Hasse Principle” for Finite p-Groups with Cyclic Subgroups of Index p2

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“HASSE PRINCIPLE” FOR FINITE $p$-GROUPS WITH CYCLIC SUBGROUPS OF INDEX $p^2$

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1. Introduction

Let $G$ be a group. A map $f : G \rightarrow G$ satisfying $f(xy) = f(x)f(y)^x$ for every $x, y \in G$, where $f(y)^x = xf(y)x^{-1}$, is called a cocycle of $G$. Let $f$ be a cocycle of $G$. If, for every $x \in G$, there exists $a \in G$ such that $f(x) = a^{-1}ax$ then $f$ is called a local coboundary, and if there exists $a \in G$ such that $f(x) = a^{-1}ax$ for every $x \in G$ then $f$ is called a (global) coboundary. $G$ is said to enjoy “Hasse principle” if every local coboundary of $G$ is a coboundary. Abelian groups trivially enjoy “Hasse principle”. It is known that a finite group $G$ enjoys “Hasse principle” if and only if every conjugacy preserving automorphism of $G$ is an inner automorphism ([6], Theorem 3.1).

Some types of groups enjoying “Hasse principle” are known ([1], [2], [3], [5], [6], [7], [8], [9]). For finite $p$-groups, it is known that the following groups enjoy “Hasse principle”.

1. finite $p$-groups with cyclic subgroups of index $p$ ([1]);
2. extraspecial $p$-groups ([1]);
3. finite $p$-groups of order $p^4$ ([2]).

Among the known results, the following are useful for our study:

Theorem 1 ([2]). Metacyclic groups enjoy “Hasse principle”.

Theorem 2 ([3]). Let $H$ be a central subgroup of $G$. If $G/H$ is generated by $xH$ and $yH$ ($x, y \in G$) and every element of $G/H$ can be written as $x^ry^sH$, then $G$ enjoys “Hasse principle”.

Recently, M. Kumar and L. R. Vermani [3] proved that for an odd prime $p$, every non-abelian finite $p$-group of order $p^m$ having a normal cyclic subgroup of order $p^{m-2}$ but having no element of order $p^{m-1}$ enjoys “Hasse principle”. Further they have described that there are fourteen $2$-groups (up to isomorphism) of order $2^m$ of the above type and they showed that twelve of them enjoy “Hasse principle” but remaining two do not enjoy “Hasse principle”. In [4], for any prime $p$, all finite non-abelian $p$-groups of order $p^m$ having cyclic subgroups of order $p^{m-2}$ but having no element of order $p^{m-1}$ are classified. From the result we see that there is a missing group in a description in [3], which is given by

$$\langle a, b \mid a^{2^{m-2}} = 1, b^4 = a^{2^{m-3}}, b^{-1}ab = a^{-1} \rangle$$
(see [4], Remark 3 (1)). This group is metacyclic, and so enjoys “Hasse principle”. Further, two groups given in [3], Theorem 3.4 are isomorphic (see [4], Remark 3 (2)).

In this note we report that every non-abelian $p$-group of order $p^m$ having a cyclic subgroup of order $p^{m-2}$ but having no normal cyclic subgroup of order $p^{m-2}$ and no element of order $p^{m-1}$ enjoys “Hasse principle”. From now on suppose that $G$ is a non-abelian $p$-group of this type.

(I) For an odd prime $p$, there are seven possibilities about $G$. Using notation given in [4], we here list these groups:

$$G_1 = \langle x, y, z | x^{p^{m-2}} = 1, y^p = z^p = 1, xy = yx, z^{-1}xz = xy, yz = zy \rangle \quad (m \geq 3);$$

$$G_5 = \langle x, y, z | x^{p^{m-2}} = 1, y^p = z^p = 1, xy = yx, z^{-1}xz = xy, z^{-1}yz = x^{p^{m-3}}y \rangle \quad (m \geq 4);$$

$$G_6 = \langle x, y, z | x^{p^{m-2}} = 1, y^p = z^p = 1, xy = yx, z^{-1}xz = xy, z^{-1}yz = x^{p^{m-3}}y \rangle \quad (m \geq 4),$$

where $r$ is a quadratic nonresidue mod $p$.

