On minimal surfaces in a Riemannian manifold of constant curvature

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ON MINIMAL SURFACES IN A RIEMANNIAN MANIFOLD OF CONSTANT CURVATURE

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In the present paper, we shall study minimal surfaces in a Riemannian manifold $\hat{M}$ of (non-zero) constant curvature $\kappa$. When $\hat{M}$ is a sphere $S(1)$ of constant curvature 1, we find the conditions such that the immersion is uniquely determined up to a rigid motion of $S(1)$. In §2, we define the $(n+1)$th shape operator ($n$th torsion operator) and the $n$th torsion index ($T_n$-index) for minimal surfaces in $\hat{M}$. In §3, we study complete flat minimal surfaces in $\hat{M}$. When $\hat{M}$ is $S(1)$, we find the conditions such that the immersion is uniquely determined up to a rigid motion of $S(1)$. In §4, we give examples of minimal immersions of the Euclidean plane into a sphere. In the last section, we study compact minimal surfaces of non-negative curvature ($\kappa \geq 0$) in $\hat{M}$ and prove that they are generalized Veronese surfaces if they are immersed in $\hat{M} = S(1)$ by the immersions with full torsion.

§ 1. Preliminaries. Let $\hat{M}$ be a $(2+\nu)$-dimensional Riemannian manifold of constant curvature $\kappa$, and $\hat{M}$ a 2-dimensional Riemannian manifold which is isometrically immersed in $\hat{M}$ by an immersion $x : M \rightarrow \hat{M}$. The geodesic codimension of $M$ in $\hat{M}$ is defined to be the minimum of codimension of $M$ in totally geodesic submanifolds of $\hat{M}$, see [7]. We denote by $\nabla$ (resp. $\hat{\nabla}$) the covariant differentiation on $M$ (resp. $\hat{M}$). Then the (first) shape operator (second fundamental form) $\varphi$, of the immersion is given by $\varphi_1(X, Y) = \hat{\nabla}_XY - \nabla_XY$ for any tangent vector fields $X$ and $Y$ of $M$. There holds then $\varphi_1(X, Y) = \varphi(Y, X)$.

$F(\hat{M})$ and $F(M)$ denote the orthonormal frame bundles over $\hat{M}$ and $M$, respectively. Let $B$ be the set of all elements $b = (\rho, e_1, e_2, \cdots, e_{2+\nu}) \in F(\hat{M})$ such that $(\rho, e_1, e_2) \in F(M)$, identifying $\rho \in M$ with $x(\rho)$ and $e_i$ with $dx(e_i)$, $i = 1, 2$. Then $B$ is a smooth submanifold of $F(\hat{M})$. Let $\hat{\omega}_A$, $\hat{\omega}_{AB} = -\hat{\omega}_{BA}$, $A, B = 1, 2, \cdots, 2+\nu$, be the basic and connection forms of $\hat{M}$ on $F(\hat{M})$ which satisfy the following structure equations

$$
\begin{align}
\sum_B \hat{\omega}_{AB} \wedge \hat{\omega}_B, \\
\sum_C \hat{\omega}_{AC} \wedge \hat{\omega}_{CB} - c\hat{\omega}_A \wedge \omega_B.
\end{align}
$$

(1.1)
In this paper, we use the following convention on the range of indices
\[ i, j, \ldots = 1, 2, \quad \alpha, \beta, \ldots = 3, 4, \ldots, 2 + \nu. \]
Deleting the hats of \( \hat{\omega}_{AB} \) on \( B \), as is well known, we have
\[
\begin{align*}
\omega_{\alpha} &= 0, \\
\omega_{ij} &= \sum_{\ell} h_{i\ell} \omega_{\ell j}, \quad h_{ij} = h_{ji}, \\
d\omega_i &= \omega_{ij} \wedge \omega_j, \quad i \neq j,
\end{align*}
\]
(1.2)
\[
\begin{align*}
d\omega_{ij} &= \sum_k \omega_{ik} \wedge \omega_{kj} - \Omega_{ij}, \quad \Omega_{ij} = \frac{1}{2} \sum R_{ijkl} \omega_k \wedge \omega_l, \\
R_{ijkl} &= c(\partial_i \partial_j \partial_k - \partial_i \partial_k \partial_j + \sum_{\alpha} (h^{a}_{ik} h^{j\alpha}_{k} - h^{a}_{i\alpha} h^{j\alpha}_{k})), \\
d\omega_{\alpha\beta} &= \sum_{\ell} \omega_{\ell\alpha} \wedge \omega_{\ell\beta} - \Omega_{\alpha\beta}, \quad \Omega_{\alpha\beta} = \frac{1}{2} \sum R_{\alpha\beta\ell i} \omega_{\ell} \wedge \omega_i, \\
R_{\alpha\beta ij} &= \sum_k (h^{a}_{ik} h^{\beta}_{jk} - h^{a}_{i\alpha} h^{\beta}_{jk}).
\end{align*}
\]

\( M \) is said to be minimal if its mean curvature vector \( \frac{1}{2} \sum_{j, \alpha} h_{ij}^a e_{\alpha} \) vanishes identically, i.e., if trace \( H_{\alpha} = 0 \) for all \( \alpha \), \( H_{\alpha} = (h_{ij}^a) \). The minimal index at \( p \in M \) (m-index, \( m \)-index) \( M \) is defined to be the dimension of the linear space of all second fundamental forms corresponding to normal vectors at \( p \) with vanishing trace. We easily have \( m \)-index \( p M \leq 2 \) at every point \( p \in M \). We denote the square of the norm of the second fundamental form by
\[
S = \frac{1}{2} \sum h_{ij}^a h_{ij}^a
\]
The normal scalar curvature \( K_N \) of \( M \) in \( \hat{M} \) is defined by
\[
K_N = \sum_{i<j} (R_{\alpha\beta ij})^2 = \sum_{i<j} \{ \sum_{k} (h_{ik}^a h_{jk}^\alpha - h_{i\alpha} h_{jk}^a) \}^2.
\]

\( \S \ 2. \) The \((n + 1)\)-th shape operator and the \( n \)-th torsion index (\( T_n \)-index). In this section, we assume that \( M \) is minimal in \( \hat{M} \). We define the \((n + 1)\)-th shape operator (the \( n \)-th torsion operator) and the \( n \)-th torsion index for minimal surfaces in \( \hat{M} \) by induction on \( n \).

The (first) shape operator \( \varphi_1 \) can be written as
\[
\varphi_1(X, Y) = \sum_{i,j,a} h_{ij}^a \omega_j(X) \omega_i(Y) e_a.
\]
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Since \( \sum_{j=1}^{2} h_{ij}^2 = 0 \), for \( X = e_1 \cos \theta + e_2 \sin \theta \), we have

\[
\varphi_s(X) : = \varphi_s(X, X) = \cos 2\theta \cdot F_1 + \sin 2\theta \cdot G_1,
\]

where \( F_1 = \sum_{e} h_{1e}^2 e_e \) and \( G_1 = \sum_{e} h_{2e}^2 e_e \). Let \( S_p^1 \) be the unit circle in a tangent space \( M_p \) to \( M \) at \( p \in M \). It is clear from (2.1) that the image of \( S_p^1 \) under \( \varphi_s \) is a point, a segment, or an ellipse (a circle) according as \( m_{-1} \)\( M = 0, 1, \) or \( 2 \). We easily see that

\[
S^j - K_N = (\|G_1\|^2 - \|F_1\|^2)^2 + 4 \langle G_1, F_1 \rangle^2,
\]

which is geometrically stated as follows:

\[
\begin{align*}
S^j - K_N &= (\text{length of major axis})^2 - (\text{length of minor axis})^2 \quad &\text{if } m_{-1} \ M = 2, \\
S^j - K_N &= (\text{length of the segment})^4 \quad &\text{if } m_{-1} \ M = 1, \\
S^j &= K_N = 0 \quad &\text{if } m_{-1} \ M = 0.
\end{align*}
\]

If \( m_{-1} \ M = 0 \) at every point \( p \in M \), then \( M \) is totally geodesic in \( \hat{M} \). If \( m_{-1} \ M = 1 \) at every point \( p \in M \), then the geodesic codimension of \( M \) is 1.

