Families of geodesics which distinguish flat tori

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FAMILIES OF GEODESICS WHICH DISTINGUISH FLAT TORI

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0. Introduction. The flat tori play distinguished roles in the study of manifolds with nonpositive sectional curvature and of nonnegative sectional (or Ricci) curvature. Namely, the Riemannian metrics on a torus having one of these curvature conditions are flat. And, the behavior of geodesics on these manifolds are very different. This suggests to us that some behavior of geodesics distinguishes flat tori from any other Riemannian manifold. In the present paper we study the conditions (0.1) – (0.4) on surfaces.

Throughout the paper let $M$ be a complete Riemannian manifold which is homeomorphic to a plane and $N$ a complete Riemannian manifold with dimension 2. We denote by $N = M/D$ a quotient manifold, where $D$ is the group of isometries acting freely on $M$ which is properly discontinuous. All geodesics are always parametrized by arc-length, unless otherwise stated. Let $SN$ be the unit tangent bundle of $N$ and $\pi : SN \to N$ the projection. For any $v \in SN$ let $\gamma_v : (-\infty, \infty) \to N$ be the geodesic with $\dot{\gamma}_v(0) = v$, but the interval may be restricted if stated.

(0.1) For any $v \in SN$ there exists a section $\tau_v : N \to SN$, $\tau_v(\pi(v)) = v$, such that $\tau_v \circ g^t \circ \tau_v$ is a flow on $N$, where $g^t : SN \to SN$ is the geodesic flow of $N$.

(0.2) Each non-trivial homotopy class of closed curves in $N$ contains a family of geodesics which cover $N$ simply.

(0.3) The set of all tangent vectors of geodesics which are dense in $N$ is dense in $SN$.

(0.4) For any non-contractible closed curve $K$ and for any point $p$ in $N$ there exists at most two geodesic ray emanating from $p$ which does not intersect $K$.

The flat tori $T^2$ satisfy these conditions. We shall show that the converse of (0.1), (0.2) and (0.4) are true in Section 1, 3 and 5, resp.. In Section 2 we provide the properties of axial isometries on $M$ which is used in Section 3, 4, 5. We discuss the condition (0.1) and (0.2) in Section 4 and obtain some results concerning (0.3). The results are sometimes obtained by applying a theorem of E. Hopf([9]) that the tori $T^2$ without conjugate points are flat. And, the study of Busemann-Pedersen([6]) concerning axial isometries would be very useful to the investigation here. In the paper they
have studied $G$-spaces (see [4]) which are much more general than complete Riemannian manifolds. Hence, if the readers are interested in the theory of $G$-spaces, all conclusions should be understood that the universal covering $G$-surfaces are straight and satisfy the parallel axiom (for the definitions of "straight" and "parallel axiom" see [4]).

1. Surfaces having flows whose orbits are geodesics. The important property of surfaces is that a curve can separate the space locally and sometimes globally. This is the principal tools of the investigation here. Let $A$ be an open domain in $N$. We say that $A$ is covered simply by a family $\Gamma$ of geodesics if the geodesic in $\Gamma$ through $p$ uniquely exists locally for each point $p \in A$ and is maximal in $A$, i.e., any extension does not stay in $A$. In the definition, "locally" means "globally" also. If $A$ is simply connected, then we can give all geodesics of the family an orientation so that the tangent vectors $v(x)$ of all geodesics depend continuously on their foot points $x$ in $A$. And, each geodesic of the family decomposes $A$ into two components.

Lemma 1.1. Let $A$ be a simply connected open domain in $N$ which is covered simply by a family $\Gamma$ of geodesics. Then, all subarcs of all geodesics in $\Gamma$ are the unique geodesic in $A$ connecting their endpoints. In particular, all geodesics of $\Gamma$ are minimizing in $A$ and the vectors $v(x)$ depend differentiably on the points $x \in A$.