$$G_7 = \langle x, y, z | x^{p^{m-2}} = 1, y^p = z^p = 1, y^{-1}xy = x^{1+p^{m-3}}, z^{-1}xz = xy, yz = zy \rangle \quad (m \geq 4);$$

$$G_9 = \langle x, y | x^{p^{m-2}} = 1, y^2 = 1, y^{-1}xy = x^{1+p} \rangle \quad (m \geq 5);$$

$$G_{10} = \langle x, y | x^{p^{m-2}} = 1, x^{p^{m-3}} = y^2, y^{-1}xy = x^{1-p} \rangle \quad (m \geq 6);$$

$$G_{11} = \langle x, y, z | x^9 = 1, y^3 = 1, z^3 = x^3, xy = yx, z^{-1}xz = xy, z^{-1}yz = x^6y \rangle \quad (m \geq 4).$$

By Theorem 2, $G_1$ enjoys “Hasse principle”, and because $G_9$ and $G_{10}$ are metacyclic by Theorem 1, they also enjoy “Hasse principle”.

(II) For $p = 2$, there are twelve possibilities about $G$. Again, using notation in [4], we list these groups:

$$G_5 = \langle x, y, z | x^{2^{m-2}} = 1, y^2 = z^2 = 1, xy = yx, z^{-1}xz = xy, yz = zy \rangle \quad (m \geq 4);$$

$$G_9 = \langle x, y | x^{2^{m-2}} = 1, y^4 = 1, x^{-1}yx = y^{-1} \rangle \quad (m \geq 5);$$

$$G_{13} = \langle x, y, z | x^{2^{m-2}} = 1, y^2 = z^2 = 1, xy = yx, z^{-1}xz = x^{-1}y, yz = zy \rangle \quad (m \geq 5);$$

$$G_{14} = \langle x, y, z | x^{2^{m-2}} = 1, y^2 = 1, z^2 = x^{2^{m-3}}, xy = yx, z^{-1}xz = x^{-1}y, yz = zy \rangle \quad (m \geq 5);$$
Proof. Let $f \in \text{Aut}_c G_5$ such that $f(z) = z$. Then there exist $a = x^i y^j z^k$, $b = x^r y^s z^t \in G_5$ with $0 \leq i, r < p^{m-2}$, $0 \leq j, s, t < p$ such that $f(x) = a^{-1} x a, f(y) = b^{-1} y b$, and so

\[ f(x) = z^{-k} y^{-j} x^{-i} \cdot x \cdot x^i y^j z^k = z^{-k} x z^k, \]

\[ f(y) = z^{-t} y^{-s} x^{-r} \cdot y \cdot x^r y^s z^t = z^{-t} y z^t. \]
As \( z^{-1}xz = xy \) and \( z^{-1}yz = x^{p^m - 3}y \) we have

\[
z^{-k}xz^k = x^{1 + (1 + 2 + \cdots + (k - 1))p^m - 3}y^k = x^{1 + \frac{k(k - 1)}{2}p^m - 3}y^k.
\]

We also have \( z^{-t}yz^t = x^{tp^m - 3}y \). Therefore \( f(x) = x^{1 + \frac{k(k - 1)}{2}p^m - 3}y^k, \ f(y) = x^{tp^m - 3}y \). Since \( f \) is an automorphism,

\[
f(z)^{-1}f(x)f(z) = f(z^{-1}xz) = f(xy) = f(x)f(y).
\]

We have

\[
f(z)^{-1}f(x)f(z) = z^{-1}(x^{1 + \frac{k(k - 1)}{2}p^m - 3}y^k)z = (z^{-1}xz)^{1 + \frac{k(k - 1)}{2}p^m - 3}(z^{-1}yz)^k = x^{1 + (k + \frac{k(k - 1)}{2})p^m - 3}y^{1 + k},
\]

\[
f(x)f(y) = x^{1 + \frac{k(k - 1)}{2}p^m - 3}y^kx^{tp^m - 3}y = x^{1 + (t + \frac{k(k - 1)}{2})p^m - 3}y^{1 + k}.
\]

Therefore the following congruence holds:

\[
1 + \left( k + \frac{k(k - 1)}{2} \right)p^m - 3 \equiv 1 + \left( t + \frac{k(k - 1)}{2} \right)p^m - 3 \pmod{p^m - 2}.
\]

From this it follows that \( k \equiv t \pmod{p} \). Then because \( 0 \leq k, t < p \), we have \( k = t \). Thus we have \( f(x) = z^{-k}xz^k, \ f(y) = z^{-k}xz^k, \ f(z) = z^{-k}xz^k \). This shows that \( f \in \text{Inn} \ G_5 \), and so \( G_5 \) enjoys “Hasse principle”. By an analogous argument we can show that \( G_6 \) enjoys “Hasse principle”. \( \square \)

In the rest of the paper, we proceed with a similar argument as above. Given \( f \in \text{Aut}_c G \), the image \( f(g) \) of \( g \in G \) will be denoted by \( \overline{g} \).