Henceforth, we consider the case that \( m_{-1} \ M = 2 \) at every point \( p \in M \). Then, we can choose a local frame field \( b \in B \) such that

\[
\omega_{j1} \neq 0, \quad \omega_{j\beta} = 0, \quad \alpha_1 \in \{3, 4\}, \quad \text{for } \beta > 4.
\]

From (1.1), (1.2) and (2.3) we may write

\[
\sum_{e} h_{ij}^e \omega_{e, \beta} = \sum_{e} h_{ij}^e \omega_{e, \beta}, \quad \beta > 4,
\]

where \( h_{ij}^e \) are symmetric in the indices \( i, j, k \) and \( \sum_{j} h_{ij}^e = 0 \). Then, we can consider the 3-linear mapping from \( M_p \times M_p \times M_p \) into the normal space \( N_p \) at \( p \) as follows

\[
\varphi_s(X_1, X_2, X_3) = \sum_{\beta > 4} h_{ij}^e \omega_i(X_1) \omega_j(X_2) \omega_\beta(X_3) e_\beta, \quad X_i \in M_p.
\]

We call this mapping \( \varphi_s \) the second shape operator (first torsion operator) of \( M \) in \( \hat{M} \). Putting \( \varphi_s(X) = \varphi_s(X, X, X) \) for \( X \in M_p \), we get the mapping \( \varphi_s \) from \( M_p \) into \( N_p \). For \( X = e_1 \cos \theta + e_2 \sin \theta \), we have

\[
\varphi_s(X) = \cos 3\theta \cdot F_2 + \sin 3\theta \cdot G_2
\]

where \( F_2 = \sum_{\beta > 4} h_{11}^e e_\beta \) and \( G_2 = \sum_{\beta > 4} h_{11}^e e_\beta \). We call the dimension of the
image of $M_p$ under $\varphi_2$, the first torsion index of $M$ in $\hat{M}$ at $p \in M$ and denote it by $T_1$-index$_p M$. It is clear from (2.4) that the image of $S_p$ under $\varphi_2$ is a point, a segment, or an ellipse (a circle) according as $T_1$-index$_p M = 0$, 1, or 2. We easily see that

$$\left(\|G_2\|^2 - \|F_2\|^2\right)^2 + 4\langle G_2, F_2 \rangle^2 = \begin{cases} (\text{length of major axis})^2 - (\text{length of minor axis})^2)^2 & \text{if } T_1\text{-index}_p M = 2, \\ (\text{length of the segment})^4 & \text{if } T_1\text{-index}_p M = 1, \\ 0 & \text{if } T_1\text{-index}_p M = 0. \end{cases}$$

Thus, $\left(\|G_2\|^2 - \|F_2\|^2\right)^2 + 4\langle G_2, F_2 \rangle^2$ is invariant under the rotation of frames $\{e_i, e_s\}$ and $\{e_i, e_s\}$, so it is a differentiable function on $M$. If $T_1$-index$_p M = 0$ at every point $p \in M$, then we see that the geodesic codimension of $M$ is 2. If $T_1$-index$_p M = 1$ at every point $p \in M$, then we see that the geodesic codimension of $M$ is 3, this fact was proved in Theorem 4 in [7].

We consider the case that $T_1$-index$_p M = 2$ at every point $p \in M$. Then, we can choose a local frame field satisfying (2.3) and

$$\omega_{x_2} \neq 0, \quad \omega_{x_2} = 0, \quad \alpha_1 \in \{3, 4\}, \quad \alpha_2 \in \{5, 6\}, \quad 6 < \gamma.$$

It follows from (1.1), (1.2) and (2.5) that we may write

$$\sum_{\alpha} h_{ijkl} \omega_{x_2} = \sum_{\alpha} h_{ijkl} \omega_i, \quad 6 < \gamma,$$

where $h_{ijkl}$ are symmetric in the indices $i, j, k, l$ and $\sum_{\alpha} h_{ijkl} = 0$. Then, we can consider the 4-linear mapping from $M_p \times \cdots \times M_p$ into $N_p$ as follows: for $X_j \in M_p$, $j = 1, 2, 3, 4$,

$$\varphi_2(X_1, X_2, X_3, X_4) = \sum_{\alpha, \beta} h_{ijkl} \omega_i(X_1) \omega_j(X_2) \omega_k(X_3) \omega_l(X_4) e_\alpha.$$

We call this mapping $\varphi_2$ the third shape operator (second torsion operator) of $M$ in $\hat{M}$. Putting $\varphi_2(X) = \varphi_2(X, X, X, X)$ for $X \in M_p$, we get the mapping $\varphi_2$ from $M_p$ into $N_p$. Particularly, for a unit tangent vector $X = e_i \cos \theta + e_s \sin \theta$, we have

$$\varphi_2(X) = \cos 4\theta \cdot F_3 + \sin 4\theta \cdot G_3,$$

where $F_3 = \sum_{\alpha, \beta} h_{i1\beta} e_\beta$ and $G_3 = \sum_{\alpha, \beta} h_{i2\beta} e_\beta$. We call the dimension of the image of $M_p$ under $\varphi_2$ the second torsion index of $M$ in $\hat{M}$ at $p$ and denote it by $T_2$-index$_p M$. From (2.6), we see that the image of $S_p$ under
\( \varphi \) is a point, a segment, or an ellipse (a circle) according as \( T_\gamma \text{-index}_pM = 0, 1, \) or 2. By the same reason as the case of \( \varphi \), we easily see that 
\[
(\|G_n\|^3 - \|F_n\|^3)^2 + 4 < F_n, \ G_n > ^2
\]
is a differentiable function on \( M \).

Now, we assume that \( T_\gamma \text{-index}_pM = 2 \) at every point \( p \in M, \ n \geq 2 \). Then, we can choose a local frame such that
\[
(2.7) \quad \left\{ \begin{array}{l}
\omega_{t+1} = 0, \\
\omega_{t} = 0, \\
\alpha_{t+1} = \in I_t, \\
\alpha_t \in I_{t+1}, \quad 2t + 4 \leq \gamma, \quad t = 0, 1, 2, \ldots, n - 1.
\end{array} \right.
\]

From (1.1), (1.2) and (2.7), we may write
\[
\sum_{t=1}^{\gamma} h_{t+1}^{j_1 \ldots j_{t+2}} \omega_{t+1} \partial_{t+1} \omega_{t} = \sum_{t=3}^{\gamma} h_{t+1}^{j_1 \ldots j_{t+2}} \omega_{t+1} \partial_{t+1} \omega_{t}, \quad 2t + 4 \leq \gamma,
\]
where \( h_{t+1}^{j_1 \ldots j_{t+2}} \) are symmetric in the indices \( j_1, j_2, \ldots, j_{t+2} \) and \( \sum_{t=0}^{\gamma} h_{t+1}^{j_1 \ldots j_{t+2}} \partial_{t+1} \omega_{t} = 0 \) for \( t = 0, 1, 2, \ldots, n - 1 \). Hence, for \( t = n - 1 \), we can consider the \((n+2)\)-linear mapping \( \varphi_{n+1} \) from \( M_p \times \cdots \times M_p \) into \( N_p \) as follows: for \( X_j \in M_p, \ j = 1, 2, \ldots, n + 2 \),
\[
\varphi_{n+1}(X_1, \ldots, X_{n+2}) = \sum_{j=1}^{n+2} h_{j_1 j_2 \ldots j_{n+2}} \omega_{j_1}(X_1) \cdots \omega_{j_{n+2}}(X_{n+2}) e_j.
\]

Putting \( \overline{\varphi}_{n+1}(X) = \varphi_{n+1}(X, \ldots, X) \) for \( X \in M_p \), we get the mapping \( \overline{\varphi}_{n+1} \) from \( M_p \) into \( N_p \). We call this mapping \( \varphi_{n+1} \) (or \( \overline{\varphi}_{n+1} \)) the \((n+1)\text{-th shape operator} \) \((n\text{-th torsion operator})\) of \( M \) in \( \hat{M} \). Particularly, for a unit tangent vector \( X = e_\theta \cos \theta + e_\varphi \sin \theta \), we have
\[
(2.8) \quad \overline{\varphi}_{n+1}(X) = \cos(n+2)\theta \cdot F_{n+1} + \sin(n+2)\theta \cdot G_{n+1},
\]
where \( F_{n+1} = \sum_{j=3}^{n+2} h_{j_1 j_2 \ldots j_{n+2}} \) and \( G_{n+1} = \sum_{j=3}^{n+2} h_{j_1 j_2 \ldots j_{n+2}} \). We call the dimension of the image of \( M_p \) under \( \overline{\varphi}_{n+1} \) the \( n\text{-th torsion index} \) of \( M \) in \( \hat{M} \) at \( p \in M \) and denote it by \( T_{n}\text{-index}_pM \). It is clear from (2.8) that the image of \( S_p \) under \( \overline{\varphi}_{n+1} \) is a point, a segment, or an ellipse (a circle) according as \( T_{n}\text{-index}_pM = 0, 1, \) or 2. By the same reason as the case of \( \varphi \), we see easily that 
\[
(\|G_{n+1}\|^3 - \|F_{n+1}\|^3)^2 + 4 < F_{n+1}, \ G_{n+1} > ^2
\]
is a function on \( M \).