Proof. Let $v(x)$ be the tangent vector at $x \in A$ of the geodesic of $\Gamma$ determined by $x$. We take the signed angle of unit vectors at $x$ from $v(x)$ for any $x \in A$. Suppose there is a geodesic $\alpha : [0, L] \to N$ joining $p$ and $q$ in $A$ both of which are in a geodesic $\gamma$ of $\Gamma$ such that $\alpha$ is not a subarc of $\gamma$. If $\alpha(L_0), 0 < L_0 \leq L_1$, is the first intersection point of $\alpha$ and $\gamma$, then the signs of $\alpha(L_0)$ and $\alpha(0)$ are different. Hence, the angle of $\alpha(L_0)$ is either zero or $\pi$ for some $L_0, 0 < L_0 \leq L_1$. This implies that $\alpha$ or the reversed geodesic of $\alpha$ is a subarc of a geodesic of $\Gamma$ because of the simplicity of the family of geodesics, a contradiction.

The existence of a family of geodesics which entirely covers $N$ simply would be controlled by the topological structure of $N$.

Lemma 1.2. Suppose $N$ is compact. If $N$ is covered simply by a family $\Gamma$ of geodesics, then $N$ is topologically either the torus or one-sided torus.
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Proof. It suffices to prove that the statement is true if \( N \) is orientable. Let \( N = M/D \), where \( M \) is the universal covering space of \( N \). The lift \( \tilde{\Gamma} \) of \( \Gamma \) to \( M \) also covers \( M \) simply. Define an orientation of \( \tilde{\Gamma} \) and denote by \( \nu(x) \) the tangent vector at \( x \) of the geodesic of \( \tilde{\Gamma} \) passing through \( x \). Since \( N \) is orientable, the vector field \( \nu \) is invariant under \( D \). This implies that \( \nu \) induces the vector field on \( N \). Since it has no singular points, it follows from the Hopf-Lefschetz fixed point theorem that \( N \) is topologically the torus.

It should be noted that \( N \) is orientable if \( N \) has two families of geodesics which cover \( N \) simply in such a way that the tangent vectors at each point of \( N \) of the geodesics are linearly independent.

**Theorem 1.3.** Suppose \( N \) is compact. If the geodesic flow on \( N \) satisfies the property (0.1), then \( N \) is a flat torus.

Proof. For any unit vector \( v \in SN \) the section \( \tau_v \) induces a family \( \Gamma \) of geodesics which cover \( N \) simply with \( \gamma_v \in \Gamma \). By Lemma 1.2, \( N \) is topologically either the torus or one-sided torus. To apply the theorem of E. Hopf we must prove that \( N \) has no conjugate points. Since the lift of \( \gamma_v \) to the universal covering space \( M \) of \( N \) is minimizing, there is no conjugate points along \( \gamma_v \). Thus, \( N \) is flat. This completes the proof.

2. Axial isometries and the structure of surfaces. Let \( \varphi \) be an isometry of \( M \). The function \( d_\varphi : M \to R \) given by \( d_\varphi(p) = d(p, \varphi(p)) \) for any \( p \in M \) is called the displacement function of \( \varphi \). We say that \( \varphi \) is axial if there is a minimizing geodesic \( \gamma : (-\infty, \infty) \to M \) such that \( \varphi\gamma(t) = \gamma(t + a) \) for any \( t \in (-\infty, \infty) \) and some constant \( a > 0 \). The \( \gamma \) is called an axis of \( \varphi \). We can find important results on axial isometries in a paper of Busemann-Pedersen ([6], and also see [5] pp. 64–67).

(2.1) If \( \gamma \) is an axis of \( \varphi \), then \( a = \min d_\varphi \) and \( \gamma \) is also an axis of \( \varphi^n, n \geq 1 \).

(2.2) Let \( \varphi \) be an isometry which preserves an orientation of \( M \). If \( \gamma : (-\infty, \infty) \to M \) is an axis of \( \varphi^n, n > 1 \), then \( \gamma \) is also an axis of \( \varphi \).

