The group of Galois H-dimodule algebras

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THE GROUP OF GALOIS H-DIMODULE ALGEBRAS

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Let $R$ be a commutative ring with identity, and let $H$ be a commutative, cocommutative, finite Hopf $R$-algebra with antipode. Using the notion of $H$-dimodule algebra, given by Long in [3], Nakajima defines in [4] Galois $H$-dimodule algebras generalizing graded Galois algebras and $H$-objects of Galois in the sense of Chase-Sweedler. In the same paper, he shows that the set of isomorphism classes of Galois $H$-dimodule algebras admits a structure of monoid provided that the Hopf algebra is a free $R$-module of finite type.

The purpose of the present paper is to show that for any commutative, cocommutative, finite Hopf $R$-algebra with antipode the above mentioned monoid is a group which contains the Galois group of $H$ (Chase-Sweedler [2], Beattie [1]), as a subgroup.

0. Preliminaries. In what follows, $R$ is assumed to be a commutative ring with identity, and $H$ a commutative, cocommutative, finite Hopf algebra with antipode. The counit, comultiplication and antipode of $H$ will be denoted by $\varepsilon : H \rightarrow R$, $\delta : H \rightarrow H \otimes H$ and $\lambda : H \rightarrow H$, respectively, where $\otimes$ stands for $\otimes_R$.

For $A$, any $R$-algebra, we denote the unit and multiplication with $\eta_A : R \rightarrow A$ and $\mu_A : A \otimes A \rightarrow A$, respectively.

$\phi_A : H \otimes A \rightarrow A$ will denote a structure of $H$-module and $\chi_B : B \rightarrow B \otimes B$ of $H$-comodule.

An $R$-algebra $A$ is called $H$-module (respectively, $H$-comodule) algebra if $\eta_A$ and $\mu_A$ are $H$-module (respectively, $H$-comodule) morphisms.

An $H$-module algebra $A$ which is also $H$-comodule algebra and satisfies $\chi_A \cdot \phi_A = (\phi_A \otimes A) \cdot (H \otimes \chi_A) : H \otimes A \rightarrow A \otimes H$, is called $H$-dimodule algebra.

For any $H$-module algebra $A$, and $H$-comodule algebra $B$, the smash product $A \# B$, is a $R$-algebra: the $R$-module support is $A \otimes B$, and the multiplication is given by

$$(a \# b) \cdot (c \# d) = \sum_{i,j} a_i \cdot b_{i1}(c) \# b_{i0} \cdot d$$

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**Definition 1.1.** An $H$-dimodule algebra $A$ is called a Galois $H$-dimodule algebra if the following conditions are satisfied:

i) $A$ is a faithfully flat $R$-module

ii) $\gamma_A = (\mu_A \otimes H) \cdot (A \otimes \chi_A) : A \otimes A \rightarrow A \otimes H$ is an isomorphism.

**Proposition 1.2.** For $A$ and $B$ any two Galois $H$-dimodule algebras, the following statements are true:

a) $A'$ is a Galois $H$-dimodule algebra, where $A'$ has the same $H$-module algebra structure of $A$ and has $\chi_{A'} = (A \otimes \lambda) \cdot \chi_A$ as $H$-comodule structure.

b) $A^{op}$ is a Galois $H$-dimodule algebra.

c) $\bar{A}$ is a Galois $H$-dimodule algebra, where $\bar{A}$ has the same $H$-module structure of $A$, and its algebra structure is given by $\mu_{\bar{A}} = \mu_A \cdot \tau \cdot (A \otimes \phi_A) \cdot (\chi_A \otimes A)\cdot \tau$ being the twisting isomorphism.

d) The smash products, $A \# B$, is an Galois $H \otimes H$-dimodule algebra. The structure of $H \otimes H$-module is given by

$$(h_1 \otimes h_2)(a \# b) = h_1(a) \# h_2(b)$$

and the one of $H \otimes H$-comodule by

$$\chi_{A \# B}(a \# b) = \sum_{(a \# b)} a_{(0)} \# b_{(0)} \otimes a_{(1)} \otimes b_{(1)}.$$  

e) $H$ is a Galois $H$-dimodule algebra with the trivial structure of $H$-module, $\phi_H = \varepsilon \otimes H$, and with its comultiplication as $H$-comodule structure.