§ 3. Complete flat minimal surfaces in \( \hat{M} \). In this section, we assume that \( M \) is a complete, connected and oriented 2-dimensional Riemannian manifold which is minimally immersed in a Riemannian manifold \( \hat{M} \) of constant curvature \( c \neq 0 \) and that the Gaussian curvature \( K \) of \( M \) is identically zero. Then, as is well known, \( M \) may be considered as a Riemann surface. Since \( K \equiv 0 \) and \( M \) is complete, as is stated in [1], \( M \) is parabolic, i.e., a negative subharmonic function.
on $M$ must be constant. We first have

**Lemma 1.** On $M$ we have only one of the following cases:

- **S$_c$-case** $S = c$ and $K_N = 0$,
- **C$_c$-case** $S = c$ and $K_N = S^2 = c^2$,
- **E$_c$-case** $S = c$, $K_N = \text{constant} > 0$ and $S^2 - K_N > 0$.

**Proof.** Since $K = 0$ and $K = c - S$, we have

$$S = c = \text{constant} > 0 \text{ on } M.$$  

We shall prove that $K_N$ is constant on $M$. Since $K = 0$ on $M$, we can choose a neighborhood $U$ of a point $p \in M$ in which there exist isothermal coordinates $(u, v)$ and a frame field $b \in B$ such that

$$ds^2 = du^2 + dv^2, \quad \omega_i = du, \quad \omega_3 = dv,$$

where $ds$ is the line element of $M$. We may write $\omega_\alpha$ $(i = 1, 2$ and $3 \leq \alpha)$ as follows

$$\omega_\alpha = f_\alpha \omega_1 + g_\alpha \omega_2, \quad \omega_3 = g_\alpha \omega_1 - f_\alpha \omega_2,$$

where $f_\alpha$ and $g_\alpha$ are differentiable functions on $U$. Using the structure equations, we can see that the complex-valued function

$$w(z, \bar{z}) = \|G_1\|^2 - \|F_1\|^2 + 2i \langle F_1, G_1 \rangle$$

is holomorphic in $z = u + iv$, where $F_1 = \sum_{\alpha=2} f_\alpha \theta_\alpha$ and $G_1 = \sum_{\alpha=2} g_\alpha \theta_\alpha$.

Hence, $|w(z, \bar{z})|^2$ is a subharmonic function on $M$. Since $S^2 - K_N = |w(z, z)|^2$ and $S = c$, we see that $K_N$ is a non-negative superharmonic function on $M$, so it must be constant on $M$, because $M$ is parabolic. Thus, we have only one of three cases in Lemma.

By Lemma 1, at every point $p \in M$, the image of $S_p^1$ under $\varphi_1$ is a segment of the constant length or an ellipse with axes (a circle with radius) of the constant lengths according as $S_c$-case or $E_c$-case ($C_c$-case).

From now on, by $S$-case, $C$-case, or $E$-case we mean the case where the image of $S_p^1$ under the $t$-th shape operator (the $(t-1)$-th torsion operator) $\varphi_t$ is a segment, a circle, or an ellipse respectively.

In the $S_c$-case, $m$-index$_p M = 1$ at every point $p \in M$, so the geodesic codimension of $M$ is 1 by Theorem 1 in [6].

In the $C_c$-case and the $E_c$-case, $m$-index$_p M = 2$ at every point $p \in M$. In the $E_c$-case, we can choose a local frame field $b \in B$ such that
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\[ (3.1) \quad H_3 = \begin{pmatrix} k_1 & 0 \\ 0 & -k_1 \end{pmatrix}, \quad H_4 = \begin{pmatrix} 0 & \sigma_1 k_1 \\ \sigma_1 k_1 & 0 \end{pmatrix}, \quad H_\beta = 0, \quad 4 < \beta, \]

where \( k_1 \) and \( \sigma_1 \) are real constants \((\neq 0)\) on \( M \), \( \sigma_1 \neq 1 \). From (3.1) we have

\[ \omega_{12} = \omega_{34} = 0, \]

which implies that on a neighborhood \( U \) of a point \( p \in M \) we can choose isothermal coordinates \((u, v)\) such that

\[ (3.2) \quad ds^2 = du^2 + dv^2, \quad \omega_1 = du, \quad \omega_2 = dv. \]

In the \( C_\sigma \)-case, since \( K = 0 \), on a neighborhood of a point of \( M \) we can choose isothermal coordinates and a frame field satisfying (3.1) and (3.2), where \( \sigma_1 = 1 \). Thus, we may consider that the \( C_\sigma \)-case is a special case of the \( E_\sigma \)-case. It follows from (3.1) and (3.2) that we have

\[ (3.3) \quad \omega_{13} + i\omega_{23} = k_1 d\bar{z}, \quad z = u + iv, \]
\[ \omega_{14} + i\omega_{24} = i\sigma_1 k_1 d\bar{z}, \quad \omega_1 + i\omega_2 = dz. \]

Since \( \omega_{1\beta} = 0 \) \((4 < \beta)\), we may write

\[ (3.4) \quad \omega_{3\beta} + i\sigma_1 \omega_{4\beta} = (f_\beta + ig_\beta) d\bar{z}, \quad \beta < 4, \]

where \( f_\beta \) and \( g_\beta \) are differentiable functions on \( U \). Then, for a unit tangent vector \( X = e_1 \cos \theta + e_2 \sin \theta \), the second shape operator \( \vec{\varphi}_z \) is written as

\[ \vec{\varphi}_z(X) = \cos 3\theta \cdot F_3 + \sin 3\theta \cdot G_3, \]

where \( F_3 = k_1 \sum_{\beta \neq 4} f_\beta e_\beta \) and \( G_3 = k_1 \sum_{\beta \neq 4} g_\beta e_\beta \) are normal vector fields on \( U \).

Using the structure equations, from (3.2) and (3.4) we see that the complex-valued function

\[ w_i(z, \bar{z}) = -k_1 \sum_{\beta} (f_\beta - ig_\beta)^i = \|G_2\|^2 - \|F_3\|^2 + 2i \langle G_3, F_3 \rangle \]

is holomorphic in \( z \), because \( k_1 \) is constant on \( M \). Since, as be stated in \( \S \) 2, \( |w_1(z, \bar{z})|^2 \) is a differentiable function on \( M \), \( |w_1(z, \bar{z})|^2 \) is a subharmonic function on \( M \). Since \( \omega_{12} = \omega_{34} = 0 \), \( |w_1(z, \bar{z})|^2 \leq k_1 \{ \sum_{\beta \neq 4} (f_\beta^2 + g_\beta^2) \} = 4\sigma_1 k_1^2 \) is constant \((> 0)\). Hence \( |w_1(z, \bar{z})|^2 \) must be constant on \( M \), because \( M \) is parabolic. Then, for the second shape operator we have \( S \)-case, \( C \)-case or \( E \)-case on \( M \).

In the \( S \)-case for \( \vec{\varphi}_z \) on \( M \), \( T_i \)-index on \( M = 1 \) at every point \( p \in M \),
so the geodesic codimension of \( M \) is 3, see § 2.

We next consider the \( C \)-case and \( E \)-case for \( \overline{\varphi}_2 \). Since the image of \( S_p \) under \( \overline{\varphi}_2 \) is an ellipse with axes (a circle with radius) of constant length at every point \( p \in M \), we can choose a neighborhood \( U \) of a point \( p \in M \) in which there exist isothermal coordinates \((u, v)\) and a local frame field \( b \in B \) satisfying (3.3) and

\[
\begin{align*}
\omega_{38} + i\sigma_{1}\omega_{48} &= k_z d\overline{z}, \\
\omega_{37} &= 0, \\
\omega_{36} + i\sigma_i\omega_{46} &= i\sigma_i k_z d\overline{z}, \\
\omega_{47} &= 0, \\
6 < \gamma,
\end{align*}
\]

where \( k_z \) is a non-zero complex constant on \( M \) and \( \sigma_z \) is a non-zero real constant on \( M \). In the \( C \)-case for \( \overline{\varphi}_2 \), we may assume that \( k_z \) is a non-zero real constant on \( M \) and \( \sigma_z = 1 \). From (3.3) and (3.5), we have \( \omega_{38} = 0 \) and we may write

\[
\omega_{37} + i\sigma_i\omega_{47} = (f_t + ig_t) d\overline{z}, \quad 6 < \gamma.
\]

Hence, for \( X = e_1 \cos \theta + e_2 \sin \theta \in M_p \), we write the third shape operator \( \overline{\varphi}_3 \) as

\[
\overline{\varphi}_3(X) = \cos 4\theta \cdot F_3 + \sin 4\theta \cdot G_3,
\]

where \( F_3 \) and \( G_3 \) are real normal vector fields such that \( F_3 + iG_3 = k_z \sum \) (terms involving \( f_t + ig_t \)). Continuing this way, we have

**Lemma 2.** If the image of \( S_p \) under the \( t \)-th shape operator \( \overline{\varphi}_t \) is an ellipse with axes (a circle with radius) of constant length at every point \( p \in M \), for \( t = 1, 2, \ldots, s, \ 2 \leq s \), then the image of \( S_p \) under the \((s+1)\)-th shape operator \( \overline{\varphi}_{s+1} \) is a segment of constant length or an ellipse with axes (a circle with radius) of constant length at every point \( p \in M \).