\( G_7 \) enjoys “Hasse principle”.

**Proof.** Let \( f \in \text{Aut}_c G_7 \) such that \( \overline{z} = z \). Then there exist \( a = x^iy^jz^k \), \( b = x^iy^sz^t \in G_7 \) with \( 0 \leq i, r < p^m - 2, 0 \leq j, k, s, t < p \) such that \( \overline{a} = a^{-1}xa, \ \overline{b} = b^{-1}yb \). We then have \( \overline{x} = x^{1 + jp^m - 3}y^k, \ \overline{y} = x^{-rp^m - 3}y, \ \overline{z} = z \). Since \( f \) is an automorphism, \( \overline{z}^{-1}\overline{x}\overline{z} = \overline{x}\overline{y} \). Because \( \overline{x}\overline{y} = x^{1 + (j - r)p^m - 3}y^{k + 1}, \ \overline{z}^{-1}\overline{x}\overline{z} = x^{1 + jp^m - 3}y^{k + 1} \), we have

\[
\overline{z}^{-1}\overline{x}\overline{z} = \overline{x}\overline{y} \iff rp^m - 3 \equiv 0 \pmod{p^m - 2}.
\]

Thus we have \( \overline{x} = x^{1 + jp^m - 3}y^k, \ \overline{y} = y, \ \overline{z} = z \). Therefore setting \( u = y^iz^k \), we have

\[
f(x) = u^{-1}xu, \ f(y) = u^{-1}yu, \ f(z) = u^{-1}zu,
\]

and so \( f \in \text{Inn} \ G_7 \). \( \square \)

\( G_{11} \) enjoys “Hasse principle”. 
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Proof. Let $f \in \text{Aut}_c G_{11}$ such that $z = z$. Then there exist $a = x^i y^j z^k$, $b = x^r y^s z^t \in G_{11}$ with $0 \leq i, r < 2^{m-2}$, $0 \leq j, k, s, t < 2$ such that $x = x^{-1} xa$, $y = b^{-1} yb$. We then have $x = x^{1+3k(k-1)} y^k$, $y = x^{6t} y$, $z = z$. Since $f$ is an automorphism, $z^{-1} x z = x y$. Because $x y = x^{1+6t+3k(k-1)} y^{k+1}$, $z^{-1} x z = x^{1+6k+3k(k-1)} y^{k+1}$, we have $z^{-1} x z = x y$, $k = t$. Thus we have $x = x^{1+3k(k-1)} y^k$, $y = x^{6k} y$, $z = z$. Therefore setting $u = z^k$, we have

$$f(x) = u^{-1} xu, \quad f(y) = u^{-1} yu, \quad f(z) = u^{-1} zu,$$

and so $f \in \text{Inn} G_{11}$. 

3. THE CASE $p = 2$

We here show that the groups $G_{17}, G_{18}, G_{22}, G_{24}, G_{25}$ and $G_{26}$ given in (II) enjoy “Hasse principle”.

$G_{17}$ enjoys “Hasse principle”.

Proof. Let $f \in \text{Aut}_c G_{17}$ such that $z = z$. Then there exist $a = x^i y^j z^k$, $b = x^r y^s z^t \in G_{17}$ with $0 \leq i, r < 2^{m-2}$, $0 \leq j, k, s, t < 2$ such that $x = x^{-1} xa$, $y = b^{-1} yb$. We then have $x = x^{1+j+2^m-3} y^k$, $y = x^{j+2^m-3} y$, $z = z$. Since $f$ is an automorphism, we have $z^{-1} x z = x y$. Because $x y = x^{1+(j+r)2^m-3} y^{k+1}$, $z^{-1} x z = x^{1+2^m-3} y^{k+1}$, we have $z^{-1} x z = x y$, $k = 0 \pmod{2}$. Thus we have $x = x^{1+j+2^m-3} y^k$, $y = y$, $z = z$. Therefore setting $u = y^j z^k$, we have $x = u^{-1} xu$, $y = u^{-1} yu$, $z = u^{-1} zu$, and so $f \in \text{Inn} G_{17}$. 

$G_{18}$ enjoys “Hasse principle”.