(2.3) Let \( \varphi \) be an isometry which preserves an orientation of \( M \). If \( 0 < d_\varphi(p) = \min d_\varphi \) for some \( p \in M \), then the geodesic \( \gamma : (-\infty, \infty) \to M \), with \( \gamma(0) = p \) and \( \gamma(d_\varphi(p)) = \varphi(p) \), is minimizing and hence an axis of \( \varphi \).

We see in a book of Cheeger-Ebin ([7] p. 156) that

(2.4) If \( N = M/D \) is compact, where \( D \) is the group of isometries, then all \( d_\varphi, \varphi \in D \), assume their minimums on \( M \).
Lemma 2.1. Let $M$ be oriented and let $\varphi$ be an axial isometry of $M$. If the displacement function $d_\varphi$ of $\varphi$ is bounded, then $\varphi$ preserves the orientation of $M$.

Proof. Let $\gamma : (-\infty, \infty) \to M$ be an axis of $\varphi$. Suppose $\varphi$ dose not preserve the orientation of $M$. Then, it follows that

$$d_\varphi(p) \geq 2d(p, \gamma(-\infty, \infty))$$

for any $p \in M$, since all minimizing geodesics joining $p$ and $\varphi p$ intersect $\gamma(-\infty, \infty)$. This contradicts that $d_\varphi$ is bounded on $M$.

Now we characterize the torus by a property of displacement functions. The idea of the proof is seen in a paper of Eberlein-O'Neill ([8], Proposition 6.8).

Proposition 2.2. Suppose $N = M/D$ is compact and oriented. If the displacement function of an isometry in $D \setminus \{1\}$ is bounded on $M$, then all the displacement functions of $D$ are bounded on $M$ and $D$ is abelian. In particular, $N$ is topologically the torus.

Proof. Suppose $d_\varphi, \varphi \in D \setminus \{1\}$, is bounded on $M$. Let $1 \neq \Psi \in D$ be an arbitrary isometry. Then, $\Psi$ has an axis $\gamma : (-\infty, \infty) \to M$, i.e., $\Psi \gamma(t) = \gamma(t+a)$ for all $t \in (-\infty, \infty)$, where $a = \min d_\varphi > 0$. If we put $p = \gamma(0)$ and $\gamma_1 = \varphi \gamma$, then

$$d(p, \Psi^{-n}\varphi\Psi^\varphi\Psi^{-1}(\varphi p)) = d(\Psi^n p, \varphi\Psi^n p) \leq C$$

for all integers $n$ and some constant $C > 0$, since $d_\varphi$ is bounded on $M$. Thus, the set $\Psi^{-n}\varphi\Psi^\varphi\Psi^{-1}(\varphi p); n \in \mathbb{Z}$ is bounded. Since $D$ acts freely and is properly discontinuous, there exist $m > n \in \mathbb{Z}$ such that

$$\Psi^{-n}\varphi\Psi^\varphi\Psi^{-1} = \Psi^{-m}\varphi\Psi^\varphi\Psi^{-1}.$$  

Therefore,

$$\Psi^m = \varphi\Psi^{-m}\varphi^{-1}.$$  

It follows from (2.1) that $\gamma_1$ is also an axis of $\Psi^{-m}\varphi$. And, $\Psi$ has an axis $\gamma_1$ because of (2.2). Thus,

$$\Psi \varphi = \gamma_1(a) = \varphi \gamma(a) = \varphi \varphi p.$$  

This implies that $\Psi \varphi = \varphi \Psi$, since $D$ acts freely on $M$.  

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Next we prove that $d_r$ is bounded on $M$. Let $\alpha : [0, L] \to M$ be a minimizing geodesic joining $p$ and $\varphi p$, where $L = d(p, \varphi p)$. Let $A$ be the compact domain bounded by $\gamma[0, a] \cup \Psi[0, L] \cup \gamma(0, a) \cup a[0, L]$. If $D_i \subset D$ is the abelian subgroup generated by $\varphi$ and $\Psi$, then $D_i A = M$. Therefore,

$$d_r \leq \max d_r(A),$$

since $d_\sigma(q) = d_r(x)$ for any $q \in M$ and a point $x \in A$ such that $\sigma q = x$ for some $\sigma \in D_i$.