**Proof.** By induction, from the assumption we can verify that on a neighborhood \( U \) of a point \( p \in M \) there exist isothermal coordinates \((u, v)\) and a frame field \( b \in B \) such that

\[
\begin{align*}
\omega_{13} + i\sigma_1\omega_{23} &= k_{1+1} d\overline{z}, \\
\omega_{23} &= d\overline{u} + id\overline{v} = \omega_1 + i\omega_2, \\
\omega_{14} + i\sigma_i\omega_{24} &= i\sigma_i k_{t+1} d\overline{z}, \\
\omega_{47} &= 0, \\
\alpha_t &= 2t + 1, \\
\alpha_s &= 2t + 2, \\
\beta_t &= 2t + 3, \\
\beta_s &= 2t + 4, \\
2t + 4 < \gamma, \\
t &= 0, 1, 2, \ldots, s - 1,
\end{align*}
\]

where \( k_t (2 \leq t \leq s) \) are non-zero complex constant on \( M \) and \( k_1 \) and \( \sigma_1 \) \((1 \leq t \leq s, \ \sigma_0 = 1)\) are non-zero real constant on \( M \). Using the structure equations, from (3.6), we obtain
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(3.7), \quad \omega_{\eta \gamma} = 0, \quad \alpha_t = 2t+1, \quad \alpha_s = 2s+2 \quad \text{for} \quad t = 0, 1, 2, \ldots, s

and we may write

\begin{equation}
(3.8) \quad \omega_{\eta \gamma} + i\omega_{\eta \gamma} = (f_r + ig_r)d\bar{z}, \quad a_1 = 2s+1, \quad a_2 = 2s+2 < \gamma.
\end{equation}

Let \( F_{s+1} \) and \( G_{s+1} \) be real normal vector fields on \( U \) such that \( F_{s+1} + iG_{s+1} = k_1k_2\cdots k_s \sum_{r > s+2} (f_r + ig_r)e_r \), then for a unit tangent vector \( X = e_1\cos\theta + e_2\sin\theta \in M_p \), the \((s+1)\)-th shape operator \( \bar{\varphi}_{s+1} \) is written as

\begin{equation}
\bar{\varphi}_{s+1}(X) = \cos(s + 2\theta) \cdot F_{s+1} + \sin(s + 2\theta) \cdot G_{s+1}.
\end{equation}

From (3.6), (3.7), and (3.8), we see that the complex-valued function

\begin{equation}
\omega_{s}(z, \bar{z}) = -(k_1k_2\cdots k_s) \sum_{r > s+2} (f_r - ig_r)^2 = |G_{s+1}|^2 - |F_{s+1}|^2 + 2i < G_{s+1}, F_{s+1} >
\end{equation}

is holomorphic in \( z \), because \( k_i \) \((i = 1, 2, \ldots, s)\) are constant on \( M \). Since, as previously stated in § 2, \( \omega(z, \bar{z}) \) is a differentiable function on \( M \), \( |\omega(z, \bar{z})|^2 \) is a subharmonic function on \( M \). On the other hand, from (3.7), we have

\begin{equation}
|\omega_{s}(z, \bar{z})|^2 \leq |k_1k_2\cdots k_s|^4 \left( \sum (f_r^2 + g_r^2) \right)^2 = |k_1k_2\cdots k_s|^4 \left( 1 + \sigma_{r-1}^2 \right)^2 = \text{constant} (> 0) \text{ on } M.
\end{equation}

Hence the subharmonic function \( |\omega_{s}(z, \bar{z})|^2 \) must be constant on \( M \), because \( M \) is parabolic. Thus, at every point \( \rho \in M \) the image of a unit tangent circle \( S^1_{\rho} \) to \( M \) under \( \bar{\varphi}_{s+1} \) is a segment of constant length, a circle with constant length or an ellipse with axes of constant length on \( M \).

If the image of \( S^1_{\rho} \) under \( \bar{\varphi}_{s+1} \) is a segment of constant length at every point \( \rho \in M \), then \( T^*\text{index}_p M = 1 \) at every point \( \rho \) so the geodesic codimension of \( M \) is \( 2s + 1 \).

If the geodesic codimension of \( M \) is even \( 2s \), using the structure equations, from (3.6), and (3.8) we have contradiction. Hence, the geodesic codimension of \( M \) is odd \( 2m + 1 \) and the images of unit tangent circles to \( M \) under \( \bar{\varphi}_{m+1} \) are segments of constant length on \( M \). Thus, we have proved the following

**Theorem 1.** Let \( M \) be a 2-dimensional, connected, oriented and complete Riemannian manifold which is minimally immersed in a \((2 + \nu)\)-dimensional Riemannian manifold \( \hat{M} \) of non-zero constant curvature \( c \). If Gaussian curvature of \( M \) is identically zero and the image of \( M \) under the immersion is not contained in a totally geodesic submanifold of \( \hat{M} \), i.e., \( \nu \) is the geodesic codimension of \( M \), then \( \nu \) is odd \( 2m + 1 \). Furthermore, the images of unit tangent circles to \( M \)
under the t-th shape operators (1 ≤ t ≤ m) are ellipses with axes (or circles with radius) of constant length and the images of unit tangent circles to M under the (m + 1)-th shape operator are segments of constant length on M.

In view of Theorem 1, we see that on a neighborhood of a point of M there exist isothermal coordinates (u, v) and a local frame field satisfying (3.6)_{m+1}, where σ_{m+1} = 0 and k_{m+1} is a positive constant on M. In general, however, the constants k_t (1 ≤ t ≤ m + 1) and σ_t (1 ≤ t ≤ m) depend on the immersions. Using the structure equations, from (3.7)_m we get

\[(3.9)_m \quad (1 + \sigma_t^2)|k_t + 1|^2 = 2c\sigma_t^2/(1 + \sigma_t^2), \quad \sigma_0 = 1, \quad \sigma_{m+1} = 0, \quad 0 ≤ t ≤ m.\]

Since c > 0 from Lemma 1, we consider the case where \(\hat{M}\) is a \((2m+3)\)-dimensional sphere \(S^{2m+3}(c)\) of constant curvature c. We may consider \(S^{2m+3}(c) \subset E^{2m+4}\) and set \(\sqrt{c} x = e_{2m+4}\). Let \(E_t = e_{α_t} + ie_{β_t}\) and \(E^*_t = σ_t e_{α_t} - ie_{β_t}\), \(α_t = 2t + 1\), \(α_2 = 2t + 2\), where \(t = 0, 1, 2, ⋯, m\). Since \(E^*_t = ((1 + \sigma_t^2)/2σ_t)E_t - ((1 - \sigma_t^2)/2σ_t)E_t\), we have the following Frenet formulas of M

\[
dx = \frac{1}{2} (\overline{E}_0 dz + E_0 d\overline{z}), \quad z = u + iv, \\
dE_0 = -cz dz + k_1 E_1 d\overline{z}, \\
dE_1 = -((1 + \sigma_t^2)k_1/2)E_0 dz - (((1 - \sigma_t^2)k_1/2)E_0 - k_1 E_2) d\overline{z}, \\
dE_2 = -((1 + \sigma_t^2)k_2/2σ_t)E^*_1 dz - (((1 - \sigma_t^2)k_2/2σ_t)E^*_1 - k_2 E_2) d\overline{z}, \\
(3.10)_m \quad \text{........................................} \\
dE_t = -((1 + \sigma_t^2)k_1/2σ_t)E^*_t dz - (((1 - \sigma_t^2)k_1/2σ_t)E^*_t - k_1 E_{t+1}) d\overline{z}, \\
\quad \text{........................................} \\
dE_{m-1} = -((1 + σ_{m-1}^2)k_{m-1}/2σ_{m-1})E^*_{m-1} dz \\
\quad - (((1 - σ_{m-1}^2)k_{m-1}/2σ_{m-1})E^*_{m-1} - k_{m-1} E_m) d\overline{z}, \\
dE_m = -((1 + σ_m^2)k_m/2σ_m)E^*_m dz \\
\quad - (((1 - σ_m^2)k_m/2σ_m)E^*_m - k_m E_{m+1}) d\overline{z}, \\
de_{2m+3} = -(k_{m+1}/2σ_m)E^*_m d\overline{z} - (k_{m+1}/2σ_m)E^*_m d\overline{z}.
\]

From (3.9)_m and (3.10)_m we easily see that the vector fields \(E_0, E_1, E_2, ⋯, E_m, E_{2m+3}\) and \(x = e_{2m+4}\) satisfy the following equation

\[(3.11) \quad \frac{∂^3 Y}{(∂z \cdot ∂\overline{z})} = -(c/2)Y.\]