Proof. Let $f \in \text{Aut}_c G_{18}$ such that $z = z$. Then there exist $a = x^i y^j z^k$, $b = x^r y^s z^t \in G_{18}$ with $0 \leq i, r < 2^{m-2}$, $0 \leq j, k, s, t < 2$ such that $x = x^{-1} xa$, $y = b^{-1} yb$. We then have $x = x^{j+2^m-3} + (-1)^k (1+k2^m-4) + k2^m-4 y^k$, $y = x^{r+j+2^m-3} y$, $z = z$. Since $f$ is an automorphism, we have $z^2 = y$. Because $y = z^2 = x^{r+j+2^m-3} y$, $z^2 = y$, $k = 0 \pmod{2}$, we have $x = x^{j+2^m-3} + (-1)^k (1+k2^m-4) + k2^m-4 y^k$, $y = y$, $z = z$. Therefore setting $u = y^j z^k$, we have $x = u^{-1} xu$, $y = u^{-1} yu$, $z = u^{-1} zu$, and so $f \in \text{Inn} G_{18}$. 

$G_{22}$ enjoys “Hasse principle”.

Proof. Let $f \in \text{Aut}_c G_{22}$ such that $z = z$. Then there exist $a = x^i y^j z^k$, $b = x^r y^s z^t \in G_{22}$ with $0 \leq i, r < 2^{m-2}$, $0 \leq j, k, s, t < 2$ such that $x = x^{-1} xa$, $y = b^{-1} yb$. We then have $x = x^{1+2^m-4} y^k$, $y = x^{1+2^m-3} y$, $z = z$. Since $f$ is an automorphism, we have $z^{-1} x z = x^{1+2^m-4} y$. Because $x^{1+2^m-4} y = x^{1+2^m-3} + (1+k)2^m-4 y^k$, $z^{-1} x z = x^{1+k2^m-3} + (1+k)2^m-4 y^k$, $z^{-1} x z = x^{1+2^m-4} y$, $k = t$. 

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Thus we have $\overline{x} = x^{1+2^{m-4}} y^k$, $\overline{y} = x^{k} y^{2^{m-3}}, \overline{z} = z$. Therefore setting $u = z^k$, we have $\overline{x} = u^{-1} x u$, $\overline{y} = u^{-1} y u$, $\overline{z} = u^{-1} z u$, and so $f \in \text{Inn} G_{22}$. □

$G_{23}$ enjoys “Hasse principle”.

**Proof.** Let $f \in \text{Aut}_c G_{23}$ such that $z = z$. Then there exist $a = x^i y^j z^k$, $b = x^r y^s z^t \in G_{23}$ with $0 \leq i, r < 2^{m-2}$, $0 \leq j, k, s, t < 2$ such that $\overline{x} = a^{-1} x a$, $\overline{y} = b^{-1} y b$. We then have

$$
\overline{x} = \begin{cases} 
  x & (k = 0) \\
  x^{-1+2^{m-4}} y & (k = 1)
\end{cases}, \quad \overline{y} = x^{2^{m-3}} y, \quad \overline{z} = z.
$$

Since $f$ is an automorphism, we have $\overline{z}^{-1} \overline{x} \overline{z} = \overline{x}^{-1+2^{m-4}} \overline{y}$. If $k = 0$,

$$
\overline{z}^{-1} \overline{x} \overline{z} = \overline{x}^{-1+2^{m-4}} \overline{y} \iff t = 0. \quad \text{If } k = 1,
$$

$$
\overline{z}^{-1} \overline{x} \overline{z} = x^{1+(t-1)2^{m-3}}, \quad \overline{z}^{-1} \overline{x} \overline{z} = x.
$$

Therefore $\overline{z}^{-1} \overline{x} \overline{z} = \overline{x}^{-1+2^{m-4}} \overline{y} \iff t = 1$. Thus we have

$$
\overline{x} = \begin{cases} 
  x^{1+2^{m-4}} & (k = 0) \\
  x^{-1+2^{m-4}+2^{m-3}} y & (k = 1)
\end{cases}, \quad \overline{y} = x^{2^{m-3}} y, \quad \overline{z} = z.
$$

Therefore setting $u = z^k$, we have $\overline{x} = u^{-1} x u$, $\overline{y} = u^{-1} y u$, $\overline{z} = u^{-1} z u$, and so $f \in \text{Inn} G_{23}$.

$G_{24}$ enjoys “Hasse principle”.

