset G\). Then \(\{K_i\}\) covers \(H_i\) which contradicts the maximality of \(\{H_i\}\). In (f), since \(G\) is finite, its maximal cover consists of its special subgroups. A primary component of
which is cyclic is clearly contained in each special subgroup of $G$. Conversely, if $x$ is an element of $G$ not contained in the direct sum of the primary components of $G$ which are cyclic, then $G$ has a special subgroup not containing $x$. To prove (g), suppose $\{H_i\}$ and $\{K_i\}$ are two maximal covers of $G$. Let $H = \cap H_i$ and $K = \cap K_i$. Since $H$ and $K$ have finite index in $G$, so does $H \cap K$. Therefore $G/H \cap K$ is finite and has covers $C = \{H_i/ H \cap K\}$ and $C' = \{K_i/ H \cap K\}$. By the maximality of $\{H_i\}$ and $\{K_i\}$, $C$ and $C'$ are maximal covers of $G/H$. Each consists of the special subgroups of $G/H$ and $C = C'$. For some arrangement of indices, then, $H_i = K_i$ for each $i$.

2. Finite groups. Finite non-cyclic groups have maximal covers (consisting of their special subgroups) and for these groups we seek an answer to question 2 of the introduction. The number of special elements of order $m$ in a finite group $G$ is designated by $S_o(m)$. We will provide formulae to determine this number for an arbitrary finite group $G$ and positive integer $m$. We will then use these formulae to prove a theorem.

**Formula 1.** Let $G$ be a $p$-primary finite group and, for $n \geq 0$, let $r_n$ equal the number of cyclic subgroups of order $p^n$ in some decomposition of $G$ into a direct sum of cyclic subgroups. Set $r_0 = 0$. Then, for $k > 0$

$$S_o(p^k) = (\prod_{n < k} p^{(n-1)r_n} \prod_{n > k} p^{(n-k+1)r_n}) (\prod_{n < k} p^{r_n} - \prod_{n > k} p^{r_n} + 1).$$

**Proof.** Suppose $p^nG = 0$ and $G = B_0 \oplus \cdots \oplus B_n$ is the decomposition of $G$ where each $B_n$ is a direct sum of $r_n$ cyclic subgroups of order $p^n$. Suppose $g$ is a special element in $G$ of order $p^k$, $k > 0$, and let $g = x + y + z$ where $x$, $y$, $z$ are members of $\bigoplus_{n < k} B_n$, $B_n$, $\bigoplus_{n > k} B_n$, respectively. We observe that $y$ or $x$ must be special in $G$. Case 1. Assume that $y$ is special in $G$. Then $o(y) = p^k$ and the number of choices for $y$ is $|B_n| - |pB_n|$. There are $|\bigoplus_{n > k} B_n|$ choices for $x$ and, since $o(x) \leq p^k$, there are exactly $|\bigoplus_{n > k} B_n|$ choices for $z$. The number of choices for $g$ is the product of these three numbers which is, by computation,

$$\prod_{n < k} p^{r_n} (p^{r_k} - p^{r_k - r_n}) \prod_{n > k} p^{r_n}.$$ 

Case 2. Assume $y$ is not special in $G$ and hence $x$ is. Then $y$ is in $pB_k$ and $o(x) = p^k$. The number of choices for $x$, $y$, $z$ is, in turn, $\prod_{n < k} |B_n| - \prod_{n < k} |pB_n|$, $|pB_n|$, and $\prod_{n > k} p^{-k}B_n| - \prod_{n > k} |p^{n-k+1}B_n|$. The number of choices for $g$ is the product of these numbers which equals
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\[(\prod_{n<k} p^m - \prod_{n<k} p^n - \prod_{n>k} p^{m-n}) p^{m-1}k \quad (\prod_{n<k} p^m - \prod_{n>k} p^{m-n})\]

Adding the results of the two cases and simplifying, we obtain the formula.

Formula 2. If a finite group \(G\) is the direct sum of \(t\) \(P_i\)-components \(G_i\) for distinct primes \(p_1, \ldots, p_t\), and \(m = p_1^{k_1} \cdots p_t^{k_t}\), \(k_i \geq 0\), then \(S_0(m)\) equals the product \(S_{0_1}(p_1^{k_1}) \cdots S_{0_t}(p_t^{k_t})\).

Proof. This follows from the observation that an element \(g\) is special in \(G\) exactly if \(g = x_1 + \cdots + x_t\) where each \(x_i\) is special in \(G_i\).

Theorem 1. Two finite primary groups are isomorphic if and only if they are cyclic of the same order or they have isomorphic maximal covers. The restriction to primary groups is necessary.