**Remark.** In Theorem 1, it seems to be difficult to prove the rigidity for the minimal immersion of the Euclidean plane into a sphere without
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any assumption.
Under the assumption of Theorem 1, if the images of unit tangent circles to $M$ under the $t$-th shape operators ($1 \leq t \leq m$) are circles with radius of constant length on $M$, then we can choose a neighborhood of a point of $M$ in which there exist isothermal coordinates and a local frame field $b \in B$ satisfying (3.6)$_{m+1}$ such that $\sigma_i = 1$ ($1 \leq t \leq m$), $\sigma_{m+1} = 0$ and $k_i$ ($1 \leq i \leq m+1$) are non-zero real constants on $M$. These constants are independent of the immersion. When $c = 1$, in the same way as in §8 of [7], from (3.10)$_m$ we can verify that $M$ is the surface given by

$$
x = \frac{1}{\sqrt{2(m+2)}} \sum_{j=1}^{m+1} \{ A_j \exp \frac{i\sqrt{2}}{2} (u \cdot \sin \frac{2j-1+\epsilon}{2(m+2)} \pi + v \cdot \cos \frac{2j-1+\epsilon}{2(m+2)} \pi) \\
+ \bar{A}_j \exp (-i\sqrt{2}) (u \cdot \sin \frac{2j-1+\epsilon}{2(m+2)} \pi + v \cdot \cos \frac{2j-1+\epsilon}{2(m+2)} \pi) \}, \tag{3.12}
$$

where $A_1, A_2, \ldots, A_{m+2}$ are constant vectors in $C^{m+1} = E^{m+1}$ such that

$$A_j \cdot A_j = A_j \cdot A_k = \delta_j^k = 0, \quad A_j \cdot A_j = 1, \quad j \neq k,$$

and $\epsilon = 0$ or 1 according as $m$ = odd or even. Thus, we have proved the following

Theorem 2. Under the assumption of Theorem 1, if $\hat{M}$ is a unit sphere and the images of unit tangent circles to $M$ under the $t$-th shape operators ($1 \leq t \leq m$) are circles with radius of constant length on $M$, then the immersion is uniquely determined up to a rigid motion of a sphere.

We are interested in examples of flat minimal surfaces other than (3.12). In the next section, we shall find examples other than (3.12).

§ 4. Examples of minimal immersions of the Euclidean plane into a sphere of constant curvature 1. In this section, we give examples of flat minimal surfaces in $S^2(1)$ and $S^3(1)$ other than (3.12).

We first give examples in $S^2(1)$, that is, find solutions of (3.10), in the case $c = 1$. Noticing (3.11), we choose three fixed constant vectors $A_1, A_2, A_3$ in $C^3 = E^3$ such that

$$A_j \cdot A_j = A_j \cdot A_k = A_j \cdot A_k = 0, \quad \sum_{j=1}^{3} A_j \cdot \bar{A}_j = 1, \tag{4.1}$$

$$j, k = 1, 2, 3, \quad j \neq k,$$

and let

$$x = \frac{1}{\sqrt{2}} \sum_{j=1}^{3} \left( A_j \exp \frac{1}{\sqrt{2}} (z \epsilon^{a_j} - \bar{z} \epsilon^{a_j}) + \bar{A}_j \exp \frac{1}{\sqrt{2}} (-z \epsilon^{\bar{a}_j} + \bar{z} \epsilon^{\bar{a}_j}) \right), \tag{4.2}$$
where the bar denotes the conjugate and $\alpha_j$ ($j = 1, 2, 3$) are real constant numbers. It is clear that $x = \bar{x}$ and $x \cdot x = 1$ by (4.1). In this case, we set
\begin{align*}
E_s &= \frac{\partial E}{\partial z} = - \sum_{j=1}^{3} e^{-i\alpha_j} \{ A_j \exp \left( \frac{1}{\sqrt{2}} (z e^{i\alpha_j} - \bar{z} e^{-i\alpha_j}) \right) - \overline{A}_j \exp \left( - \frac{1}{\sqrt{2}} (\bar{z} e^{-i\alpha_j} - z e^{i\alpha_j}) \right) \}, \\
E_1 &= \frac{\partial E}{k_1 \partial z} = \frac{1}{k_1} \frac{1}{\sqrt{2}} \sum_{j=1}^{3} e^{-i\alpha_j} \{ A_j \exp \left( \frac{1}{\sqrt{2}} (z e^{i\alpha_j} - \bar{z} e^{-i\alpha_j}) \right) + \overline{A}_j \exp \left( - \frac{1}{\sqrt{2}} (\bar{z} e^{-i\alpha_j} + z e^{i\alpha_j}) \right) \}, \\
e_3 &= \frac{\partial E}{k_2 \partial z} + \frac{1}{2k_2} \sum_{j=1}^{3} e^{-i\alpha_j} \left[ \frac{1 - \sigma_1}{1 + \sigma_1} e^{i\alpha_j} - e^{-i\alpha_j} \right] \cdot \left\{ A_j \exp \left( - \frac{1}{\sqrt{2}} (z e^{i\alpha_j} - \bar{z} e^{-i\alpha_j}) \right) - \overline{A}_j \exp \left( \frac{1}{\sqrt{2}} (\bar{z} e^{-i\alpha_j} + z e^{i\alpha_j}) \right) \right\},
\end{align*}

where $k_1, k_2$ and $\sigma_1$ are non-zero real constant numbers satisfying (3.9). Then, we easily see that these vectors satisfy the equation (3.10), in the case $c = 1$. From the above equations we have
\begin{align*}
x \cdot E_0 &= E_0 \cdot E_1 = E_0 \cdot E_1 = E_1 \cdot e_3 = e_3 \cdot x = 0, \\
E_0 \cdot \bar{E}_0 &= 2, \quad E_1 \cdot \bar{E}_1 = 1 + \sigma_1^2.
\end{align*}
Hence, we see that (4.2) is a solution of (3.10), if and only if there hold the following equalities
\begin{align*}
x \cdot E_1 &= E_1 \cdot E_0 = E_1 \cdot e_3 = 0, \quad E_1 \cdot E_1 = 1 - \sigma_1^2, \\
e_3 &= \bar{e}_3, \quad e_3 \cdot e_3 = 1.
\end{align*}
By the above definitions of $x$, $E_0$, $E_1$ and $e_3$, these equations are equivalent to the following
\begin{align}
(4.3) & \sum_{j=1}^{3} A_j \cdot \bar{A}_j e^{-2i\alpha_j} = 0, \\
(4.4) & \sum_{j=1}^{3} k_i^2 (1 - \sigma_1^2), \\
(4.5) & \sum_{j=1}^{3} A_j \cdot \bar{A}_j e^{-2i\alpha_j} = -2k_1^2 k_2^2, \\
(4.6) & \cos 3\alpha_j - (1 - \sigma_1^2)(1 + \sigma_1^2) \cos \alpha_j = 0 \quad \text{for} \quad j = 1, 2, 3.
\end{align}
Now, we shall find constant vectors $A_j$ ($j = 1, 2, 3$) and real constant numbers $\alpha_j$ ($j = 1, 2, 3$) satisfying (4.3) ~ (4.6) under the condition (3.9). We consider the following special case:
\begin{align}
(4.7) & A_1 \cdot \bar{A}_1 = A_2 \cdot \bar{A}_2, \quad -\alpha_1 = \alpha = \alpha_3, \quad 2\alpha_3 = \pi.
\end{align}
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Then, from (4.1), (4.3) and (4.7) we have

\[ A_1 \cdot \bar{A}_1 = A_2 \cdot \bar{A}_2 = \frac{1}{2(1 + \cos 2\alpha)} , \quad A_3 \cdot \bar{A}_3 = \frac{\cos 2\alpha}{1 + \cos 2\alpha} . \]

Hence, in this special case, if there exist \( A_j \) and \( \alpha_j \) \((j = 1, 2, 3)\) satisfying (4.3) \(\sim\) (4.6), the solutions (4.2) are determined only by one parameter \( \alpha \). Using \((1 + \sigma_j^0)k_i^1 = 1\), from (4.4) and (4.8) we have

\[ \cos 2\alpha = [(1 - \sigma_j^0)/(1 + \sigma_j^0) + 1]/2 = 1/(1 + \sigma_j^0) = k_i^1 . \]

Since \( k_1 \neq 0, \sigma_j \neq 0 \) and \((1 + \sigma_j^0)k_i^1 = 1\), we have \(0 < k_i^1 < 1\), which implies \(0 < \cos 2\alpha < 1\) by (4.9), so that we may assume that \(0 < \alpha < \pi/4\). Then, from (4.9) we have

\[ \cos \alpha = \sqrt{(k_i^1 + 1)/2}, \quad \sin \alpha = \sqrt{(1 - k_i^1)/2} . \]

and from (4.8) we have

\[ A_1 \cdot \bar{A}_1 = A_2 \cdot \bar{A}_2 = \frac{1}{2(1 + k_i^1)} , \quad A_3 \cdot \bar{A}_3 = \frac{k_i^1}{1 + k_i^1} . \]

We can easily see that these constants satisfy (4.5) and (4.6). Thus, we obtain examples given by

\[
x = \frac{1}{\sqrt{2}} \left[ A_1 \exp(i \sqrt{k_i^1 + 1} - u \sqrt{1 - k_i^1}) + \bar{A}_1 \exp(u \sqrt{1 - k_i^1} - v \sqrt{1 + k_i^1}) \right.
\]

\[ + A_2 \exp(v \sqrt{1 + k_i^1} + u \sqrt{1 - k_i^1}) + \bar{A}_2 \exp(-i(v \sqrt{1 + k_i^1} + u \sqrt{1 - k_i^1})) \]

\[ + A_3 \exp(\sqrt{2} iu) + \bar{A}_3 \exp(-\sqrt{2} iu) \],

where \( A_1, A_2, A_3 \) are fixed constant vectors in \( C^4 = E^4 \) satisfying (4.1) and (4.10) and \( k_i^1 \) is a positive constant smaller than 1.