**Proof.** Let $f \in \text{Aut}_c G_{24}$ such that $z = z$. Then there exist $a = x^i y^j z^k$, $b = x^r y^s z^t \in G_{24}$ with $0 \leq i, r < 2^{m-2}$, $0 \leq j, k, s, t < 2$ such that $\overline{x} = a^{-1} x a$, $\overline{y} = b^{-1} y b$. We then have

$$
\overline{x} = \begin{cases} 
  x^{1+2^{m-4}} & (k = 0) \\
  x^{-1+2^{m-4}+2^{m-3}} y & (k = 1)
\end{cases}, \quad \overline{y} = x^{2^{m-3}} y, \quad \overline{z} = z.
$$

Since $f$ is an automorphism, we have $\overline{z}^{-1} \overline{x} \overline{z} = \overline{x}^{-1+2^{m-4}} \overline{y}$. If $k = 0$,

$$
\overline{z}^{-1} \overline{x} \overline{z} = \overline{x}^{-1+2^{m-4}} \overline{y} \iff r \equiv 0 \pmod{2}. \quad \text{If } k = 1,
$$

$$
\overline{z}^{-1} \overline{x} \overline{z} = x^{1+(r-j)2^{m-3}}, \quad \overline{z}^{-1} \overline{x} \overline{z} = x^{2^{m-3}} x = x^{1+j2^{m-3}}.
$$

Therefore $\overline{z}^{-1} \overline{x} \overline{z} = \overline{x}^{-1+2^{m-4}} \overline{y} \iff r \equiv 0 \pmod{2}$. Thus we have

$$
\overline{x} = \begin{cases} 
  x^{1+2^{m-4}} & (k = 0) \\
  x^{-1+2^{m-4}+2^{m-3}} y & (k = 1)
\end{cases}, \quad \overline{y} = y, \quad \overline{z} = z.
$$
Therefore setting $u = y^j$, we have $\overline{x} = u^{-1} x u$, $\overline{y} = u^{-1} y u$, $\overline{z} = u^{-1} z u$, and so $f \in \text{Inn} G_{24}$.

$G_{25}$ enjoys “Hasse principle”.

Proof. Let $f \in \text{Aut}_c G_{25}$ such that $\overline{z} = z$. Then there exist $a = x^i y^j z^k$, $b = x^r y^s z^t \in G_{25}$ with $0 \leq i, r < 2^{m-2}$, $0 \leq j, k, s, t < 2$ such that $\overline{x} = a^{-1} x a$, $\overline{y} = b^{-1} y b$. We then have

$$\overline{x} = \begin{cases} x^{1+2^{m-3}} & (k = 0) \\ x^{-1+2^{m-4}+j2^{m-3}} y & (k = 1) \end{cases}, \quad \overline{y} = x^{r2^{m-3}} y, \quad \overline{z} = z.$$ 

Since $f$ is an automorphism, we have $\overline{z}^{-1} x \overline{z} = \overline{x}^{-1+2^{m-4}} \overline{y}$. If $k = 0$, $\overline{x}^{-1+2^{m-4}} \overline{y} = x^{-1+2^{m-4}+(j+r)2^{m-3}} y$, $\overline{z}^{-1} x \overline{z} = x^{-1+2^{m-4}+j2^{m-3}} y$.

Therefore $\overline{z}^{-1} x \overline{z} = \overline{x}^{-1+2^{m-4}} \overline{y} \iff r \equiv 0 \pmod{2}$. If $k = 1$, $\overline{x}^{-1+2^{m-4}} \overline{y} = x^{1+(r-j)2^{m-3}} = x^{1+(r+j)2^{m-3}}$, $\overline{z}^{-1} x \overline{z} = x^{1+j2^{m-3}}$.

Therefore setting $u = y^j$, we have $\overline{x} = u^{-1} x u$, $\overline{y} = u^{-1} y u$, $\overline{z} = u^{-1} z u$, and so $f \in \text{Inn} G_{25}$.

$G_{26}$ enjoys “Hasse principle”.

Proof. Let $f \in \text{Aut}_c G_{26}$ such that $\overline{z} = z$. Then there exist $a = x^i y^j z^k$, $b = x^r y^s z^t \in G_{26}$ with $0 \leq i, r < 8$, $0 \leq j, k, s, t < 2$ such that $\overline{x} = a^{-1} x a$, $\overline{y} = b^{-1} y b$. We then have $\overline{x} = x^{1+4j y^k}$, $\overline{y} = x^{4r y}$, $\overline{z} = z$. Since $f$ is an automorphism, we have $\overline{x} = x^{1+4j y^k}$, $\overline{y} = x^{4r y}$, $\overline{z} = z$. Therefore setting $u = y^j z^k$, we have $\overline{x} = u^{-1} x u$, $\overline{y} = u^{-1} y u$, $\overline{z} = u^{-1} z u$, and so $f \in \text{Inn} G_{26}$.

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