Proof. Isomorphic finite groups are cyclic of like order or have isomorphic maximal covers by Fact (g). Suppose, then, \(G\) and \(G'\) are two finite \(p\)-primary groups with isomorphic maximal covers \(C\) and \(C'\), respectively. Let \(M\) be the least positive integer such that \(p^M G = 0 = p^M G'\). Choose a set of ranks \(\{r_i\}\) for \(G\) as in Formula 1, and, similarly, a set of ranks \(\{s_i\}\) for \(G'\). We will show \(r_i = s_i\) for each \(i\). Since \(C\) and \(C'\) consist of the special subgroups in \(G\) and \(G'\), these groups have the same number of special subgroups of order \(p^k\) for each \(k\). Since a special subgroup of order \(p^k\) contains exactly \(p^k - p^{k-1}\) special elements, \(S_0(p^k) = S_0(p^k)\) for each \(k\). We now use these equations to show \(r_i = s_i\) for all \(i\). We use induction beginning with \(n = M\). By our choice of \(M\), \(S_0(p^n) = S_0(p^n) \neq 0\). If we make the substitutions in this equation indicated by Formula 1, we obtain an equation of the form \(p^k(p^n-1) = p^k(p^n-1)\) for some integers \(a\) and \(b\). Thus, \(r_N = s_N\). Assume \(r_n = s_n\) for \(n > N\) and that \(S_0(p^n) = S_0(p^n) \neq 0\) (otherwise \(r_n = s_n = 0\)). By Formula 1, we have

\[p^k(\prod_{n<N} p^n + \prod_{n>N} p^n + 1) = p^k(\prod_{n<N} p^n - \prod_{n>N} p^n + 1)\]

for some integers \(a\) and \(b\). Now, if \(S_0(p^n) = S_0(p^n) = 0\) for all \(n < N\), then \(\prod_{n<N} p^n = 1 = \prod_{n>N} p^n\) and \(\prod_{n<N} p^n - \prod_{n>N} p^n + 1 = 1 = \prod_{n<N} p^n - \prod_{n>N} p^n + 1\). We cancel the two right-hand terms on each side and obtain \(\prod_{n<N} p^n = \prod_{n<N} p^n = 1 = \prod_{n<N} p^n = \prod_{n<N} p^n - \prod_{n<N} p^n + 1\).

Thus \(\sum_{N}^M r_n = \sum_{N}^M s_n\) and \(r_n = s_n\). By induction, \(r_n = s_n\) for all \(n\) and \(G \cong G'\).
We verify the second sentence of the theorem by an example. For positive integers \(n\) and \(m\), let \(Z_n^m\) be the direct sum of \(m\) copies of the integers modulo \(n\). Let \(G = Z_2^3 \oplus Z_3^2\) and \(G' = Z_1^3 \oplus Z_2^2\). By Formulae 1 and 2, \(S_0(10) = (2^5 - 1)(5 - 1) = 124 = (2 - 1)(5^3 - 1) = S_0(10)\), yet \(G\) is not isomorphic to \(G'\).

**Remarks.** 1) In Formula 1, \(S_0\) is evaluated in terms of a particular direct sum decomposition of \(G\), but \(S_0\) is, by definition, independent of this decomposition. 2) The proof of Theorem 1 yields a new (albeit cumbersome) proof of the invariance of ranks of a finite primary group under different decompositions into direct sums of cyclic subgroups. Suppose \(G\) and \(G'\) are finite \(p\)-primary groups and that we have obtained particular sets of rank \([r_n]\) and \([s_n]\) for \(G\) and \(G'\), respectively. If \(G = G'\), then, for each \(n\), \(S_0(p^n) = S_0(p^n)\) and, by our proof, \(r_n = s_n\). 3) The example we gave relies on the fact that \(1 + 2 + 2^2 + 2^3 + 2^4 = 31 = 1 + 5 + 5^2\). There is no other known example of a prime number which can be expressed as a finite power series in two different ways.

3. Abelian groups in general. We now respond to the first two questions of the introduction for groups in general.

**Theorem 2.** An abelian group \(G\) has a maximal cover iff it has a decomposition \(G = H \oplus A\) where the order of \(A\) is positive but finite and, for every prime \(p\), either \(pA = A\) and \(H/pH\) is cyclic or \(pH = H\) and \(A/pA\) is not cyclic.