Next, we shall give examples in \( S^1(1) \), that is, find solutions of (3.10)\(\bar{\sigma} \in \) in the case where \( c = 1 \) and \( k_2 = \) real constant \( \neq 0 \). Noticing (3.11), we choose four fixed vectors \( A_1, A_2, A_3, A_4 \) in \( C^4 = E^4 \) such that

\[ A_j \cdot \bar{A}_j = A_j \cdot A_k = A_j \cdot \bar{A}_k = 0, \quad \sum_{j=1}^{4} A_j \cdot \bar{A}_j = 1, \quad j, k = 1, 2, 3, 4, \quad j \neq k. \]

Let

\[ x = \frac{1}{\sqrt{2}} \sum_{j=1}^{4} \left[ A_j \exp \left( \frac{1}{\sqrt{2}} (\alpha \sigma_j^0 - \bar{\alpha} \sigma_j^0) \right) + \bar{A}_j \exp \left( -\frac{1}{\sqrt{2}} (\alpha \sigma_j^0 + \bar{\alpha} \sigma_j^0) \right) \right], \]

where \( \alpha_j \) \((j = 1, 2, 3, 4)\) are real constant numbers. It is clear from (4.12) and (4.13) that \( x = \bar{x} \) and \( x \cdot x = 1 \). We define vectors \( E_0, E_1, E_2 \)
and $e_\tau$ as follows

$$
E_0 = 2 \frac{\partial X}{\partial z} = -\sum_{j=1}^4 e^{-u_j}B_j^*, \\
E_1 = \frac{\partial E_0}{k_1 \frac{\partial z}{\partial x}} = \frac{1}{k_1 \sqrt{2}} \sum_{j=1}^4 e^{-u_j}B_j, \\
E_2 = \frac{\partial E_1}{k_2 \frac{\partial z}{\partial x}} + \frac{(1-\sigma_1^2)k_1 E_1}{2k_2} = \frac{1}{2k_1 k_2} \sum_{j=1}^4 \left[ \frac{(1-\sigma_1^2)}{1+\sigma_1^2} e^{-u_j} \right] B_j^*, \\
e_\tau = \frac{\partial E_2}{k_3 \frac{\partial z}{\partial x}} + \frac{(1-\sigma_1^2)k_2 E_1}{2k_3 \sigma_1} = \frac{1+\sigma_1^2}{2\sigma_1} E_1 - \frac{1-\sigma_1^4}{2\sigma_1} E_1, \\
\frac{1}{2\sqrt{2}} k_1 k_2 k_3 \sum_{j=1}^4 \left[ e^{-u_j} \frac{1-\sigma_1^2+1-\sigma_2^2 e^{2iu_j}-(1-\sigma_1^2)(1-\sigma_2^2) e^{-2iu_j}}{1+\sigma_1^2(1+\sigma_2^2)} (1-\sigma_1)(1-\sigma_2) e^{-2iu_j} \right] B_j,
$$

where $B_j = A_j \exp \left( (z e^{iu_j} - \bar{z} e^{-iu_j}) \sqrt{\frac{i}{2}} \right) + \bar{A}_j \exp \left( (-z e^{iu_j} + \bar{z} e^{-iu_j}) \sqrt{\frac{i}{2}} \right)$ and $B_j^* = A_j \exp \left( (z e^{iu_j} - \bar{z} e^{-iu_j}) \sqrt{\frac{i}{2}} \right) - \bar{A}_j \exp \left( (-z e^{iu_j} + \bar{z} e^{-iu_j}) \sqrt{\frac{i}{2}} \right)$ and $k_1$, $k_2$, $k_3$, $\sigma_1$, and $\sigma_2$ are real constant numbers satisfying (3.9). From (4.14) we easily see that

$$
x \cdot E_0 = E_1 \cdot E_1 = E_0 \cdot \bar{E}_1 = E_0 \cdot e_\tau = E_1 \cdot E_2 = E_1 \cdot \bar{E}_2 = E_2 \cdot E_2 = E_2 \cdot e_\tau = 0, \\
E_0 \cdot \bar{E}_0 = 2, \quad E_1 \cdot \bar{E}_1 = 1 + \sigma_1, \quad E_2 \cdot \bar{E}_2 = 1 + \sigma_2.
$$

Hence, (4.13) is a solution of (3.10) if and only if the following conditions are satisfied:

$$
x \cdot E_1 = x \cdot e_\tau = E_0 \cdot \bar{E}_0 = E_0 \cdot E_0 = E_0 \cdot \bar{E}_2 = E_1 \cdot e_\tau = 0, \\
E_1 \cdot E_1 = 1 - \sigma_1^2, \quad E_2 \cdot E_2 = 1 - \sigma_2^2,
$$

(4.16) \hspace{1cm} e_\tau \text{ is a unit real vector.}

Using (4.12), we see that the conditions (4.15) are equivalent to the following equalities:

$$
\sum_{j=1}^4 A_j \cdot \bar{A}_j e^{-2iu_j} = 0, \\
\sum_{j=1}^4 A_j \cdot \bar{A}_j e^{-4iu_j} = k_i^2(1 - \sigma_i^2), \\
\sum_{j=1}^4 A_j \cdot \bar{A}_j e^{-6iu_j} = -2k_i^2k_j^2(1 - \sigma_j).
$$

In order to find constant vectors $A_j$ in $C^4$ and constant numbers $\alpha_j$ \hspace{1cm} \hspace{1cm} (j = 1, 2, 3, 4) satisfying (4.16) and (4.17), we consider the following special case:

$$
A_1 \cdot \bar{A}_1 = A_2 \cdot \bar{A}_2, \quad -\alpha_1 = \alpha_3 = \alpha, \quad \alpha_2 = 0, \quad 2\alpha_4 = \pi.
$$
By means of (3.9) and (4.12), from (4.17) we have

\[
\begin{align*}
\cos 2\alpha &= k^i(1 - \sigma_1^2)/(2\sigma_1^2) = k^i(1 - k^i) =: k, \\
A_1 \cdot \bar{A}_1 &= A_2 \cdot \bar{A}_2 = (k^1 - 1)/(2k^2 - 2), \\
A_2 \cdot \bar{A}_2 &= (k - k^1)/(2k - 2), \\
A_4 \cdot \bar{A}_4 &= (k + k^1)/(2k + 2).
\end{align*}
\]

Therefore, we have seen that if there exist \( A_j \) and \( \alpha_j (j = 1, 2, 3, 4) \) satisfying (4.12), (4.16) and (4.17), then (4.13) is determined only by two parameters \( k_1 \) and \( k_3 \). In this case, we can show that \( e_t \) is a unit real vector. Since \( k^2 = 2\sigma_1^2/(1 + \sigma_1^2) \), \( k^3(1 + \sigma_1^3) = 1 \), \( \sigma_1 \neq 0 \) and \( \sigma_2 \neq 0 \), we have

\[
0 < k_1 < 1 \quad \text{and} \quad 0 < k_3 < 2,
\]

which implies \(|\cos 2\alpha| < 1\) by (4.19). Hence, we may assume that \( 0 < \alpha < \pi/2 \). Thus, we obtain examples of minimal immersions of the Euclidean plane into a sphere \( S^1(1) \) given by

\[
\begin{align*}
x &= \frac{1}{\sqrt{2}} \left[ A_1 \exp i(v\sqrt{1 + k} - u\sqrt{1 - k}) + A_3 \exp i(u\sqrt{1 - k} - v\sqrt{1 + k}) \right. \\
&\quad + A_2 \exp (v\sqrt{2}i) + A_3 \exp (-v\sqrt{2}i) \\
&\quad + A_2 \exp (u\sqrt{1 + k} - u\sqrt{1 - k}) + A_3 \exp (-u\sqrt{1 + k} - u\sqrt{1 - k}) \right] \\
&\quad + A_4 \exp (\sqrt{2}iu) + A_4 \exp (-\sqrt{2}iu),
\end{align*}
\]

where \( A_1, A_2, A_3, A_4 \) are fixed constant vectors in \( C^4 = E^4 \) satisfying (4.12) and (4.20) and \( k \) is a constant real number such that

\[
k := k^i(1 - k^i), \quad 0 < k_1 = \text{const.} < 1, \quad 0 < k_3 = \text{const.} < 2.
\]

Remark. We have obtained many examples of the Euclidean plane minimally immersed into the Euclidean unit spheres \( S^4 \) and \( S^7 \) other than Otsuki's surfaces. When \( m = 1 \) and \( m = 2 \), Otsuki's surfaces (3.12) are included in these examples as the special case where \( \sigma_1^3 = 1 \) and \( \sigma_2^3 = 1 \).