If we apply the first part of the argument above to any isometry of $D$, then we conclude that $D$ is abelian.

3. Surfaces having families of closed geodesics. In the present section we shall prove the converse of (0.2). We first prove

Theorem 3.1. Suppose $N = M/D$ is compact. If all the displacement functions of $D$ are constant on $M$, then $N$ is a flat torus.

Proof. By Lemma 2.1, all $\varphi \in D$ preserve an orientation of $M$, so $N$ is orientable. It follows from Proposition 2.2 that $N$ is topologically the torus. Hence, to apply the theorem of E. Hopf, we must prove only that $N$ or $M$ has no conjugate points. We first of all should remark that if $x$ and $y$ in $M$ are over the same point in $N$, then there is the unique geodesic joining $x$ and $y$ which is a subarc of an axis of some isometry of $D$. This is shown from Lemma 2.1, (2.3), and Lemma 1.1.

Assume that $\varphi$ and $\Psi$ generate $D$. Let $z$ be an arbitrary point of $M$. We define a coordinate on $M$ as follows. Let $\alpha : (-\infty, \infty) \to M$ and $\beta : (-\infty, \infty) \to M$ be axes of $\varphi$ and $\Psi$ with $\alpha(0) = z = \beta(0)$ and with speed $c = \min d_\alpha, d = \min d_\beta$, resp. For any point $x$ in $M$ the axis of $\Psi(\varphi)$ through $x$ intersects $\alpha(-\infty, \infty)(\beta(-\infty, \infty))$ at exactly one point, say $\alpha(u)$ and $\beta(v)$, resp.). We give the $x$ a coordinate $(u, v)$. In the coordinate, all curves $u = \text{const.}$ and $v = \text{const.}$ are minimizing geodesics. And, the $u$- and $v$-coordinates of other geodesics are strictly monotone for their parameters. Thus, we represent as a curve a geodesic by $v = f(u)$ for any $u \in (-\infty, \infty)$, where $f$ is strictly monotone. Since all $v = \text{const.}$ are parallel to each other in the sense of Busemann ([5] p. 65 (2)), $|f(u)| \to \infty$ as $|u| \to \infty$ ([4] p. 216). Hence, as seen in [4] p. 216, we can define the "slope" for any geodesic $v = f(u)$ by $\lim_{u \to \pm \infty} \frac{f(u)}{u}$. For the slope, we have from the same argument as Busemann's (see [4] pp. 216–219).
Assertion. For any $\mu \in (-\infty, \infty)$ there passes the unique minimizing geodesic through each point $p$ of $M$ with slope $\mu$.

Although he proved the assertion under the hypothesis that $M$ is straight, i.e., in the Riemannian case it is equivalent to having no conjugate points, his proof is valid in our situation by virtue of the above remark.

Let $S_z M$ be the set of all unit vectors at $z$ and $A$ the set of all vectors $w \in S_z M$ such that the geodesic $\gamma_w : (-\infty, \infty) \to M$ with $\dot{\gamma}_w(0) = w$ is minimizing. The set $A$ is closed in $S_z M$ and contains the initial vectors of geodesics from $z = (0,0)$ to $(m,n)$ for all $(m,n) \in \mathbb{Z} \times \mathbb{Z}$. We want to prove that $A = S_z M$. Suppose for an indirect proof that $S_z M - A \neq \emptyset$. Let $w_0$ and $w_1$ be the boundary vectors of a component of $S_z M - A$. Then, we may assume that the geodesics with initial vectors $w_0$ and $w_1$ have representations of the forms

$$\gamma_{w_0} : v = f(u), \quad \gamma_{w_1} : v = g(u), \text{ with } g(u) > f(u),$$

for $u > 0$. Then for any integer $n > 0$

$$0 < g(u) - f(u) < 1,$$

since, otherwise the geodesic $S_n$ joining $(n,f(n))$ and $(n,g(n))$ contains a point of the form $(n,m)$ with integer $m$, and this contradicts the choice of $w_0$ and $w_1$. However, if so, then

$$\lim_{u \to +\infty} \frac{f(u)}{u} = \lim_{u \to +\infty} \frac{g(u)}{u},$$

contradicting Assertion. Thus, $S_z M = A$. Therefore, by the theorem of Hopf, $N$ is flat. This completes the proof.