**Proof.** b) After [3, Prop 1.7 and Prop 2.10], enough to show that $\gamma_{A^{op}}$ is an isomorphism. But this follows from

$$(\gamma_A \otimes H) \cdot (A \otimes \gamma_{A^{op}})$$

(definitions of $\gamma_A$, $\gamma_{A^{op}}$, and $A$ is $H$-comodule algebra)

$$= (\mu_A \otimes H \otimes H) \cdot (A \otimes \mu_A \otimes \mu_H \otimes H) \cdot (A \otimes A \otimes \chi_A \otimes \delta) \cdot (A \otimes \tau \otimes H) \cdot (A \otimes A \otimes \chi_A)$$

$$= (\mu_A \otimes H \otimes H) \cdot (A \otimes A \otimes \mu_H \otimes H) \cdot (\mu_A \otimes \chi_A \otimes \delta) \cdot (A \otimes A \otimes \tau) \cdot (A \otimes \chi_A \otimes A) \cdot (A \otimes \tau)$$

(definitions of $\gamma_A$ and $\gamma_H$)

$$= (A \otimes \gamma_H) \cdot (A \otimes H) \cdot (A \otimes \tau) \cdot (A \otimes H) \cdot (A \otimes \tau).$$

c) $\gamma_{\bar{A}} = (\mu_{\bar{A}} \otimes H) \cdot (A \otimes \chi_{\bar{A}}) = \gamma_{A^{op}} (A \otimes \phi_A) \cdot (\chi_A \otimes A).$

From, $\gamma_{A^{op}}$ and $(A \otimes \phi_A) \cdot (\chi_A \otimes A)$ isomorphisms it follows $\gamma_{\bar{A}}$ isomor-
2. The group $\text{Gal}_H(R, H)$. Two Galois $H$-dimodule algebras are said to be isomorphic if they are so as $R$-algebras, as $H$-modules and as $H$-comodules. The set of isomorphism classes of Galois $H$-dimodule algebras is denoted by $\text{Gal}_H(R, H)$.

From now on, $A$ and $B$ denote two arbitrary Galois $H$-dimodule algebras.

**Definition 2.1.** The product $A \cdot B$ is defined by

$$A \cdot B = \left\{ \sum_{i,t} a_i \# b_t \in A \# B \mid \sum_{i,t} a_{i,m} \# b_{it} \otimes a_{t,i} = \sum_{i,t} a_i \# b_{im} \otimes b_{ti} \right\}$$

and therefore we have the equalizer

$$A \cdot B \xrightarrow{i_{AB}} A \# B \xrightarrow{A \otimes \chi_B} A \# B \otimes H \xrightarrow{(A \otimes \tau)(\chi_A \otimes B)}$$

**Proposition 2.2.** $A \cdot B$ is a Galois $H$-dimodule algebra.

**Proof.** The algebra structure is induced by the one of $A \# B$.

The $H$-module structure $\phi_{A,B}$ is given by the factorization, through the equalizer $i_{AB}$, of the morphism

$$H \otimes A \cdot B \xrightarrow{\delta \otimes i_{AB}} H \otimes H \otimes A \otimes B$$

$$H \otimes \tau \otimes B \xrightarrow{} H \otimes A \otimes H \otimes B \xrightarrow{\phi_A \otimes \phi_B} A \otimes B$$

i.e.,

$$h(\sum_{i,t} a_i \# b_t) = \sum_{i,t} h_0(a_i) \# h_1(b_t).$$

The $H$-comodule structure is given by the map

$$\chi_{A,B} : A \cdot B \longrightarrow A \cdot B \otimes H,$$

$$\chi_{A,B}(\sum_{i,t} a_i \# b_t) = \sum_{i,t} a_{i,m} \# b_{it} \otimes a_{t,i} = \sum_{i,t} a_i \# b_{im} \otimes b_{ti},$$

which is the only one satisfying

$$(i_{AB} \otimes H) \cdot \chi_{A,B} = (A \otimes \tau) \cdot (\chi_A \otimes B) \cdot i_{AB} = (A \otimes \chi_B) \cdot i_{AB}.$$
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\[ A \# B \otimes A \cdot B \xrightarrow{A \# B \otimes i_{AB}} A \# B \otimes A \# B \]
\[ A \# B \otimes A \otimes \chi_{B} \xrightarrow{A \# B \otimes (\langle A \otimes \tau \rangle (\chi_{A} \otimes B))} A \# B \otimes A \# A \otimes H \]
\[ A \# B \otimes B \xrightarrow{A \# B \otimes \delta} A \# B \otimes H \otimes H \]
\[ A \# B \otimes H \otimes \delta \xrightarrow{(H \otimes \tau)(\delta \otimes H)} A \# B \otimes H \otimes H \otimes H \]

are diagrams of equalizer.