§ 5. Compact minimal surfaces of non-negative (\( \neq 0 \)) curvature in \( \hat{M} \). In this section, we shall consider connected compact minimal surfaces of non-negative curvature \( K (\neq 0) \) in a \((2 + \nu)\)-dimensional Riemannian manifold \( \hat{M} \) of constant curvature \( c \).

Let \( U \) be a neighborhood of a point \( p \in M \) in which there exist isothermal coordinates \((u, v)\) and a frame field \( b \in B \) such that

\[
ds^2 = E\left(du^2 + dv^2\right), \quad \omega_1 = \sqrt{E} \, du, \quad \omega_2 = \sqrt{E} \, dv,
\]
where \( ds \) is the line element of \( M \) and \( E = E(u, v) \) is a positive function on \( U \). In this case, we may write

\[
\omega_{1\alpha} = f_\alpha \omega_1 + g_\alpha \omega_2, \quad \omega_{2\alpha} = g_\alpha \omega_1 - f_\alpha \omega_2, \quad 3 < \alpha,
\]

where \( f_\alpha \) and \( g_\alpha \) are functions on \( U \). Then, using the structure equations, we can verify that the complex valued function

\[
(5.2) \quad w(z, \bar{z}) = E^2 (|G_1|^2 - |F_1|^2) + 2iE^2 \langle F_1, G_1 \rangle, \quad F_1 = \sum f_\alpha \varepsilon_\alpha, \quad G_1 = \sum g_\alpha \varepsilon_\alpha,
\]
is holomorphic in \( z = u + iv \). Then, we have the following

**Lemma 3.** We have \( S^2 - K_N = 0 \) on \( M \).

**Proof.** By an easy computation, we see that \( S^2 - K_N = |w(z, \bar{z})|^2 / E^4 \) on \( M \). If \( S^2 - K_N = 0 \) does not hold identically on \( M \), \( S^2 - K_N \) takes its positive maximum \( A \) at some \( p \in M \). Let \( U \) be a neighborhood of \( p \) in which \( S^2 - K_N > 0 \) and there exist isothermal coordinates \( (u, v) \) and a frame field \( b \in B \) satisfying (5.1). Then, from (5.2) we have

\[
(5.3) \quad \Delta \log (S^2 - K_N) = -4 \Delta \log E = 8EK, \quad \Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2},
\]
because the Gaussian curvature \( K \) is given by \( K = -(1/2E) \Delta \log E \). If \( K \geq 0 \), the function \( \log(S^2 - K_N) \) is a subharmonic function on \( U \), so it must be constant \( A \) on \( U \). Therefore, the closed set \( \{ p \in M | S^2 - K_N = A \text{ at } p \} \) of \( M \) is open in \( M \). Since \( M \) is connected, \( S^2 - K_N \) is identically a positive constant \( A \) on \( M \). It follows from this fact and (5.3) that \( K \) is identically zero, which contradicts \( K \neq 0 \) on \( M \).

By Lemma 3, if \( m \)-index \( p ; M \neq 0 \) at every point \( p \in M \), then we can choose a neighborhood \( U \) of a point \( p \in M \) in which there exist isothermal coordinates and frame fields satisfying (5.1) and

\[
(5.4) \quad \begin{cases}
\omega_{13} = k_1 \omega_1 = \omega_{24}, \\
\omega_{14} = \omega_{23} = -k_1 \omega_2 = -\omega_{14}, \\
\omega_{23} = -k_1 \omega_2, \\ 4 < \beta,
\end{cases}
\]

where \( k_1 \) is a positive differentiable function \( M \). Using the structure equations, from (5.4) we have

\[
\omega_{24} = 2 \omega_{13} - (\log k_1)_4 \omega_1 + (\log k_1)_4 \omega_2,
\]

where \( d(\log k_1) = \sum_{j=1}^2 (\log k_1)_{i} \omega_j \). Furthermore, from (5.4) we may write

\[
(5.5) \quad \omega_{23} = f_\beta \omega_1 + g_\beta \omega_2, \quad \omega_{48} = g_\beta \omega_1 - f_\beta \omega_2, \quad 4 < \beta,
\]

and define two normal vector fields \( F_2 = \sum f_\beta \varepsilon_\beta \) and \( G_2 = \sum g_\beta \varepsilon_\beta \) on \( U \).
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Then, we can write the second shape operator \( \overline{\varphi}_2 \) as

\[
\overline{\varphi}_2(X) = k_1 \{ \cos 3\theta \cdot F_3 + \sin 3\theta \cdot G_3 \}, \quad X = e_1 \cos \theta + e_2 \sin \theta \in M_p.
\]

Using the structure equations, from (5.5) and (5.6) we see that the complex-valued function

\[
w_1(z, \bar{z}) = E^3 k_1^3 \left( \|G_3\|^2 - \|F_3\|^2 \right) + 2i E^3 k_1^3 \langle F_3, G_3 \rangle
\]

is holomorphic in \( z \). As stated in §2, we see that \( |w_1(z, \bar{z})|^2/E^8 = k_1^2 \left( \|G_3\|^2 - \|F_3\|^2 \right)^2 + 4 \langle G_3, F_3 \rangle^2 \) is a differentiable function on \( M \). Hence, by the same reason as the proof of Lemma 3, we can prove

**Lemma 4.** If \( m\)-index, \( \mu M \neq 0 \) at every point \( p \in M \), then, at each point \( p \in M \), the image of \( S_0 \) under the second shape operator is a point \( p \) or a circle according as \( T_{r,p} \)-index, \( \mu M = 0 \) or \( \neq 0 \).

By Lemma 3 and Lemma 4, if \( m\)-index, \( \mu M \neq 0 \) and \( T_{r,p} \)-index, \( \mu M \neq 0 \) at every point \( p \in M \), then we can choose a neighborhood \( U \) of a point \( p \in M \) in which there exist isothermal coordinates \((u, v)\) and a frame field \( b \in B \) satisfying (5.1), (5.4) and

\[
\begin{align*}
\omega_{35} &= k_2 \omega_1 = \omega_{46}, & \omega_{34} &= \omega_{45} = 0, \\
\omega_{36} &= k_2 \omega_2 = -\omega_{45}, & 6 < \gamma,
\end{align*}
\]

where \( k_2 \) is a positive differentiable function on \( M \). Let \( \lambda_2 = k_1 k_2 \) and \( d(\log \lambda_2) = \sum_{j=1}^{3} (\log \lambda_2)_{\omega_j} \), then from (5.7) we have

\[
(5.8) \quad \omega_{48} = 3 \omega_{12} - (\log \lambda_2)_{\omega_1} + (\log \lambda_2)_{\omega_2}
\]

and we may write

\[
\omega_{57} = f_{1} \omega_1 + g_{1} \omega_2, \quad \omega_{56} = g_{1} \omega_1 - f_{1} \omega_2, \quad 6 < \gamma.
\]

Hence, for a unit tangent vector \( X = e_1 \cos \theta + e_2 \sin \theta \in M_p \), the third shape operator \( \overline{\varphi}_3 \) is written as

\[
\overline{\varphi}_3(X) = \lambda_2 \{ \cos 4\theta \cdot F_3 + \sin 4\theta \cdot G_3 \},
\]

where \( F_3 = \sum_{i \geq 4} f_i e_i \) and \( G_3 = \sum_{i \geq 4} g_i e_i \) are normal vector fields on \( U \). Continuing this way, we have the following

**Theorem 3.** Let \( M \) be a 2-dimensional, connected and compact Riemannian manifold of non-negative curvature \( (\equiv 0) \) which is minimally immersed in a \((2 + \nu)\)-dimensional Riemannian manifold \( \hat{M} \) of constant curvature.
curvature $c$. If we have

(A) the image of $M$ is not contained in a totally geodesic submanifold of $\hat{M}$, i.e., $\nu$ is the geodesic codimension of $M$,

(B) $m$-index $pM \neq 0$ at every point $p \in M$,

(C) $T_n$-index $pM (n = 1, 2, \cdots)$ are defined at every point $p \in M$ and

$T_n$-index $pM \neq 0$ at every point $p \in M$ for $n = 1, 2, \cdots, [\frac{1}{2}] - 1$,

then $\nu$ must be equal to $2m$ and the image of unit tangent circles to $M$
under the $n$-th shape operators $\bar{\varphi}_n (1 \leq n \leq m)$ are circles.