Busemann could prove that $S_z M = A$ in the course of showing Assertion, but it was not stated because his interest was different from ours. We express Theorem 3.1 in terms of closed geodesics.

Corollary 3.2. Suppose $N = M/D$ is compact. If each non-trivial homotopy class of closed curves in $N$ contains a family of closed geodesics which cover $N$ simply, then $N$ is a flat torus.

Proof. Let $H$ be a non-trivial homotopy class of closed curves in $N$ and $\varphi \in D$ correspond to $H$. Let $\Gamma$ be a family of closed geodesics in $H$ which cover $N$ simply. The lift $\tilde{\Gamma}$ of $\Gamma$ to $M$ covers $M$ simply also. It follows from Lemma 1.1 that the geodesic connecting $p$ and $\varphi p$ uniquely exists and is a
subarc of the geodesics in \( \widetilde{\Gamma} \) through \( p \) for any point \( p \in M \). Thus, if a point \( p \) in \( M \) is over \( p_0 \) in \( N \), then \( d_\varphi(p) \) is the length of the closed geodesic in \( \Gamma \) through \( p_0 \). Since, because of the first variation formula, all closed geodesics in \( \Gamma \) have the same length, \( d_\varphi \) is constant on \( M \). Corollary 3.2 follows from Theorem 3.1.

4. Example and discussion. In the present section we show that the assumptions of Theorem 1.3 and Theorem 3.1 is best in some sense. The examples and discussion here are due to Bliss([3]), Busemann-Pedersen([6]), Busemann([5]) and Innami([10]). After studying examples we investigate the condition (0.3).

Let \( f : \mathbb{R} \to \mathbb{R} \) be a positive periodic function with period 1. Define a Riemannian metric on \( \mathbb{R}^2 \) by \( ds^2 = dx^2 + f(x)^2 dy^2 \). Then, the affine translations \( \varphi : \mathbb{R}^2 \to \mathbb{R}^2 \) given by \( \varphi(x, y) = (x+1, y) \) and \( \Psi_t : \mathbb{R}^2 \to \mathbb{R}^2 \) given by \( \Psi_t(x, y) = (x, y+t) \) for each \( t \in \mathbb{R} \) are isometries. Hence, if \( D \) is the group of isometries generated by \( \varphi \) and \( \Psi_t \), then \( (\mathbb{R}^2, ds^2)/D = T^2 \) is a torus. And, if \( f(x_0) = \min f \), then \( (x_0, y) \) is a pole for any \( y \in (-\infty, \infty) \). Let \( D_t \) be the isometry group generated by \( \{ \varphi \} \cup \{ \Psi_t : t \in (-\infty, \infty) \} \). Then, \( d_\eta \) assumes its minimum at all poles for any \( \eta \in D_t \). Hence, for any unit vector \( v \) except \( \frac{\partial}{\partial y} \) at any pole we construct a section \( \tau_v \) from \( T^2 \) to \( ST^2 \) such that \( f^v = \pi \circ \exp \circ \tau_v \) is a flow on \( T^2 \). If we choose the function \( f \) such that the length of the minimum set of \( f \) is sufficiently greater than the complement, we have

\[(4.1) \quad \text{For any } \epsilon > 0 \text{ there is a non-flat Riemannian metric } G \text{ on } T^2 \text{ such that } \lambda > \mu \geq (1-\epsilon)\lambda, \text{ where } \lambda \text{ is the volume of } ST^2 \text{ and } \mu \text{ is the volume of the set of all } v \in ST^2 \text{ such that } f^v \text{ is a flow on } T^2.\]

On the assumption of Theorem 3.1 and Corollary 3.2 we have

\[(4.2) \quad \text{For any } \eta \in D \text{ except for } D_t = \{ \Psi^n \ ; \ n \in \mathbb{Z} \} \text{ the displacement function } d_\eta \text{ is constant on } (\mathbb{R}^2, ds^2).\]

Each non-trivial homotopy class of closed curves in \( T^2 \) except for ones corresponding to \( D_t \) contains a family of closed geodesics which cover \( T^2 \) simply.