The isomorphism \( \gamma_{A \# B} : (A \# B) \otimes (A \# B) \rightarrow A \# B \otimes H \otimes H \) factors through \( A \# B \otimes \delta \) yielding, this way, morphisms \( h \) and \( h' \), inverse to each other, making the diagram

\[
\begin{array}{ccc}
(A \# B) \otimes A \cdot B & \xrightarrow{A \# B \otimes i_{AB}} & (A \# B) \otimes (A \# B) \\
\uparrow h & & \uparrow \gamma_{A \# B} \\
(A \# B) \otimes H & \xrightarrow{A \# B \otimes \delta} & (A \# B) \otimes H \otimes H \\
\end{array}
\]

\[
\begin{array}{ccc}
\gamma_{A \# B}^{-1} & & \gamma^{-1}_{A \# B} \\
\downarrow & & \downarrow \\
(A \# B) \otimes H & \xrightarrow{A \# B \otimes \delta} & (A \# B) \otimes H \otimes H \\
\end{array}
\]

to commute.

It follows, then, that

\[
((A \# B) \otimes \delta \otimes H) \cdot (h \otimes H) \
= ((A \# B) \otimes \delta \otimes H) \cdot ((A \# B) \otimes (\gamma_{H} \cdot \tau)) \cdot \\
(h \otimes H) \cdot ((A \# B) \otimes \tau) \cdot (h \otimes A \cdot B)
\]

and therefore \( \gamma_{A \# B} \) is an isomorphism, because \( (A \# B) \otimes \delta \otimes H \) is an equalizer and the involved morphisms are isomorphisms.

**Corollary 2.3.** The product of Galois \( H \)-dimodule algebras induces an operation on \( \text{Gal}_{\ast}(R, H) \) whose identity element is the class of \( H \).

**Proposition 2.4.** \( \text{Gal}_{\ast}(R, H) \) is a group.

**Proof.** We will show that \( |A|^{-1} = |\overline{A}| \).

Indeed, since \( A \) is a Galois \( H \)-object [2],

\[
H \eta_{A} \otimes H \rightarrow A \otimes H \text{ is the equalizer of } A \otimes H \xrightarrow{\chi_{A} \otimes H} A \otimes H \otimes H
\]
On the other hand, the morphism

$\alpha \cdot \overline{A} \xrightarrow{i_A,\overline{A}} A \# \overline{A} \xrightarrow{\gamma_A} A \otimes H$

equalizes this pair because

$(\chi_A \otimes H) \cdot \gamma_A \cdot i_A,\overline{A} = (\chi_A \otimes H) \cdot (\mu_A \otimes H) \cdot (A \otimes \chi_A) \cdot i_A,\overline{A}$

$(\chi_A \otimes H)$ is a morphism of algebras

$= (\mu_A \otimes \mu_H \otimes H) \cdot (A \otimes \tau \otimes H \otimes \lambda) \cdot (A \otimes H \otimes A \otimes \delta) \cdot (\chi_A \otimes \chi_A) \cdot i_A,\overline{A}$

$(A \otimes H \otimes \chi_A)$ is an $H$-comodule (definition 2.1 and $\tau^2 = 1$)

$= (\mu_A \otimes \mu_H \otimes H) \cdot (A \otimes \tau \otimes H \otimes \lambda) \cdot (A \otimes H \otimes A \otimes \delta) \cdot (A \otimes H \otimes \chi_A) \cdot (A \otimes \tau) \cdot (A \otimes \chi_A) \cdot i_A,\overline{A}$

$(A \otimes H \otimes \chi_A)$ is an $H$-comodule

$= (\mu_A \otimes \mu_H \otimes H) \cdot (A \otimes A \otimes \tau \otimes \lambda) \cdot (A \otimes A \otimes H \otimes \tau) \cdot (A \otimes A \otimes H \otimes \lambda) \cdot (A \otimes A \otimes H \otimes \delta) \cdot (A \otimes H \otimes \chi_A) \cdot i_A,\overline{A}$

(cocommutativity of $H$)

$= (A \otimes \chi_H \otimes H) \cdot (\mu_A \otimes \lambda) \cdot (A \otimes \chi_A) \cdot i_A,\overline{A} = (A \otimes \chi_H \otimes H) \cdot \gamma_A \cdot i_A,\overline{A}$.