**Proof.** 1) Suppose \(G\) has a maximal cover \(\{H_i\}\) and \(H = \bigcap H_i\). We wish to show that \(H\) is a direct summand of \(G\). Since \(G/H\) is a finite direct sum of cyclic groups, it will suffice to show that \(H\) is pure in \(G\) (\(p^kG \cap H = p^kH\) for every prime \(p\) and positive integer \(k\)). Let \(\tilde{G} = G/H\). Suppose \(p\) is a prime such that \(p\tilde{G} = \tilde{G}\). Suppose \(p^k \in H\), for some \(g \in G\) and positive \(k\). Since \(\{\tilde{G}\}, p = 1\), there is an integer \(r\) such that \(rg \in H\) and \((r_p) = 1\). Let \(a, b\) be integers so that \(ar + bp = 1\). Then \(g = (ar + bp)g = arg + bp^k g \in H\) and \(p^kG \cap H = p^kH\). Suppose, then, \(p\) is a prime such that \(p\tilde{G} \neq \tilde{G}\). We observe that \(\tilde{G} = G/H\) has a maximal cover \(\{H_i/H\}\), so that \(\bigcap (H_i/H) = 0\), and, by Fact (f), that the \(p\)-component of \(G/H\) is non-cyclic. Suppose \(H \neq pH\). Then \(H\) has a subgroup \(K\), where \(|H/K| = p^k\), \(k > 0\), and \(G/K\) has a maximal cover \(\{H_i/K\}\) with \(\bigcap (H_i/K) = H/K\). But, since \(|H/K|\) is a power of \(p\), \(\bigcap (H_i/K) = 0\) by Fact (f). Therefore \(pH = H\) and, as a result,
\( p^k G \cap H = p^k H \) for all \( k \). Since \( H \) is pure in \( G \), for some subgroup \( A, G = H \oplus A. \) If \( p \) is a prime such that \( pG \neq \overline{G} \), then \( A/pA \) is not cyclic and \( pH = H. \) If \( pG = \overline{G} \), then \( pA = A \) and we now show \( H/pH \) is cyclic. If not, \( H \) has a subgroup \( K \) such that \( |H/K| = p^k \) for some positive \( k \), and \( H/K \) is non-cyclic. Now \( G/K \) has a maximal cover \( \{H_i/K\} \) with \( \cap (H_i/K) = H/K. \) But \( H/K \) is non-cyclic and, by Fact (f), not in \( \cap (H_i/K) \). Therefore, \( H/pH \) is cyclic. The conditions of the theorem are satisfied.

2) Suppose \( G = H \oplus A \) with the properties stated in the theorem. For some subgroups \( H_i \subseteq G, G/H \) has a maximal cover \( \{H_i/H\}. \) Each \( H_i = H \oplus C \) for some cyclic subgroup \( C \) of \( A \). For each prime \( p, H_i/pH_i \cong H/pH + C/pC \) which is cyclic. Therefore, no \( H_i \) has a cover and the cover \( \{H_i\} \) of \( G \) is maximal.

From Theorems 1 and 2, we obtain the following:

**Theorem 3.** If groups \( G \) and \( G' \) have maximal covers and \( G/pG \) and \( G'/pG' \) are cyclic for all primes \( p \) except one, then \( G \cong G' \) iff their maximal covers are isomorphic. The restriction on primes is necessary.

**Proof.** It suffices to show that, if \( G \) and \( G' \) have the given restrictions on primes and isomorphic maximal covers \( \{H_i\} \) and \( \{H'_i\} \), then \( G \cong G' \). Let \( G = H \oplus A, G' = H' \oplus A' \) with appropriate properties, as indicated in Theorem 2. Then \( H_i = H \oplus C_i \) and \( H'_i = H' \oplus C'_i \) with \( C_i \subseteq A, C'_i \subseteq A' \). Now for fixed prime \( q, C_i \) and \( C'_i \) are \( q \)-primary and finite while \( qH = H \) and \( qH' = H' \). Since \( H_i \cong H'_i \) for each \( i \), it follows that \( C_i \cong C'_i \) and \( H \cong H' \). Therefore \( G/H \) and \( G'/H' \) have isomorphic maximal covers \( \{H_i/H\} \) and \( \{H'_i/H'\}. \) By Theorem 1, then, \( G/H \cong G'/H', A \cong A', \) and \( G \cong G' \).