Let \( f(x_0) = \min f \) and \( f(x_i) > \min f, x_0 < x_i < x_0 + 1 \). Let \( a : (-\infty, \infty) \to (\mathbb{R}^2, ds^2) \) be a geodesic with \( a(t) = (x(t), y(t)) \) and \( a(0) = (x_i, y_i) \). It follows from a theorem of Clairaut that \( f(x(t))\cos \theta(t) = f(x(0))\cos \theta(0)\)
for all $t \in (-\infty, \infty)$, where $\theta(t)$ is the angle of $\hat{a}(t)$ and $\frac{\partial}{\partial y}$ at $a(t)$. Hence, if $f(x)\cos \theta(0) > \min f$, and if $x_0 < x_1 \leq x_1 < x_3 < x_0 + 1$ with $f(x_1) = f(x_3) = f(x_0)\cos \theta(0)$, then $a$ lies in the strip bounded by the lines $(x_2, R)$ and $(x_3, R)$. In particular, we have

**Proposition 4.1.** Let $T^2$ be a torus with metric $ds^2 = dx^2 + f^2(x)dy^2$. If the set of all tangent vectors of geodesics which are dense in $T^2$ is dense in $ST^1$, then $f(x) = \text{const.}$, and, in particular, $T^2$ is flat.

The similar situation arises in the case of tori having poles. Let $T^1 = R^2 / D$ be a torus having poles. Suppose a point $p \in R^2$ lies over a pole in $T^2$. Then, $d_\varphi$ assumes its minimum at $p$ for all $\varphi \in D$(see[10]). Let $\varphi$ be an isometry of $D$ and $\gamma : (-\infty, \infty) \rightarrow R^2$ be the axis of $\varphi$ with $\gamma(0) = p$. Then, there passes the unique asymptote to $\gamma$ through each point $q$, since if $d_\varphi(p) = c$, $\gamma(nc)$ is a pole for each $n \in Z$(see[4],[5],[10]). Let $p_1$ be another pole in $R^2$ and let $\gamma_1 : (-\infty, \infty) \rightarrow R^2$ be the axis of $\varphi$ through $p_1$. In general, the asymptotes to $\gamma$ and $\gamma_1$ through a point $q$ in the strip bounded by $\gamma$ and $\gamma_1$ are different. Suppose $\alpha, \beta : (-\infty, \infty) \rightarrow R^2$ are the asymptotes to $\gamma$ and $\gamma_1$ with $\alpha(0) = \beta(0) = q$, $\alpha \neq \beta$, resp.. Let $A$ be the domain bounded by $\alpha[0, \infty)$ and $\beta[0, \infty)$ which does not contain both $\gamma$ and $\gamma_1$ and $S$ the set of all $v \in SR^2$ inward $A$. We assert