**Proof.** For the (first) shape operator and the second shape operator,
we have proved our latter assertion in Lemma 3 and Lemma 4. By the
induction on $n$, we shall prove that the image of a unit tangent circle $S_p$
under the $n$-th shape operators ($1 \leq n \leq [\frac{1}{2}]$) are circles for every $p \in M$.

Now, we assume that the above assertion holds for all $t \leq s - 1$. Then,
we can choose a neighborhood $U$ of a point $p \in M$ in which there exist
isothermal coordinates $(u, v)$ and a frame field $b \in B$ satisfying (5.1) and

$$
\begin{align*}
\omega_{a,b_1} &= k_1 \omega_1 = \omega_{a,b_2}, & \omega_{a,t} &= \omega_{a,B} = 0, \\
\omega_{a,b_2} &= k_0 \omega_2 = -\omega_{a,B}, & 2t + 2 < \gamma, \\
\alpha_1 &= 2t - 1, & \alpha_2 &= 2t, & \beta_1 &= 2t + 1, & \beta_2 &= 2t + 2, \\
& & t &= 1, 2, \cdots, s - 1,
\end{align*}
$$

(5.9)

where $k_t (1 \leq t \leq s - 1)$ are positive differentiable functions on $M$. Using
the structure equations, from (5.9) we have

$$
\omega_{a,b_2} = (t + 1) \omega_{a_2} - (\log \lambda_2) \omega_1 + (\log \lambda_1) \omega_2,
$$

(5.10)_{s-1}

where $\beta_1 = 2t + 1, \beta_2 = 2t + 2, \lambda_i := k_1 \cdot k_2 \cdots \cdot k_t$ and $d(\log \lambda_i) = \sum_{j=2}^s (\log \lambda_j) \omega_j$ for $t \leq 1, 2, \cdots, s - 1$. From (5.9), we may write

$$
\begin{align*}
\omega_{a,t} &= f_t \omega_1 + g_t \omega_2, & a_1 &= 2s - 1, \\
\omega_{a,B} &= g_t \omega_1 - f_t \omega_2, & a_2 &= 2s, & 2s < \gamma.
\end{align*}
$$

Hence, for a unit tangent vector $X = e_1 \cos \theta + e_2 \sin \theta \in M_p$, the $s$-th
shape operator $\bar{\varphi}_s$ is written as

$$
\bar{\varphi}_s (X) = \lambda_{s-1} \{ \cos (s + 1) \theta \cdot F_s + \sin (s + 1) \theta \cdot G_s \},
$$

where $F_s = \sum_{t<s} f_t e_t$ and $G_s = \sum_{t>s} g_t e_t$ are normal vector fields on a neighborhood $U$ of $p \in M$. Using the structure equations, from (5.9) and (5.10)_{s-1}
we can verify the complex-valued function

http://escholarship.lib.okayama-u.ac.jp/mjou/vol17/iss1/2

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(5.11) \( w_{t-1}(z, \bar{z}) = E^{t+1} \lambda_{t-1}^1 \left( \|G\|^2 - \|F_t\|^2 \right) + 2iE^{t+1} \lambda_{t-1}^2 < G_n, F_t > \)

is holomorphic in \( z = u + iv \). Since \( \lambda_{t-1} \) is a differentiable function on \( M \), as be stated in §2, \( |w_{t-1}(z, \bar{z})|^2/E^{t+2} = \lambda_{t-1}^1 \left( \|G\|^2 - \|F_t\|^2 \right)^2 + 4 < G_n, F_t > \) is a differentiable function on \( M \). Therefore, by the same way as the proof of Lemma 3, we see that at each point \( p \in M \) the image of a unit tangent circle \( S_p \) to \( M \) under the \( s \)-th shape operator \( \tilde{\nabla}_s \) is a point \( p \) or a circle according as \( T_{s+1} \)-index, \( M \neq 0 \) or \( \neq 0 \). Since \( T_{s+1} \)-index, \( M \neq 0 \) at every point \( p \in M \) if \( s \leq \left[ \frac{\nu}{2} \right] \) from (C), at each point \( p \in M \) we can choose a neighborhood \( U \) of a point \( p \) in which there exist isothermal coordinates \( (u, v) \) and a frame field \( b \in B \) satisfying (5.1), (5.4), (5.9) for every \( t \leq s \) and

\[
\begin{align*}
\omega_{u_1 b_1} &= k_1 \omega_{u_1} = \omega_{u_2 b_2}, \\
\omega_{u_1 t} &= 0, \\
\omega_{u_2 b_2} &= k_2 \omega_{u_2} = -\omega_{u_1 b_2}, \\
b_1 &= 2s + 1, \\
b_2 &= 2s + 2, \\
2s + 2 &< \nu,
\end{align*}
\]

where \( k_t \) is a positive differentiable function on \( M \).

Thus, it is clear that the geodesic codimension \( \nu \) of \( M \) is even \( 2m \) (\( m \) a positive integer).

Using the structure equations, from (5.4), (5.9) and (5.10), we obtain

(5.12) \[
\begin{align*}
\Delta(\log \lambda_t) &= E \left( (t+1)K - 2k_t^1 + 2k_t^{1+1} \right), \\
\Delta(\log \lambda_m) &= E \left( (m+1)K - 2k_m^1 \right),
\end{align*}
\]

where \( \Delta = \partial^2/\partial u^2 + \partial^2/\partial v^2 \). From (5.12) we have

(5.13) \[
\Delta \log (\lambda_1 \cdot \lambda_2 \cdots \lambda_m) = E \left\{ \frac{m(m+3)}{2} K - 2k_1^1 \right\} = E \left\{ \frac{(\nu+2)(\nu+4)}{8} K - c \right\},
\]

because \( K = c - 2k_1^1 \) and \( \nu = 2m \). Therefore, if \( (\nu+2)(\nu+4) K - 8c \) does not change its sign, \( \log (\lambda_1 \cdot \lambda_2 \cdots \lambda_m) \) is a subharmonic or superharmonic function on \( M \), so it must be constant, because \( M \) is compact. Hence, \( K = 8c/(\nu+2)(\nu+4) \) is constant \( \geq 0 \) and so \( k_t (1 \leq t \leq m) \) are constant on \( M \). Supposing \( K = 1 \), from (5.12) we get

\[
k_t^1 = (m-t+1)(m+t+2)/4 \quad \text{for} \quad t = 1, 2, \ldots, m,
\]

\[
c = (m+1)(m+2)/2.
\]

Let \( E_t = e_{2t+1} + ie_{2t+2}, \ t = 0, 1, 2, \ldots, m \). Then, the Frenet formulas of \( M \) can be written as follows

\[
dx = \frac{1}{h} \left( E_0 dz + E_0 d\bar{z} \right), \quad z = u + iv,
\]
T. ITOH

\[ DE_0 = \frac{1}{h} E_0 (\ddz - zd\z) + \frac{2k_1}{h} E_1 d, \]
\[ DE_1 = -\frac{2k_1}{h} E_1 dz + \frac{2}{h} E_1 (\ddz - zd\z) + \frac{2k_2}{h} E_2 d\z, \]
\[ DE_t = -\frac{2k_t}{h} E_{t-1} dz + \frac{t+1}{h} E_t (\ddz - zd\z) + \frac{2k_{t+1}}{h} E_{t+1} d, \]
\[ DE_m = -\frac{2k_m}{h} E_{m-1} dz + \frac{m+1}{h} E_m (\ddz - zd\z), \]

where \( D \) denotes the covariant differentiation of \( \hat{M} \) and \( h = 1 + z\z \). Now, let \( \hat{M} \) be a \((2m + 2)\)-dimensional sphere \( S^{2m+2}(R) \) with radius \( R = 1/\sqrt{c} = \sqrt{2/(m+1)(m+2)} \). We may consider \( S^{2m+2}(R) \subseteq E^{2m+3} \) and put \( x = Re_{m+1} \). By an computation analogous to the one in [7], we can verify that \( M \) is a generalized Veronese surface. Thus, we have proved

**Theorem 4.** If the assumptions of Theorem 3 are satisfied and \((\nu+2)(\nu+4)K-8c\) does not change its sign, then \( K \) is a positive constant on \( \hat{M} \). Let \( K = 1 \) and \( \hat{M} \) be a \((\nu+2)\)-dimensional sphere of constant curvature \((m+1)(m+2)/2\). Then \( M \) is a generalized Veronese surface.

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