4. **Groups without covers.** A group \( G \) is without cover iff \( G/pG \) is cyclic for each prime \( p \). We now characterize these groups further. If \( G \) is such a group, \( G = D \oplus R \) where \( D \) is divisible (\( pD = D \) for each \( p \)), \( R \) is reduced (it has no non-zero divisible subgroups), and \( R/pR \) is cyclic for each prime \( p \). Let us assume, then, that \( G \) is reduced and \( G/pG \) is cyclic for each \( p \). Let \( \Pi = \prod \) (\( \Pi \) \( G/p^n G \)), the complete direct sum of the groups \( G/p^n G \) where \( p \) and \( n \) range over all primes and positive integers respectively. Let \( \phi \) be the natural map from \( G \) to \( \Pi \). We claim \( \phi \) is injective and that \( \phi(G) \) is pure in \( \Pi \). We first show that the kernel \( \phi, \bigcap p^n G, \) equals 0. Suppose \( x \neq 0 \) is a member of this subgroup. For fixed \( p, \) since \( G \) is reduced and \( G/pG \) is cyclic, the \( p \)-primary component of the torsion subgroup of \( G \) is cyclic, and \( G = A \oplus B \) where \( p^n A = 0 \) for some \( N \geq 0 \) and
B has no elements of order \( p \). Since \( x \in \bigcap_n p^nG, \ x \in B \) and \( B \) contains a set of elements \( \{x_n\} \) such that \( x = p^n x_n \) for all positive \( n \). Since \( p^n(x_n - p x_{n-1}) = 0 \) and \( B \) has no elements of order \( p \), \( x_n = p x_{n-1} \) for each \( n \). We can find a similar set of elements for each prime \( p \) and the subgroup generated by the elements in these sets is isomorphic to the rational numbers. However, \( G \) is reduced has no such subgroup. Therefore, \( \bigcap p^nG = 0 \) and \( \phi \) is injective. The proof that \( \phi(G) \) is pure in \( II \) is straightforward (see, for example, Lemma 30.3 or Theorem 39.5 of [2]). We now examine more closely how \( \phi(G) \) sits in \( II \). For each prime \( p \), let \( \pi_p \) be the projection of \( II \) onto \( \Pi G / p^nG \). Suppose first that \( p \) is a prime for which \( G \) has elements of order \( p \). Then \( G = A \bigoplus B \), as above, with \( A \) cyclic and \( p \)-primary, and \( pB = B \). Therefore, \( B \) is the kernel of \( \pi_p \phi \) and \( \pi_p \phi(G) \cong A \). Secondly, let \( p \) be a prime such that \( G \not\cong pG \) but \( G \) has no elements of order \( p \). We claim that \( \Pi G / p^nG \) contains a copy of the \( p \)-adic integers which, in turn, contains \( \pi_p \phi(G) \). For \( n \leq m \), let \( \pi^n \) map \( G/p^nG \) onto \( G/p^nG \) by sending \( g + p^nG \) to \( g + p^nG \). Then \( \{(G/p^nG, \pi^n)\} \) forms an inverse system and the inverse limit, \( \lim G/p^nG \) consists of all vectors \( (\ldots, a_m, \ldots) \) in \( \Pi G/p^nG \) such that \( \pi^n a_m = a_n, \ n \leq m \) (see [2], Vol. I, page 60, for details). Since \( G/pG \) is cyclic of order \( p \) and \( G \) has no elements of order \( p \), each \( G/p^nG \) is cyclic of order \( p^n \). As a result, the subgroup \( \lim G/p^nG \) of \( \Pi G/p^nG \) is isomorphic to the \( p \)-adic integers (see [2], Vol. I, page 62, for a proof). Now, if \( x \in G \) and \( \pi_p \phi(x) = (\ldots, x_m, \ldots) \), then \( \pi^n x_m = x_n, \ n \leq m \), since \( x_i = x + p^i G \) for each \( i \). Therefore, \( \pi_p \phi(G) \subseteq \lim G/p^nG \). We have proved half of the following theorem.

**Theorem 4.** A group \( G \) has the property: \( G/pG \) is cyclic for each prime \( p \) iff \( G \) is a pure subgroup of a group of the form \( D \bigoplus \bigcap p \in J K_p \), \( J \) a set of distinct primes, where \( D \) is divisible and each \( K_p \) is either cyclic and \( p \)-primary or isomorphic to the \( p \)-adic integers.

**Proof.** Necessity was established above. Suppose, then, \( G \) is a pure subgroup of some \( K = D \bigoplus \bigcap p \in J K_p \). We must show \( G/qG \) is cyclic for each prime \( q \). Now \( K_q / qK \cong K / qK \cong (G, qK) / qK \cong G / G \cap qK = G / qG \) by the purity of \( G \) in \( K \). If \( K_q \) is cyclic, then \( G/qG \) is cyclic. If \( K_q \) is isomorphic to the \( q \)-adic integers, it is well known that \( K_q / qK_q \) is cyclic (e.g., see Th. 88.1 of [2]). Again, \( G/qG \) is cyclic, and the proof is complete.

A further discussion of torsion-free groups of finite rank with the
property of Theorem 4 may be found in [6].

BIBLIOGRAPHY