**Lemma 4.2. All geodesics emanating from $q$ with initial tangent vectors in $S$ lie entirely in $A$.**

**Proof.** Suppose there is a $v \in S$ such that the geodesic $\gamma_v : [0, \infty) \rightarrow R^2$ with $\gamma_v(0) = v$ intersects $\alpha$ or $\beta$, say $q_1 = \alpha(s_0) = \gamma_v(t_0)$ for the first intersection point. Since $\alpha$ is an asymptote to $\gamma$, the geodesics $\alpha_n : [0, L_n] \rightarrow R^2$ connecting $q_1$ to $\gamma(nc)$ converges to $\alpha[0, \infty)$ as $n$ goes to infinity. This implies that the domain $B$ surrounded by $\beta[0, \infty) \cup \alpha[0, s_0] \cup \alpha_n[0, L_n] \cap \gamma(nc, \infty)$ is locally convex for sufficiently large $n$, where "locally convex" means that the angle of the domain at each vertex is less than or equal to $\pi$. Hence, a shortest curve from $q$ to $\gamma(nc)$ in $A$ is a geodesic $\gamma_2 : [0, L] \rightarrow R^2$, which intersects $\alpha[0, \infty)$ twice, at $\alpha(0)$ and another point $q_2$. Since $\gamma(nc)$ is a pole, $\gamma_2$ is minimizing in $R^2$. However, since $\alpha$ is minimizing, the minimizing geodesic connecting $\alpha(0)$ to $q_2$ must exist uniquely, a contradiction. This completes the proof.

We do not know whether $d(\gamma_v[0, \infty), \gamma(-\infty, \infty)) > 0$ and $d(\gamma_v[0, \infty), \gamma(-\infty, \infty)) > 0$ and $d(\gamma_v[0, \infty), \gamma(-\infty, \infty)) > 0$.
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γ₁(−∞, ∞)) > 0 for any geodesic γᵥ : [0, ∞) → ℜ² with ̇γ(0) = v ∈ S. So we need an additional assumption for analogous result to Proposition 4.1.

**Proposition 4.3.** Let T² = M/D be a torus having two poles which do not lie simultaneously in any closed geodesic. If the set of all tangent vectors of geodesics which are dense in T² is dense in ST², then T² is flat.

**Proof.** Let ϕ ∈ D. We shall prove that the displacement function dϕ is constant on M. We may assume that no η ∈ D exists such that ϕ = ηⁿ, n ≠ ± 1, since dϕ is so if dη is constant on M. Let Ψ be an isometry of D such that ϕ and Ψ generate D. If γ, α : (−∞, ∞) → M are axes of ϕ and Ψ, resp., through a pole p = γ(0) = α(0) in M, then the compact domain surrounded by γ(−∞, ∞), α(−∞, ∞), Ψγ(−∞, ∞) and Ψα(−∞, ∞) is the closure F of a fundamental domain of T². Another pole p₁ exists in the interior of F. If γ₁ : (−∞, ∞) → M is the axis of ϕ through p₁, then γ₁(−∞, ∞) decomposes the strip B bounded by γ(−∞, ∞) and Ψγ(−∞, ∞) into two components B₁ and B₂. Suppose there is a point q, in B₁ say, such that the asymptotes to two axes of ϕ which consist of the boundary of B₁ are different. Since the angular domain in B₁ as in Lemma 4.2 can be constructed, the set of all tangent vectors of geodesics which are dense in T² is not dense in ST², a contradiction. Thus, B is covered simply by a family of asymptotes to both γ and Ψγ. Suppose for an indirect proof that dϕ is not constant. Then, from Lemma 5.1, there exists a geodesic τ : (−∞, ∞) → M contained in the interior of B which is not an axis of ϕ. Put x = τ(0). Since τ does not have any self-intersection point, it decomposes B into two components C₁ and C₂ also. By Lemma 1.1, the asymptote to both γ and Ψγ from x is entirely contained in one of them, say C₁ and C₂ ⊃ γ(−∞, ∞). A shortest curve from x to γ(nc) in C₂ must be the unique minimizing geodesic in M, since γ(nc) is a pole for any n ∈ ℤ, where c = min dϕ. This implies that the asymptote to γ from x lies in C₁, a contradiction. Thus, dϕ is constant on M. Proposition 4.3 follows from Theorem 3.1.

5. Geodesic rays which avoid crossing closed curves. In the present section we investigate the condition (0.4). We first prepare a lemma. The idea of the proof was used by Bangert ([1], [2]).