Therefore, there exists a morphism $f : A \cdot \overline{A} \longrightarrow H$ so that $(\eta_A \otimes H) \cdot f = \gamma_A \cdot i_A,\overline{A}$ (i.e. calling $q$ to $f(a \# a)$, $1_A \neq q = \sum_{(a)} a a_{\delta 0} \neq \lambda(a_{\delta 1})$)

This $f$ is an $H$-module morphism, i.e., the diagram

$\xymatrix{H \otimes A \cdot \overline{A} \ar[r]^{H \otimes f} \ar[d]_{\phi_A,\overline{A}} & H \otimes H \ar[d]_{\epsilon \otimes H} \\
A \cdot \overline{A} \ar[r]^{f} & H}$

commutes. Indeed.

Since $\eta_A \otimes H$ is an equalizer,

$(\eta_A \otimes H) \cdot f \cdot \phi_A,\overline{A} = \gamma_A \cdot i_A,\overline{A} \cdot \phi_A,\overline{A}$

(definition of $\phi_A,\overline{A}$)

$= (\mu_A \otimes H) \cdot (A \otimes \chi_A) \cdot (\phi_H \otimes \phi_A) \cdot (H \otimes \tau \otimes A) \cdot (\delta \otimes A \otimes A) \cdot (H \otimes i_A,\overline{A})$

$(A \cdot \overline{A})$ is an $H$-comodule

$= (\mu_A \otimes H) \cdot (\phi_H \otimes \phi_H \otimes H) \cdot (H \otimes \tau \otimes \chi_A) \cdot (\delta \otimes A \otimes A) \cdot (H \otimes i_A,\overline{A})$

$(A)$ is an $H$-module algebra
$= (\phi_\alpha \otimes H) \cdot (H \otimes \mu_\alpha \otimes H) \cdot (H \otimes A \otimes \chi_\alpha) \cdot (H \otimes i_{A, A})$

$(\eta: \text{R} \rightarrow A \text{ is an } H\text{-module morphism})$

$= (\eta_\alpha \otimes H) \cdot (\epsilon \otimes H) \cdot (H \otimes f)$.

$f$ is a morphism of $H$-comodules or, what is the same, the diagram

\[
\begin{array}{ccc}
A \cdot \overline{A} & \xrightarrow{\chi_{A, A'}} & A \cdot \overline{A} \otimes H \\
\downarrow f & & \downarrow f \otimes H \\
H & \xrightarrow{\delta} & H \otimes H
\end{array}
\]

commutes. Indeed.

Since $\eta_\alpha \otimes H \otimes H$ is an equalizer,

$(\eta_\alpha \otimes H \otimes H) \cdot (f \otimes H) \cdot \chi_{A, A'} = (\eta_\alpha \otimes f \otimes H) \cdot \chi_{A, A'}$

(definition of $f$)

$= (\gamma_\alpha \otimes H) \cdot (i_{A, A'} \otimes H) \cdot \chi_{A, A'}$

(definition of $\chi_{A, A'}$)

$= (\gamma_\alpha \otimes H) \cdot (A \otimes \chi_{A'}) \cdot i_{A, A'}$

(definition of $\gamma_\alpha$)

$= (\mu_\alpha \otimes H \otimes H) \cdot (A \otimes \chi_{A'} \otimes H) \cdot (A \otimes \chi_{A'}) \cdot i_{A, A'}$

($A'$ is an $H$-comodule)

$= (\mu_\alpha \otimes H \otimes H) \cdot (A \otimes A \otimes \delta) \cdot (A \otimes \chi_{A'}) \cdot i_{A, A'}$

$= (A \otimes \delta) \cdot (\mu_\alpha \otimes H) \cdot (A \otimes \chi_{A'}) \cdot i_{A, A'}$

(definition of $\gamma_\alpha$)

$= (A \otimes \delta) \cdot \gamma_{A'} \cdot i_{A, A'} = (A \otimes \delta) \cdot (\eta_\alpha \otimes f) = (\eta_\alpha \otimes H \otimes H) \cdot \delta \cdot f.$

Analogously $f$ is a morphism of $R$-algebras and by ([1], Lemma 1.1., p. 688) is an isomorphism.

**Corollary 2.5.** The map

\[ \text{Gal}(R, H) \longrightarrow \text{Gal}_*(R, H) \]

obtained by regarding any Galois $H$-object as a Galois $H$-dimodule algebra with the trivial action([1],[2]) is a monomorphism of groups, in general not surjective([4], Remark 2.5, p. 174).
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