**Lemma 5.1.** Let ϕ be an axial isometry of M having two axes and B the strip bounded by the axes. If the displacement function dϕ is not constant in B, then there is a geodesic γ : (−∞, ∞) → M contained in the interior of
B which is a lift of a closed geodesic in B/D, but not an axis of \( \varphi \), where \( D = |\varphi^n; n \in \mathbb{Z}| \).

Proof. Take a strip \( B_1 \subset B \) whose boundary consists axes \( \gamma \) and \( \gamma_1 \) of \( \varphi \) such that \( d_\varphi(q) > \min d_\varphi(M) \) for any \( q \) in the interior of \( B_1 \). We shall find a closed geodesic in the interior of \( B_1/D = B_0 \). Let \( h_0 \) and \( h_1 : [0, 1] \to B_0 \) be over \( \gamma \) and \( \gamma_1 \) and \( h_s : [0, 1] \to B_0 \), \( s \in [0, 1] \), a homotopy from \( h_0 \) to \( h_1 \). Let \( 2c \) be the minimum of the convexity radii of all points in \( B_0 \). Take a partition \( 0 = t_0 < t_1 < \cdots < t_n = 1 \) of \( [0, 1] \) such that for each \( s \in [0, 1] \) the lengths of \( h_s[t_i, t_{i+1}] \) are less than \( c \) for \( i = 0, 1, \ldots, 2n-1 \). Define a new homotopy \( Dh_s \) from \( h_s \) to \( h_1 \) as follows. \( Dh_s \) is a closed and broken geodesic which consists of minimizing geodesics connecting \( h_s(t_{2i}) \) to \( h_s(t_{2i+1}) \), \( i = 0, 1, \ldots, n-1 \). Again, define a homotopy \( D^2h_s \) from \( h_0 \) to \( h_1 \) as follows. \( D^2h_s \) consists of minimizing geodesics from \( Dh_s(t_{2i+1}) \) to \( Dh_s(t_{2i+2}) \), \( i = 0, 1, \ldots, n-1 \). Repeating the process we have a sequence \( |D^n h_s ; n = 0, 1, \ldots| \) of homotopies. For each \( n \), since \( D^n h_s \) is a homotopy from \( h_0 \) to \( h_1 \), the set \( J_n = \{ s \in [0, 1] : \text{the length of } D^n h_s \text{ is greater then or equal to max } d_\varphi(B_1) \text{ is closed and not empty. Since } J_n \supset J_{n+1} \text{ for all } n, \text{ a real } s_0 \in \bigcap_{n=1} J_n \text{ exists. Thus, the limit of a converging subsequence of } |D^n h_s| \text{ is a closed geodesic whose length is greater than or equal to max } d_\varphi(B_1) \text{, since the number of vertices of each } D^n h_s \text{ is } n \). This completes the proof.

We now prove the converse of (0.4).

**Theorem 5.2.** If a torus \( T^2 \) satisfies the condition (0.4), then \( T^2 \) is flat.

Proof. Let \( T^2 = M/D \) and \( \varphi \in D \). We may assume as in the proof of Proposition 4.3 that \( \varphi \) and some \( \psi \) can generate \( D \). Let \( \gamma : (-\infty, \infty) \to M \) be an axis of \( \varphi \) and \( B \) the strip bounded by \( \gamma \) and \( \psi \gamma \). We can consider the \( \gamma \) and \( \psi \gamma \) as two lifts of a non-contractible closed curve in \( T^2 \). By the condition (0.4), the asymptotes from any point in \( B \) to \( \gamma \) and \( \psi \gamma \) must be identified. Suppose the displacement function \( d_\varphi \) is not constant in \( B \). From Lemma 5.1 there exists a geodesic \( a : (\langle -\infty, \infty \rangle \to M \) contained in \( B \) such that \( \psi a(t) = a(t+a), a > \min d_\varphi \). The \( a \) must be an asymptote to \( \gamma \) because of (0.4). Then it follows that \( a \) is an axis of \( \varphi \) (see [5]p.65,(2)). This contradicts that \( a > \min d_\varphi \). This completes the proof.

**References**


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